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Linear symplectomorphisms as R-Lagrangian subspaces

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The graph of a real linear symplectomorphism is an R-Lagrangian subspace of a complex symplectic vector space. The restriction of the complex symplectic form is thus purely imaginary and may be expressed in terms of the generating function of the transformation. We provide explicit formulas; moreover, as an application, we give an explicit general formula for the metaplectic representation of the real symplectic group.

1. Introduction

1.1. ***Overview.*** As part of our symplectic upbringing, our ancestors impressed upon us the Symplectic Creed:

Everything is a Lagrangian submanifold [Weinstein 1981].

Obviously false if taken literally, rather than a "creed" it might be called the Maslow–Weinstein hammer, or, in French, *la déformation professionnelle symplectique*, saying that "if all you have is a [symplectic form], everything looks like a [Lagrangian submanifold]," or, in other words, to a symplectic geometer, everything should be expressed in terms of Lagrangian submanifolds. In this paper we consider a vector space endowed with *two* symplectic forms, namely the real and imaginary parts $\operatorname{Re}\omega^{\mathbb{C}}$ and $\operatorname{Im}\omega^{\mathbb{C}}$ of a complex symplectic form $\omega^{\mathbb{C}}$, and begin with the simple observation that

> *Not every Lagrangian submanifold* [with respect to $\operatorname{Re}\omega^{\mathbb{C}}$] *is a Lagrangian submanifold* [with respect to $\operatorname{Im}\omega^{\mathbb{C}}$].

We study its implications for the classification of real linear symplectomorphisms $\mathcal{H}$, as the graph of $\mathcal{H}$ is essentially by definition a Lagrangian subspace with respect to $\operatorname{Re}\omega^{\mathbb{C}}$; we ask, with some abuse of language:

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Open problem. Is every $2n \times 2n$ skew-symmetric matrix of the form $\operatorname{Im} \omega^{\mathbb{C}}|_{\operatorname{graph} \mathcal{H}}$ for some $\mathcal{H}$?

We believe that an answer would shed some light on the structure of linear symplectomorphisms. While our primary reason for writing this article is to precisely formulate the above open problem, which we do in Section 1.2, our primary technical result is to rewrite it in terms of generating functions; after all, if one guiding principle is the Symplectic Creed, another is that "symplectic topology is the geometry of generating functions" [Viterbo 1992]. Or, to go further back, while Sir William Rowan Hamilton first conceived of generating functions (or as he called them, *characteristic functions*) as mathematical tools in his symplectic formulation of optics, he later found, in his symplectic formulation of classical mechanics, that the generating function for a physical system is the least action function, in a sense that we will not make precise [Abraham and Marsden 1978; Hamilton 1834]; this gives a striking connection with the calculus of variations. Moreover, in Fresnel optics and quantum mechanics, the generating function is used as the phase function of an oscillatory integral operator; the integral operator is said to "quantize" the corresponding symplectomorphism [Grigis and Sjöstrand 1994; Guillemin and Sternberg 1984]. (Loosely speaking, when differentiating the integral, one finds that the phase function must satisfy the Hamilton–Jacobi equation.) This topic will be touched upon in Section 3. For us, the generating function corresponding to the linear symplectomorphism $\mathcal{H}$ is the scalar-valued function Φ in our main theorem:

Theorem 1. *For each $\mathcal{H} \in \operatorname{Sp}(2n, \mathbb{R})$ there exists a quadratic form $\Phi : \mathbb{C}^n_z \times \mathbb{R}^{2n}_\theta \to \mathbb{R}$ such that*

$$\operatorname{graph}_{\mathbb{C}} \mathcal{H} = \left\{ \left(z, -2\frac{\partial \Phi}{\partial z}(z, \theta) \right) : \frac{\partial \Phi}{\partial \theta}(z, \theta) = 0 \right\},$$

and the restriction of $\omega^{\mathbb{C}}$ to $\operatorname{graph}_{\mathbb{C}} \mathcal{H}$ is given by

$$\omega^{\mathbb{C}}\left(\left(z, -2\frac{\partial \Phi}{\partial z}(z, \theta) \right), \left(w, -2\frac{\partial \Phi}{\partial z}(w, \eta) \right) \right) \\ = 2\sum_{j=1}^{n} \sum_{\ell=1}^{2n} \frac{\partial^2 \Phi}{\partial z_j \partial \theta_\ell}(z_j \eta_\ell - w_j \theta_\ell) + 2 \sum_{j,m=1}^{n} \frac{\partial^2 \Phi}{\partial z_j \partial \bar{z}_m}(z_j \overline{w}_m - w_j \bar{z}_m). \quad (1)$$

Moreover, our construction provides an explicit general formula for Φ.

Our notation will be explained in the following subsection, along with the necessary background and a restatement of the open problem. We prove the theorem in Section 2, and in Section 3 we show how our construction seems to adequately answer a question of Folland [1989] regarding the metaplectic representation. We conclude with a broad indication of future work. In the Appendix we give additional linear-algebraic background and some new elementary results relevant to our problem, and also give an additional restatement of our open problem.

1.2. ***Background and restatement of the problem.*** In a real symplectic vector space there is already a natural complex structure; the model example is $\mathbb{R}^{2n}$ with the $2n \times 2n$ matrix $\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, where of course $\mathcal{J}^2 = -I$. What we mean by "complex symplectic linear algebra" is something else; we instead consider $\mathbb{C}^{2n}$ with the above matrix $\mathcal{J}$, that is, we consider

$$\omega^{\mathbb{C}} = \sum_{j=1}^{n} d\zeta_j \wedge dz_j \quad \text{on } \mathbb{C}^n_z \times \mathbb{C}^n_\zeta$$

(a nondegenerate alternating bilinear form over $\mathbb{C}$). The basic formalism of complex symplectic linear algebra is not new; indeed, complex symplectic structures naturally appear in the theory of differential equations and have been studied through that lens (see, for example, [Schapira 1981] and [Sjöstrand 1982], or [Everitt and Markus 2004] for another perspective). The point of view of this paper is that elementary linear-algebraic aspects remain unexplored in the complex case and may help us better understand the real case.

A *symplectic vector space* over a field[1] K is by definition a pair (V, ω), where V is a finite-dimensional vector space over K and ω is a nondegenerate alternating bilinear form on V. The basic example is $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ with the symplectic form $\omega = \sum_{j=1}^n d\xi_j \wedge dx_j$:

$$\omega((x, \xi), (x', \xi')) = \sum_{j=1}^{n} (\xi_j x'_j - x_j \xi'_j). \tag{2}$$

In fact, this is essentially the only example: for a general symplectic vector space (V, ω) over a field K one can find a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ for V such that

$$\omega(e_j, e_k) = 0, \quad \omega(f_j, f_k) = 0, \quad \omega(f_j, e_k) = \delta_{jk} \qquad \text{for all } j, k.$$

Such a basis is called a *symplectic basis*, and ω is of the form (2) in these coordinates. (In particular, a symplectic vector space is necessarily even-dimensional.) Note that ω vanishes on the span of the e_j, and it vanishes on the span of the f_j; such a subspace is called a *Lagrangian subspace*: a maximal subspace on which ω vanishes. (A Lagrangian subspace of V is necessarily of dimension n.)

The symplectic formalism is fundamental in Hamiltonian mechanics: the symplectic form provides an isomorphism between tangent space and cotangent space,

[1] Duistermaat's book [1996] on Fourier integral operators contains a brief treatment of symplectic vector spaces over a general field.

mapping the Hamiltonian vector field of a function f to the differential of f:

$$df = \omega(\,\cdot\,, H_f).$$

A *linear symplectomorphism* T on (V, ω) is a linear isomorphism on V such that $T^*\omega = \omega$, that is,

$$\omega(Tv, Tv') = \omega(v, v') \quad \text{for all } v, v' \in V.$$

This is equivalent to the property that a symplectic basis is mapped to a symplectic basis.

We now let (V, ω) be a real symplectic vector space. Then

$$(V \times V, \omega \oplus -\omega)$$

is a real symplectic vector space. We write $\omega_0 = \omega \oplus -\omega$ so that, by definition,

$$\omega_0((v, w), (v', w')) = \omega(v, v') - \omega(w, w').$$

The following classical result (see [Tao 2012] for a broad perspective) justifies this choice of the symplectic form:

A map $\mathcal{H} : V \to V$ is a linear symplectomorphism if and only if its graph $\{(v, \mathcal{H}(v)) : v \in V\}$ is a Lagrangian subspace of $(V \times V, \omega_0)$.

For a basic example, let

$$\mathcal{H} : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{R}^n_y \times \mathbb{R}^n_\eta, \quad (x, \xi) \mapsto (y, \eta),$$

be a linear symplectomorphism. Then graph $\mathcal{H}$ is a Lagrangian subspace for

$$\omega^{\mathbb{R}} = \sum_{j=1}^{n} d\xi_j \wedge dx_j - d\eta_j \wedge dy_j.$$

The point of view of this paper is to consider graph $\mathcal{H}$ as an $\mathbb{R}$-linear subspace of a complex symplectic vector space. After all, with $z_j = x_j + iy_j$ and $\zeta_j = \xi_j + i\eta_j$, we have the complex symplectic form

$$\omega^{\mathbb{C}} = \sum_{j=1}^{n} d\zeta_j \wedge dz_j \quad \text{on } \mathbb{C}^n_z \times \mathbb{C}^n_\zeta,$$

which induces the two *real* symplectic forms

$$\operatorname{Re}\omega^{\mathbb{C}} = \sum_{j=1}^{n} d\xi_j \wedge dx_j - d\eta_j \wedge dy_j, \quad \operatorname{Im}\omega^{\mathbb{C}} = \sum_{j=1}^{n} d\xi_j \wedge dy_j + d\eta_j \wedge dx_j$$

on $\mathbb{R}^{2n}_{x,\xi} \times \mathbb{R}^{2n}_{y,\eta}$. We then say that an $\mathbb{R}$-linear $2n$-dimensional subspace of $\mathbb{R}^{2n}_{x,\xi} \times \mathbb{R}^{2n}_{y,\eta}$ is an *R-Lagrangian subspace* if it is Lagrangian with respect to $\operatorname{Re}\omega^{\mathbb{C}}$, and an

I-Lagrangian subspace if it is Lagrangian with respect to $\operatorname{Im}\omega^{\mathbb{C}}$. Thus the graph of $\mathcal{H} : \mathbb{R}^{2n}_{x,\xi} \to \mathbb{R}^{2n}_{y,\eta}$ may be considered as an R-Lagrangian subspace of $(\mathbb{C}^n_z \times \mathbb{C}^n_\zeta, \omega^{\mathbb{C}})$.

Writing a symplectic matrix $\mathcal{H} \in \mathrm{Sp}(2n, \mathbb{R})$ as $\mathcal{H} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, we have

$$\operatorname{graph}\mathcal{H} = \{((x, \xi), (Ax + B\xi, Cx + D\xi)) : (x, \xi) \in \mathbb{R}^{2n}\};$$

or, in terms of (z, ζ), we have

$$\operatorname{graph}_{\mathbb{C}}\mathcal{H} = \{(x + i(Ax + B\xi), \xi + i(Cx + D\xi)) : (x, \xi) \in \mathbb{R}^{2n}\}.$$

Thus

$$\omega^{\mathbb{C}}|_{\operatorname{graph}_{\mathbb{C}}\mathcal{H}} = i \;\operatorname{Im}\omega^{\mathbb{C}}|_{\operatorname{graph}\mathcal{H}}$$

is given by

$$\omega^{\mathbb{C}}\big((x+i(Ax+B\xi), \xi+i(Cx+D\xi)),\ (x'+i(Ax'+B\xi'), \xi'+i(Cx'+D\xi'))\big) \\ = i\begin{pmatrix} x^T & \xi^T \end{pmatrix}\begin{pmatrix} C^T - C & -A^T - D \\ A + D^T & B - B^T \end{pmatrix}\begin{pmatrix} x' \\ \xi' \end{pmatrix}.$$

The symplectic form $\operatorname{Re}\omega^{\mathbb{C}}$ vanishes, but the symplectic form $\operatorname{Im}\omega^{\mathbb{C}}$ might *not* vanish; that is, $\operatorname{graph}_{\mathbb{C}}\mathcal{H}$ is R-Lagrangian but not necessarily I-Lagrangian.

We have thus defined a map from the group of symplectic matrices to the space of skew-symmetric matrices

$$\mathfrak{X} : \mathrm{Sp}(2n, \mathbb{R}) \to \mathfrak{so}(2n, \mathbb{R}), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} C^T - C & -A^T - D \\ A + D^T & B - B^T \end{pmatrix}.$$

We can thus restate our open problem:

Open problem. Is the map $\mathfrak{X} : \mathrm{Sp}(2n, \mathbb{R}) \to \mathfrak{so}(2n, \mathbb{R})$ a surjection?

While we do not solve this problem, the main result of the paper is Theorem 1; we can explicitly construct a generating function Φ for $\mathcal{H}$ and thus give an alternate characterization of $\omega^{\mathbb{C}}|_{\operatorname{graph}_{\mathbb{C}}\mathcal{H}}$ and hence of $\mathfrak{X}$.

2. In terms of generating functions: the proof of the theorem

Generating functions (in the sense of symplectic geometry) were discovered by Sir William Rowan Hamilton in his extensive work on optics. In modern language (and in the linear case), light rays are specified by the following data: $\mathbb{R}^2_x$ is a plane of initial positions perpendicular to the optical axis of the system, $\xi \in \mathbb{R}^2$ are the initial "directions" (multiplied by the index of refraction), $\mathbb{R}^2_y$ is a plane of terminal positions, and $\eta \in \mathbb{R}^2$ are the terminal directions. The spaces $\mathbb{R}^4_{x,\xi}$ and $\mathbb{R}^4_{y,\eta}$ are given the standard symplectic structures. Taken piece by piece, the optical system consists of a sequence of reflections and refractions for each light ray, the laws of which were long known; Hamilton's discovery was that, taken as a whole, the optical system is determined by a single function, the *generating function*, or, as Hamilton called it, the *characteristic function*, of the optical system. The transformation from initial conditions to terminal conditions is a symplectomorphism expressible in terms of a single scalar-valued function, "by which means optics acquires, as it seems to me, an uniformity and simplicity in which it has been hitherto deficient" [Hamilton 1828, Section IV, Paragraph 20].[2]

The optical framework gives an intuitive reason why, in the symplectic matrix $\mathcal{H} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, the rank of B plays a special role in characterizing $\mathcal{H}$ and thus its generating function. Again, $\mathcal{H}$ maps the initial (position, direction)-pair (x, ξ) to the terminal (position, direction)-pair

$$\begin{pmatrix} y \\ \eta \end{pmatrix} = \begin{pmatrix} Ax + B\xi \\ Cx + D\xi \end{pmatrix}.$$

The case $B = 0$ corresponds to perfect focusing: all the rays from a given position x arrive at the same position y, resulting in a perfect image. And the case $\det B \neq 0$ corresponds to *no* such focusing: two rays with initial position x but different initial directions must arrive at different positions y. (See [Guillemin and Sternberg 1984] for an exposition of symplectic techniques in optics.)

2.1. ***When B is invertible.*** We recall that

$$\operatorname{graph}_{\mathbb{C}} \mathcal{H} = \bigl\{\bigl(x + i(Ax + B\xi),\ \xi + i(Cx + D\xi)\bigr) : (x, \xi) \in \mathbb{R}^{2n}\bigr\},$$

taken over the reals, is an R-Lagrangian subspace of $(\mathbb{C}^n_z \times \mathbb{C}^n_\zeta, \omega^{\mathbb{C}})$, and we note that

$$\pi : \operatorname{graph}_{\mathbb{C}} \mathcal{H} \to \mathbb{C}^n, \quad (z, \zeta) \mapsto z,$$

is an $\mathbb{R}$-linear transformation whose kernel is given by $(x, \xi) \in \{0\} \times \ker B$. Thus it is an $\mathbb{R}$-linear isomorphism if and only if B is invertible. In this case, the general theory of symplectic geometry gives the existence of a real C^∞ function Φ defined on $\operatorname{graph}_{\mathbb{C}} \mathcal{H}$ such that

$$\operatorname{graph}_{\mathbb{C}} \mathcal{H} = \left\{\left(z, -2\frac{\partial \Phi}{\partial z}(z)\right) : z \in \mathbb{C}^n\right\}.$$

[2] There are different types of generating functions in symplectic geometry, and, as Arnold writes, "[the apparatus of generating functions] is unfortunately noninvariant and it uses, in an essential way, the coordinate structure in phase space" [Arnold 1978, Section 47]. For our purposes, we may take the term "generating function" to broadly mean a scalar-valued function which generates a symplectomorphism (or, more generally, a Lagrangian submanifold) in the same sense that a potential function generates a conservative vector field. Our generating functions are denoted by the symbol Φ below.

Hence if $\det B \neq 0$, then

$$\begin{aligned}\operatorname{graph}_{\mathbb{C}} \mathcal{H} &= \bigl\{\bigl(x+i(Ax+B\xi),\ \xi+i(Cx+D\xi)\bigr) : (x,\xi)\in\mathbb{R}^{2n}\bigr\}\\ &= \bigl\{\bigl(z, -2(\partial\Phi/\partial z)(z)\bigr) : z\in\mathbb{C}^n\bigr\}\\ &= \bigl\{\bigl(p+iq,\ B^{-1}(q-Ap)+i(Cp+DB^{-1}(q-Ap))\bigr) : p+iq\in\mathbb{C}^n\bigr\},\end{aligned}$$

where we write $z = p+iq$, so that

$$\Phi(p,q) = \tfrac{1}{2}p^T B^{-1}Ap - p^T B^{-1}q + \tfrac{1}{2}q^T DB^{-1}q. \tag{3}$$

This function appears in [Folland 1989, Equation (4.54)] and in [Guillemin and Sternberg 1984, Section 11]. (Note that $B^{-1}A$ and DB^{-1} are symmetric since $\mathcal{H}$ is symplectic.) Substituting $p = x$ and $q = Ax + B\xi$, we arrive at the following expression, with the obvious abuse of notation:

$$\Phi(x,\xi) = \tfrac{1}{2}x^T A^T Cx + x^T C^T B\xi + \tfrac{1}{2}\xi^T B^T D\xi. \tag{4}$$

Or, writing Φ with respect to z and $\bar{z}$, we have

$$\begin{aligned}\Phi(z) = {}&\tfrac{1}{8}z^T(B^{-1}A + 2iB^{-1} - DB^{-1})z\\ &+\tfrac{1}{4}\bar{z}^T(B^{-1}A - i(B^T)^{-1} + iB^{-1} + DB^{-1})z\\ &+\tfrac{1}{8}\bar{z}^T(B^{-1}A - 2iB^{-1} - DB^{-1})\bar{z}.\end{aligned}$$

Thus

$$\left(\frac{\partial^2\Phi}{\partial z_j\partial\bar{z}_k}\right) = \tfrac{1}{4}(B^{-1}A - iB^{-1} + i(B^T)^{-1} + DB^{-1}).$$

We can directly compute $\omega^{\mathbb{C}}$ restricted to $\operatorname{graph}_{\mathbb{C}}\mathcal{H}$ in terms of z and $\bar{z}$:

$$\begin{aligned}\omega^{\mathbb{C}}\left(\left(z, -2\frac{\partial\Phi}{\partial z}(z)\right),\ \left(z', -2\frac{\partial\Phi}{\partial z}(z')\right)\right) &= 4i\ \operatorname{Im}\left(\sum_{j,k} z_j\left(\frac{\partial^2\Phi}{\partial z_j\partial\bar{z}_k}\right)\bar{z}'_k\right)\\ &= 2\sum_{j,k}\frac{\partial^2\Phi}{\partial z_j\partial\bar{z}_k}(z_j\bar{z}'_k - z'_j\bar{z}_k).\end{aligned}$$

If we substitute

$$\begin{aligned} z &= x + i(Ax + B\xi),\\ z' &= x' + i(Ax' + B\xi'),\end{aligned}$$

then after a lengthy mechanical calculation we recover the expression

$$\begin{aligned}&\omega^{\mathbb{C}}\left(\left(z, -2\frac{\partial\Phi}{\partial z}(z)\right),\ \left(z', -2\frac{\partial\Phi}{\partial z}(z')\right)\right)\\ &\qquad = 2\sum_{j,k}\frac{\partial^2\Phi}{\partial z_j\partial\bar{z}_k}(z_j\bar{z}'_k - z'_j\bar{z}_k) = i\begin{pmatrix}x^T & \xi^T\end{pmatrix}\mathfrak{X}(\mathcal{H})\begin{pmatrix}x'\\ \xi'\end{pmatrix}.\end{aligned}$$

2.2. ***When B is not invertible.*** When B is *not* invertible, we seek

$$\Phi = \Phi(z,\theta) \in C^\infty(\mathbb{C}^n \times \mathbb{R}^N)$$

such that

$$\operatorname{graph}_{\mathbb{C}} \mathcal{H} = \left\{ \left(z, -2\frac{\partial \Phi}{\partial z}(z,\theta) \right) : \frac{\partial \Phi}{\partial \theta}(z,\theta) = 0 \right\}. \tag{5}$$

We follow the general method outlined by Guillemin and Sternberg [1977].

Let

$$W = \operatorname{graph}_{\mathbb{C}} \mathcal{H}, \quad X = \{(z,0); z \in \mathbb{C}^n\}, \quad Y = \{(0,\zeta); \zeta \in \mathbb{C}^n\}.$$

Since W is an R-Lagrangian subspace, we know that $W \cap Y$ and $PW \subset X$ are orthogonal with respect to $\operatorname{Re}\omega^{\mathbb{C}}$, where P is the projection onto X along Y. Indeed,

$$W \cap Y = \{(0, \xi + iD\xi) : \xi \in \ker B\}, \quad PW = \{(x + i(Ax + B\xi), 0) : (x,\xi) \in \mathbb{R}^{2n}\},$$

and we can check directly that, with $\xi \in \ker B$,

$$\omega^{\mathbb{C}}((0, \xi + iD\xi), (x' + i(Ax' + B\xi'), 0)) = i\left[\xi^T(A + D^T)x' + \xi^T B\xi'\right].$$

Since $\operatorname{graph}_{\mathbb{C}} \mathcal{H}$ is not a $\mathbb{C}$-linear subspace but an $\mathbb{R}$-linear subspace, for now we prefer to write

$$W \cap Y = \{(0, \xi; 0, D\xi) : \xi \in \ker B\},$$
$$PW = \{(x, 0; Ax + B\xi, 0) : (x,\xi) \in \mathbb{R}^{2n}\}.$$

We note that $PW \oplus (W \cap Y)$ has real dimension $2n$, hence is a Lagrangian subspace of $(\mathbb{R}^{4n}, \operatorname{Re}\omega^{\mathbb{C}})$.

We seek to write $\operatorname{graph}\mathcal{H}$ as the graph of a function from $PW \oplus (W \cap Y)$ to a complementary Lagrangian subspace; as a first step, we choose a convenient symplectic basis. We let $\{b_1, \ldots, b_k\}$ be an orthonormal basis for $\ker B$ and extend to an orthonormal basis $\{b_1, \ldots, b_n\}$ for $\mathbb{R}^n$, so that

$$\{(0, b_j; 0, Db_j) : j = 1, \ldots, k\}$$

is a basis for $W \cap Y$, and

$$\{(0, 0; Bb_j, 0) : j = k+1, \ldots, n\} \cup \{(b_j, 0; Ab_j, 0) : j = 1, \ldots, n\}$$

is a basis for PW. We then extend to the following symplectic basis for $(\mathbb{R}^{4n}, \operatorname{Re}\omega^{\mathbb{C}})$:

$$\begin{aligned} \{(0,0; Ab_j, 0) : j = 1, \ldots, k\} &\leftrightarrow \{(0, b_j; 0, Db_j) : j = 1, \ldots, k\}, \\ \{(0,0; Bb_j, 0) : j = k+1, \ldots, n\} &\leftrightarrow \{(0, A^T\beta_j; 0, \beta_j) : j = k+1, \ldots, n\}, \\ \{(b_j, 0; Ab_j, 0) : j = 1, \ldots, n\} &\leftrightarrow \{(0, -b_j; 0, 0) : j = 1, \ldots, n\}, \end{aligned} \tag{6}$$

where the $\{\beta_j\}_{j=k+1}^n$ satisfy

$$\begin{cases} A^T\beta_j \in (\ker B)^\perp = \operatorname{Im} B^T, \\ b_J \cdot B^T\beta_j = \delta_{Jj} \quad \text{for all } J \in \{k+1, \dots, n\}. \end{cases} \tag{7}$$

One advantage of using the particular symplectic basis (6) is that the vectors on the left are all "horizontal," and the vectors on the right are all "vertical". (The arrows signify the symplectically dual pairs.)

The following proposition implies the existence of $\{\beta_j\}_{j=k+1}^n$.

Proposition 2. *The set* $\{Ab_1, \dots, Ab_k, Bb_{k+1}, \dots, Bb_n\}$ *is a basis for* $\mathbb{R}^n$.

Proof. Suppose

$$\sum_{j=1}^k \alpha_j Ab_j + \sum_{j=k+1}^n \alpha_j Bb_j = 0.$$

We take the dot product with Db_J, $J \in \{1, \dots, k\}$, to get $\alpha_1 = \cdots = \alpha_k = 0$, and the rest are zero by the linear independence of $\{Bb_{k+1}, \dots, Bb_n\}$. □

Thus for $J \in \{k+1, \dots, n\}$ we can take β_J to be the unique vector orthogonal to the set

$$\{Ab_1, \dots, Ab_k, Bb_{k+1}, \dots, \widehat{Bb_J}, \dots, Bb_n\}$$

(where the wide hat denotes omission) and satisfying

$$\beta_J \cdot Bb_J = 1.$$

We will now describe graph $\mathcal{H}$ in terms of the above symplectic coordinate system: we write a general linear combination of the $4n$ vectors and find necessary and sufficient conditions on the coefficients to make the vector in graph $\mathcal{H}$. Explicitly, we write the general vector in $\mathbb{R}^{4n}$ as

$$\sum_{j=1}^k t_j'(0,0;Ab_j,0) + \sum_{j=k+1}^n t_j''(0,0;Bb_j,0) + \sum_{j=1}^n t_{n+j}''(b_j,0;Ab_j,0) + \sum_{j=1}^k \theta_j'(0,b_j;0,Db_j) + \sum_{j=k+1}^n \theta_j''(0,A^T\beta_j;0,\beta_j) + \sum_{j=1}^n \theta_{n+j}''(0,-b_j;0,0) \tag{8}$$

(the primes are not necessary but are useful for bookkeeping), and we will describe graph $\mathcal{H}$ as (t', θ'') as a function of (t'', θ').

We have the following necessary and sufficient conditions for the vector (8) to be in graph $\mathcal{H}$:

$$\sum_{j=1}^{k} t'_j A b_j - \sum_{j=k+1}^{n} \theta''_j A B^T \beta_j + \sum_{j=k+1}^{n} \theta''_{n+j} B b_j$$
$$= - \sum_{j=k+1}^{n} t''_j B b_j - \sum_{j=k+1}^{n} \theta''_j C B^T \beta_j + \sum_{j=1}^{n} \theta''_{n+j} D b_j = \sum_{j=1}^{n} t''_{n+j} C b_j.$$

In matrix form, this says:

$$\begin{pmatrix} | & & | & | & & | & | & & | \\ Ab_1 & \cdots & Ab_k & (-AB^T\beta_{k+1}) & \cdots & (-AB^T\beta_n) & Bb_1 & \cdots & Bb_n \\ | & & | & | & & | & | & & | \\ & & & | & & | & | & & | \\ & 0_{n,k} & & (-CB^T\beta_{k+1}) & \cdots & (-CB^T\beta_n) & Db_1 & \cdots & Db_n \\ & & & | & & | & | & & | \end{pmatrix} \begin{pmatrix} t' \\ \\ \theta'' \end{pmatrix}$$
$$= \begin{pmatrix} | & & | & & & \\ (-Bb_{k+1}) & \cdots & (-Bb_n) & & 0_{n,n} & \\ | & & | & & & \\ & & & | & & | \\ & 0_{n,n-k} & & Cb_1 & \cdots & Cb_n \\ & & & | & & | \end{pmatrix} \begin{pmatrix} \\ t'' \\ \\ \end{pmatrix}. \tag{9}$$

We would now like to invert the matrix on the left to get (t', θ'') as a function of (t'', θ'). Once we do that, we are close to our goal of expressing graph $\mathcal{H}$ in terms of a generating function Φ.

Letting Π denote the orthogonal projection onto $\ker B$, we find that the inverse of the matrix on the left side of (9) is

$$\begin{pmatrix} \text{---} & Db_1 & \text{---} & & & \\ & \vdots & & & 0_{k,n} & \\ \text{---} & Db_k & \text{---} & & & \\ \text{---} & D(\Pi C^T B - I)b_{k+1} & \text{---} & \text{---} & Bb_{k+1} & \text{---} \\ & \vdots & & & \vdots & \\ \text{---} & D(\Pi C^T B - I)b_n & \text{---} & \text{---} & Bb_n & \text{---} \\ \text{---} & (D\Pi A^T - I)Cb_1 & \text{---} & \text{---} & Ab_1 & \text{---} \\ & \vdots & & & \vdots & \\ \text{---} & (D\Pi A^T - I)Cb_n & \text{---} & \text{---} & Ab_n & \text{---} \end{pmatrix}.$$

Thus, defining the functions

$$f''_i(t'') = \sum_{j=k+1}^{n} [Bb_i \cdot Db_j] t''_j + \sum_{j=1}^{n} [Bb_i \cdot Cb_j] t''_{n+j} \quad \text{for } i = k+1, \dots, n,$$

$$f''_{n+i}(t'') = \sum_{j=k+1}^{n} [Cb_i \cdot Bb_j] t''_j + \sum_{j=1}^{n} [Ab_i \cdot Cb_j] t''_{n+j} \quad \text{for } i = 1, \dots, n,$$

we see that (9) is equivalent to the conditions $t' = 0,\ \theta'' = f''(t'')$. Noting that

$$\frac{\partial f''_i}{\partial t''_j} = \frac{\partial f''_j}{\partial t''_i} \quad \text{for all } i, j \in k+1, \dots, n,$$

and defining

$$F(t'') = \frac{1}{2} \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} t''_i [Bb_i \cdot Db_j] t''_j + \sum_{i=k+1}^{n} \sum_{j=1}^{n} t''_i [Bb_i \cdot Cb_j] t''_{n+j} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} t''_{n+i} [Ab_i \cdot Cb_j] t''_{n+j},$$

we conclude that the vector is in graph $\mathcal{H}$ if and only if

$$t' = 0, \quad \frac{\partial F}{\partial t''}(t'') = \theta''.$$

We now define

$$\varphi(t', t''; \theta', \theta'') = \theta' \cdot t' + F(t'') + (\theta'' - f''(t''))^2.$$

Then in $(t', t''; \theta', \theta'')$-coordinates, graph $\mathcal{H}$ is given as

$$\left\{ \left(t', t''; \frac{\partial \varphi}{\partial t'}, \frac{\partial \varphi}{\partial t''} \right) : \frac{\partial \varphi}{\partial \theta'} = 0,\ \frac{\partial \varphi}{\partial \theta''} = 0 \right\}.$$

Or, written in terms of the standard basis, graph $\mathcal{H}$ is the set of values of

$$\sum_{j=1}^{k} t'_j (0, 0; Ab_j, 0) + \sum_{j=k+1}^{n} t''_j (0, 0; Bb_j, 0) + \sum_{j=1}^{n} t''_{n+j} (b_j, 0; Ab_j, 0) + \sum_{j=1}^{k} \frac{\partial \varphi}{\partial t'_j}(t, \theta)(0, b_j; 0, Db_j) + \sum_{j=k+1}^{n} \frac{\partial \varphi}{\partial t''_j}(t, \theta)(0, A^T \beta_j; 0, \beta_j) + \sum_{j=1}^{n} \frac{\partial \varphi}{\partial t''_{n+j}}(t, \theta)(0, -b_j; 0, 0) \quad (10)$$

subject to the condition that $\frac{\partial \varphi}{\partial \theta}(t, \theta) = 0$.

We return to complex coordinates, in the standard basis; for that purpose we write the "horizontal" parts of (10) as

$$z := \sum_{j=1}^{k} i t'_j A b_j + \sum_{j=k+1}^{n} i t''_j B b_j + \sum_{j=1}^{n} t''_{n+j} (I + iA) b_j .$$

That is,

$$\operatorname{Re} z = \sum_{j=1}^{n} t''_{n+j} b_j ,$$
$$\operatorname{Im} z = \sum_{j=1}^{k} t'_j A b_j + \sum_{j=k+1}^{n} t''_j B b_j + \sum_{j=1}^{n} t''_{n+j} A b_j .$$

With the same notation as before, the inverse transformation is given by

$$\begin{aligned} t'_j &= -b_j \cdot \operatorname{Re} z + D b_j \cdot \operatorname{Im} z && \text{for } j \in \{1, \dots, k\}, \\ t''_j &= -A^T \beta_j \cdot \operatorname{Re} z + \beta_j \cdot \operatorname{Im} z && \text{for } j \in \{k+1, \dots, n\}, \\ t''_{n+j} &= b_j \cdot \operatorname{Re} z && \text{for } j \in \{1, \dots, n\}. \end{aligned} \tag{11}$$

We write the "vertical" part of (10) as:

$$\begin{aligned} \operatorname{Re} \zeta &= \sum_{j=1}^{k} \frac{\partial \varphi}{\partial t'_j}(t, \theta) b_j + \sum_{j=k+1}^{n} \frac{\partial \varphi}{\partial t''_j}(t, \theta) A^T \beta_j - \sum_{j=1}^{n} \frac{\partial \varphi}{\partial t''_{n+j}}(t, \theta) b_j , \\ \operatorname{Im} \zeta &= \sum_{j=1}^{k} \frac{\partial \varphi}{\partial t'_j}(t, \theta) D b_j + \sum_{j=k+1}^{n} \frac{\partial \varphi}{\partial t''_j}(t, \theta) \beta_j . \end{aligned} \tag{12}$$

Using $t = t(z)$ to denote the transformation (11), we define

$$\Phi(z, \theta) := \varphi(t(z), \theta),$$

so that (12) says

$$\zeta = -2 \, \frac{\partial \Phi}{\partial z} \, (z, \theta).$$

In summary, we now have the following expression for $\operatorname{graph}_{\mathbb{C}} \mathcal{H}$:

$$\operatorname{graph}_{\mathbb{C}} \mathcal{H} = \left\{ \left(z, -2 \frac{\partial \Phi}{\partial z}(z, \theta) \right) : \frac{\partial \Phi}{\partial \theta} \, (z, \theta) = 0 \right\}, \tag{13}$$

where the $\theta \in \mathbb{R}^{2n}$ are considered as auxiliary parameters, as in (5).

As for $\omega^{\mathbb{C}}|_{\operatorname{graph}_{\mathbb{C}} \mathcal{H}}$, we use the expression

$$\frac{\partial \Phi}{\partial z} \, (z, \theta) = \frac{\partial^2 \Phi}{\partial z \partial \theta} \cdot \theta + \frac{\partial^2 \Phi}{\partial z^2} \cdot z + \frac{\partial^2 \Phi}{\partial z \partial \bar{z}} \cdot \bar{z}$$

to compute

$$\begin{aligned}
&\omega^{\mathbb{C}}\Big(\Big(z, -2\frac{\partial \Phi}{\partial z}(z,\theta)\Big), \Big(w, -2\frac{\partial \Phi}{\partial z}(w,\eta)\Big)\Big) \\
&\qquad = 2z\cdot\frac{\partial \Phi}{\partial z}(w,\eta) - 2w\cdot\frac{\partial \Phi}{\partial z}(z,\theta) \\
&\qquad = 2\sum_{j=1}^{n}\sum_{\ell=1}^{2n}\frac{\partial^2\Phi}{\partial z_j\partial\theta_\ell}(z_j\eta_\ell - w_j\theta_\ell) + 2\sum_{j,m=1}^{n}\frac{\partial^2\Phi}{\partial z_j\partial\bar{z}_m}(z_j\overline{w}_m - w_j\bar{z}_m), \qquad (14)
\end{aligned}$$

where the variables are related by the conditions

$$\frac{\partial \Phi}{\partial \theta}(z,\theta) = 0 \qquad \text{and} \qquad \frac{\partial \Phi}{\partial \theta}(w,\eta) = 0.$$

Of course, from Section 1, we know that (14) is equal to

$$i\begin{pmatrix} x^T & \xi^T \end{pmatrix}\mathfrak{X}(\mathcal{H})\begin{pmatrix} x' \\ \xi' \end{pmatrix}, \qquad (15)$$

where

$$\begin{aligned}
z &= x + i(Ax + B\xi), & w &= x' + i(Ax' + B\xi'), \\
-2\frac{\partial \Phi}{\partial z}(z,\theta) &= \xi + i(Cx + D\xi), & -2\frac{\partial \Phi}{\partial z}(w,\eta) &= \xi' + i(Cx' + D\xi').
\end{aligned}$$

This completes the proof of the theorem.

We leave it as an illustrative exercise for the reader to compute Φ and its derivatives in the special cases when $B = 0$ and when B is invertible (to be compared with the generating function (3) in Section 2.1).

3. Application: the metaplectic representation

In the previous section, we showed how to associate to a linear symplectomorphism $\mathcal{H}$ a (real-valued) generating function Φ. For the purposes of Fresnel optics and quantum mechanics one then associates to the generating function Φ an oscillatory integral operator

$$\mu(\mathcal{H}): \quad \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), \quad u \mapsto a\,h^{-3n/2}\iint e^{i\Phi(x+iy,\theta)/h}u(x)\,dx\,d\theta. \qquad (16)$$

The map $\mu : \mathcal{H} \to \mu(\mathcal{H})$ is called the *metaplectic representation* of the symplectic group, and $\mu(\mathcal{H})$ is said to be the "quantization" of the classical object $\mathcal{H}$. As defined, the operator $\mu(\mathcal{H})$ maps Schwartz functions to tempered distributions, but in fact it extends to a bounded operator on $L^2(\mathbb{R}^n)$; we choose a so that $\mu(\mathcal{H})$ is unitary on $L^2(\mathbb{R}^n)$, and here $h > 0$ is a small parameter. These are the operators of “Fresnel optics,” a relatively simple model theory for optics which accounts for interference and diffraction, describing the propagation of light of wavelength h [Guillemin and Sternberg 1984]. For the analytic details we refer the reader to a text in semiclassical analysis [Dimassi and Sjöstrand 1999]; here we only show that the standard conditions are indeed satisfied.

The above (real-valued) generating function Φ, for an arbitrary $\mathcal{H} \in \mathrm{Sp}(2n, \mathbb{R})$, has the property that the 1-forms $d(\partial\Phi/\partial\theta_1), \ldots, d(\partial\Phi/\partial\theta_{2n})$ are linearly independent. Equivalently, with the notation from the previous section, the matrix

$$\begin{pmatrix} \dfrac{\partial^2\Phi}{\partial(\operatorname{Re} z)\partial\theta'} & \dfrac{\partial^2\Phi}{\partial(\operatorname{Re} z)\partial\theta''} \\ \dfrac{\partial^2\Phi}{\partial(\operatorname{Im} z)\partial\theta'} & \dfrac{\partial^2\Phi}{\partial(\operatorname{Im} z)\partial\theta''} \\ \dfrac{\partial^2\Phi}{\partial\theta'^2} & \dfrac{\partial^2\Phi}{\partial\theta'\partial\theta''} \\ \dfrac{\partial^2\Phi}{\partial\theta''\partial\theta'} & \dfrac{\partial^2\Phi}{\partial\theta''^2} \end{pmatrix} = \begin{pmatrix} | & & | & \\ (-b_1) & \cdots & (-b_k) & * \\ | & & | & \\ | & & | & \\ Db_1 & \cdots & Db_k & * \\ | & & | & \\ & 0_{k,k} & & 0_{k,(2n-k)} \\ & 0_{(2n-k),k} & & 2I_{(2n-k),(2n-k)} \end{pmatrix}$$

has linearly independent columns. (The asterisks denote irrelevant components.) This condition says that quadratic form $\Phi = \Phi(z, \theta)$ is a *nondegenerate phase function* in the sense of semiclassical analysis [Dimassi and Sjöstrand 1999].

Folland writes: “it seems to be a fact of life that there is no simple description of the operator $\mu(\mathcal{A})$ that is valid for all $\mathcal{A} \in \mathrm{Sp}$” [Folland 1989, p. 193]; however, we believe that (16), combined with our construction of Φ in the proof of Theorem 1, is such a description.

4. Conclusion

The open problem and results presented in this paper were motivated by the basic question of the relationship between real and complex symplectic linear algebra. Our approach to this question was to consider a real symplectomorphism as a Lagrangian submanifold with regard to the real part of a complex symplectic form. We believe the resulting problem of the nature of the restriction of the imaginary part of the complex symplectic form to this submanifold (formally, $\mathfrak{X}(\mathcal{H})$ for a symplectomorphism $\mathcal{H}$) is relevant to the structure of the real symplectic group. (We direct the reader to the Appendix for a list of properties of $\mathfrak{X}$ and reformulations of our open problem which lend credence to this belief.) Accordingly, we view the main result of this paper as primarily a means for further investigation of the open problem of the image of $\mathfrak{X}$. In addition to solving our open problem, we believe that, in line with our generating function formulation, it would be interesting to have a “complexified” theory of the calculus of variations. At present we only have trivial extensions of the real theory.

Appendix

A. ***Elementary properties of $\mathfrak{X}$.*** We first note some standard facts about symplectic matrices that are used throughout the paper; for further information, see, for example, [Cannas da Silva 2001] or [Folland 1989]. We write

$$\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

for the matrix representing the standard symplectic form.

Proposition 3 [Folland 1989]. *Let $\mathcal{H} \in GL(2n, \mathbb{R})$. The following are equivalent*:

(1) $\mathcal{H} \in \mathrm{Sp}(2n, \mathbb{R})$.

(2) $\mathcal{H}^T \mathcal{J} \mathcal{H} = \mathcal{J}$.

(3) $\mathcal{H}^{-1} = \mathcal{J} \mathcal{H}^T \mathcal{J}^{-1} = \left(\begin{smallmatrix} D^T & -B^T \\ -C^T & A^T \end{smallmatrix} \right)$.

(4) $\mathcal{H}^T \in \mathrm{Sp}(2n, \mathbb{R})$.

(5) $A^T D - C^T B = I,\ A^T C = C^T A$ *and* $B^T D = D^T B$.

(6) $AD^T - BC^T = I,\ AB^T = BA^T$ *and* $CD^T = DC^T$.

While $\mathfrak{X}$ may be extended to all of $\mathbb{M}^{2n}(\mathbb{R})$,

$$\mathfrak{X}: \quad \mathbb{M}^{2n}(\mathbb{R}) \to \mathfrak{so}(2n, \mathbb{R}), \quad M \mapsto \mathcal{J} M + M^T \mathcal{J}, \tag{1}$$

for purposes of our open problem the resulting linearity of $\mathfrak{X}$ does not seem to help when $\mathfrak{X}$ is restricted to $\mathrm{Sp}(2n, \mathbb{R})$.

The following proposition presents some of the most interesting elementary linear algebraic properties of $\mathfrak{X}$, which follow immediately from the definition.

Proposition 4. *Let $\mathfrak{X}$*: $\mathbb{M}^{2n}(\mathbb{R}) \to \mathfrak{so}(2n, \mathbb{R})$ *be defined as above. Then*:

(1) $\ker(\mathfrak{X}) = \mathfrak{sp}(2n, \mathbb{R})$, *the symplectic Lie algebra.*

(2) *For any* $\mathcal{H} \in \mathrm{Sp}(2n, \mathbb{R})$, $\mathfrak{X}(\mathcal{H}) = \mathcal{J}(\mathcal{H} + \mathcal{H}^{-1})$.
In particular, for $\mathcal{U} \in U(n) = \left\{ \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix} \right) \in \mathrm{Sp}(2n, \mathbb{R}) \right\}$ *we have* $\mathcal{U}^{-1} = \mathcal{U}^T$, *so* $\mathfrak{X}(\mathcal{U}) = \mathcal{J}(\mathcal{U} + \mathcal{U}^T)$.

(3) *For any* $\mathcal{H} \in \mathrm{Sp}(2n, \mathbb{R})$, $\mathfrak{X}(\mathcal{H})$ *is invertible* (*equivalently*, $\mathrm{Im}\,\omega^{\mathbb{C}}|_{\mathrm{graph}\,\mathcal{H}}$ *is nondegenerate*) *if and only if* -1 *is not a member of the spectrum of* $\mathcal{H}^2$.

(4) *For* $\mathcal{H}, \mathcal{R} \in \mathrm{Sp}(2n, \mathbb{R})$, *we have* $\mathcal{H}^T \mathfrak{X}(\mathcal{R}) \mathcal{H} = \mathfrak{X}(\mathcal{H}^{-1} \mathcal{R} \mathcal{H})$.

We now take some examples.

Examples of symplectic matrices and their images under $\mathfrak{X}$.

(1)
$$\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \mapsto \begin{pmatrix} 0 & -A^T-(A^T)^{-1} \\ A+A^{-1} & 0 \end{pmatrix}.$$

In particular,

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \mapsto \begin{pmatrix} 0 & -2I \\ 2I & 0 \end{pmatrix} = 2\mathcal{J}.$$

(2) For $B = B^T$,

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \mapsto \begin{pmatrix} 0 & -2I \\ 2I & 0 \end{pmatrix}.$$

(3) For $C = C^T$,

$$\begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \mapsto \begin{pmatrix} 0 & -2I \\ 2I & 0 \end{pmatrix}.$$

(4)
$$\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(5) For $t \in \mathbb{R}$,

$$\begin{pmatrix} (\cos t)I & (-\sin t)I \\ (\sin t)I & (\cos t)I \end{pmatrix} \mapsto \begin{pmatrix} 0 & -2(\cos t)I \\ 2(\cos t)I & 0 \end{pmatrix}.$$

(6) For any $\mathcal{H} \in \mathrm{Sp}(2n, \mathbb{R})$, we have $\mathfrak{X}(\mathcal{H}) = \mathfrak{X}(\mathcal{H}^{-1})$.

Thus in Examples (2) and (3), $\mathrm{graph}_{\mathbb{C}}\,\mathcal{H}$ is an RI-subspace (R-Lagrangian and I-symplectic). And in Example (4), $\mathrm{graph}_{\mathbb{C}}\,\mathcal{H}$ is a C-Lagrangian subspace (R-Lagrangian and I-Lagrangian).

The exact nature of the image of $\mathfrak{X}$ is an open question. The following is a partial result

Proposition 5. *For each $k \in \{0, 1, \dots, n\}$, there exists $\mathcal{H}_k \in \mathrm{Sp}(2n, \mathbb{R})$ such that* $\mathrm{rank}(\mathfrak{X}(\mathcal{H}_k)) = 2k$. *Moreover, for any $\mathcal{H} \in \mathrm{Sp}(2n, \mathbb{R})$, we have* $\ker \mathfrak{X}(\mathcal{H}) = \ker(\mathcal{H}^2 + I)$.

Proof. We fix $k \in \{0, 1, \dots, n\}$ and write

$$(x, \xi) = (x', x'', \xi', \xi''), \quad x', \xi' \in \mathbb{R}^k, \quad x'', \xi'' \in \mathbb{R}^{n-k}.$$

Let

$$\mathcal{H}_k(x', x'', \xi', \xi'') = (x', -\xi'', \xi', x'').$$

The matrix representation of $\mathcal{H}_k$ is

$$\begin{pmatrix} I_k & & 0_k & \\ & 0_{n-k} & & -I_{n-k} \\ 0_k & & I_k & \\ & I_{n-k} & & 0_{n-k} \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R}).$$

Then

$$\mathfrak{X}(\mathcal{H}_k) = \begin{pmatrix} & & -2I_k & \\ & & & 0_{n-k} \\ 2I_k & & & \\ & 0_{n-k} & & \end{pmatrix},$$

so that

$$\mathrm{rank}(\mathfrak{X}(\mathcal{H}_k)) = 2k.$$

The last statement of the proposition follows from (1). □

B. ***Restatement of the problem.*** It is sometimes convenient to work with the extension of $\mathfrak{X}$ to all of $\mathbb{M}^{2n}(\mathbb{R})$:

$$\mathfrak{X}(M) = \mathcal{J}M + M^T\mathcal{J}.$$

Then $\mathfrak{X} : \mathbb{M}(2n, \mathbb{R}) \to \mathfrak{so}(2n, \mathbb{R})$ is a linear epimorphism with kernel $\mathfrak{sp}(2n, \mathbb{R})$, the symplectic Lie algebra (see, for example, [Folland 1989, Proposition 4.2]). Thus the map $\mathfrak{X}|_{\mathrm{Sp}(2n,\mathbb{R})}$ is surjective if and only if every element of the quotient space $\mathbb{M}(2n, \mathbb{R})/\mathfrak{sp}(2n, \mathbb{R})$ contains a symplectic matrix. So our question is:

Question. Can every $M \in \mathbb{M}(2n, \mathbb{R})$ be written as $M = \mathcal{H} + \mathcal{A}$ for some $\mathcal{H} \in \mathrm{Sp}(2n, \mathbb{R})$ and some $\mathcal{A} \in \mathfrak{sp}(2n, \mathbb{R})$?

Proposition 6. *Every $M \in \mathbb{M}(2n, \mathbb{R})/\mathfrak{sp}(2n, \mathbb{R})$ has a unique representative of the form*

$$\begin{pmatrix} 0 & \mathcal{S}_2 \\ \mathcal{S}_3 & \mathcal{D} \end{pmatrix},$$

where $\mathcal{S}_2$ and $\mathcal{S}_3$ are skew-symmetric.

Proof. Existence: let

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \in \mathbb{M}(2n, \mathbb{R}).$$

Since $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathfrak{sp}(2n, \mathbb{R})$ if and only if $\delta = -\alpha^T,\ \beta = \beta^T,\ \gamma = \gamma^T$, we may replace M by

$$\widetilde{M} = M - \begin{pmatrix} M_1 & \frac{1}{2}(M_2 + M_2^T) \\ \frac{1}{2}(M_3 + M_3^T) & -M_1^T \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}(M_2 - M_2^T) \\ \frac{1}{2}(M_3 - M_3^T) & M_4 + M_1^T \end{pmatrix}.$$

Uniqueness: suppose

$$\begin{pmatrix} 0 & \mathcal{S}_2 \\ \mathcal{S}_3 & \mathcal{D} \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{S}_2' \\ \mathcal{S}_3' & \mathcal{D}' \end{pmatrix} \in \mathbb{M}(2n, \mathbb{R})/\mathfrak{sp}(2n, \mathbb{R}),$$

with the $\mathcal{S}_j$ and $\mathcal{S}_j'$ skew-symmetric. Thus

$$\begin{pmatrix} 0 & \mathcal{S}_2 - \mathcal{S}_2' \\ \mathcal{S}_3 - \mathcal{S}_3' & \mathcal{D} - \mathcal{D}' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^T \end{pmatrix} \in \mathfrak{sp}(2n, \mathbb{R}).$$

This shows that $\mathcal{S}_j - \mathcal{S}_j'$ is symmetric and skew-symmetric, hence zero, and it is clear that $\mathcal{D} = \mathcal{D}'$. □

Thinking geometrically, we are to find the projection of $\mathrm{Sp}(2n, \mathbb{R})$ onto

$$\left\{ \begin{pmatrix} 0 & \mathcal{S}_2 \\ \mathcal{S}_3 & \mathcal{D} \end{pmatrix} : \mathcal{S}_2, \mathcal{S}_3 \text{ skew-symmetric} \right\}$$

along $\mathfrak{sp}(2n, \mathbb{R})$. That is, let $\mathcal{H} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2n, \mathbb{R})$. Then

$$\pi(\mathcal{H}) = \begin{pmatrix} 0 & \frac{1}{2}(B - B^T) \\ \frac{1}{2}(C - C^T) & A^T + D \end{pmatrix}.$$

Is every

$$\begin{pmatrix} 0 & \mathcal{S}_2 \\ \mathcal{S}_3 & \mathcal{D} \end{pmatrix}$$

of this form?

For a possible simplification, the map

$$\mathcal{Y} : \mathrm{Sp}(2n, \mathbb{R}) \to \mathfrak{so}(2n, \mathbb{R}), \quad \mathcal{H} \mapsto \mathfrak{X}(-\mathcal{J}\mathcal{H}) = \mathcal{H} - \mathcal{H}^T,$$

has the same image as $\mathfrak{X} : \mathrm{Sp}(2n, \mathbb{R}) \to \mathfrak{so}(2n, \mathbb{R})$ and may be easier to understand.

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