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On symplectic capacities of toric domains

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A toric domain is a subset of $(\mathbb{C}^n, \omega_{\text{std}})$ which is invariant under the standard rotation action of $\mathbb{T}^n$ on $\mathbb{C}^n$. For a toric domain U from a certain large class for which this action is not free, we find a corresponding toric domain V where the standard action is free and for which $c(U) = c(V)$ for any symplectic capacity c. Michael Hutchings gives a combinatorial formula for calculating his embedded contact homology symplectic capacities for certain toric four-manifolds on which the $\mathbb{T}^2$-action is free. Our theorem allows one to extend this formula to a class of toric domains where the action is not free. We apply our theorem to compute ECH capacities for certain intersections of ellipsoids and find that these capacities give sharp obstructions to symplectically embedding these ellipsoid intersections into balls.

1. Introduction

Symplectic capacities, introduced by Gromov and Hofer, are symplectic invariants that assign a nonnegative real number to a subset $U \subset (\mathbb{C}^n, \omega_{\text{std}})$ and have the following properties:

(C1) Monotonicity: $c(U) \le c(V)$ if $U \hookrightarrow V$.

(C2) Conformality: $c(\lambda U) = \lambda^2 c(U)$ for $\lambda \in \mathbb{R}$.

(C3) Nontriviality: $0 < c(B^{2n}(1)) < \infty$.

Note that combining all three requires a finite capacity for any bounded U. Sometimes additional nontriviality and normalization axioms are also assumed, but we do not use them here. Many useful symplectic capacities have been defined; some are listed in [Cieliebak et al. 2007].

Define the *moment map* $\mu : \mathbb{C}^n \to \mathbb{R}^n$ of the symplectic manifold $(\mathbb{C}^n, \omega_{\text{std}})$ by

$$\mu(z_1, \dots, z_n) = (\pi |z_1|^2, \dots, \pi |z_n|^2),$$

where ω_{std} is the standard symplectic form $\omega_{\text{std}} = \sum_{i=1}^n dx_i \wedge dy_i$ on $\mathbb{C}^n$, and call $\mu(\mathbb{C}^n)$ the moment space. We call $U \subset (\mathbb{C}^n, \omega_{\text{std}})$ a *toric domain* when it can

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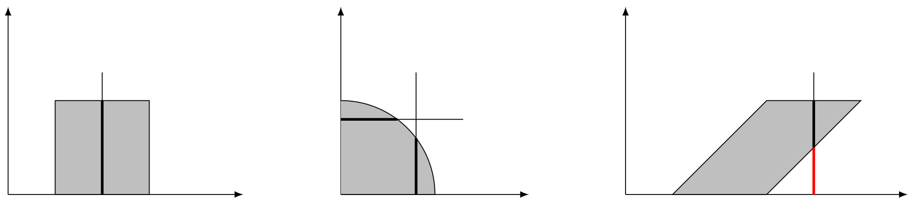

(a) (b) (c)

Figure 1. Appropriate moment regions; (a) and (b) satisfy the conditions of Criterion 1.1, but (c) does not.

be written $U = \mu^{-1}(A)$ for some *moment region* $A \subset \mathbb{R}^n_{\geq 0}$ in the moment space, or equivalently when it is invariant under the rotation action of $\mathbb{T}^n$ on $\mathbb{C}^n$. Note that this is a special case of the more general moment map associated with a Hamiltonian action of a Lie group.

Since these toric domains are uniquely represented by their moment regions, we will refer to a symplectic capacity $c(A)$ of a moment region A, and by this mean $c(\mu^{-1}(A))$. A simple calculation shows that (C2) is equivalent to $c(\lambda A) = \lambda c(A)$.

Our main theorem is that for a duly qualified toric domain U whose moment region satisfies Criterion 1.1 given below, any symplectic capacity of U is the same as the capacity of a toric domain with a free action, one whose moment region is $\mu(U)$ translated off the coordinate planes in the moment space.

Criterion 1.1. Let $A \subset \mathbb{R}^n_{\geq 0}$. If A intersects a coordinate plane

$$P_i = \{(\rho_1, \ldots, \rho_n) \in \mathbb{R}^n \mid \rho_i = 0\},$$

then any line normal to P_i has connected intersection with $A \cup P_i$.

The necessary further qualifications are given in the theorem statement below. Figure 1 illustrates this condition for $n = 2$. In this case, Criterion 1.1 ensures that the toric domain is a disk bundle over its projection to the first complex plane of $\mathbb{C}^2$; more generally, for A satisfying the other conditions below, Criterion 1.1 requires $\mu^{-1}(A)$ to be a (generalized) disk bundle over its projection to any coordinate plane P_i which it touches. Disks in the fiber space degenerate to points where A touches a coordinate plane.

Theorem 1.2. *Let $A \subset \mathbb{R}^n_{\geq 0}$ be a moment region which is compact with star-shaped interior and whose boundary intersects transversely the rays from the star-center. Assume that A satisfies Criterion 1.1. Then $c(A) = c(A + (1, 1, \ldots, 1))$ for any symplectic capacity c.*

The theorem is proved by establishing equal lower and upper bounds on $c(A)$ in terms of $c(A + (1, 1, \ldots, 1))$. The lower bound follows readily from properties of toric domains and the axioms (C1)–(C3), but for the upper bound we must combine the axioms with a nontrivial symplectic embedding. Since the proof assumes only the general axioms for capacities, this result holds for all symplectic capacities. Note that the action on a given toric domain $U = \mu^{-1}(A)$ is free if and only if U does not intersect the origin in any $\mathbb{C}$ factor; that is, its moment region does not touch any coordinate plane $P_i = \{(\rho_1, \dots, \rho_n) \in \mathbb{R}^n \mid \rho_i = 0\}$ in the moment space.

The embedded contact homology (ECH) developed by Michael Hutchings provides a natural way to define certain symplectic capacities called ECH capacities. They are defined for any subset of a symplectic 4-manifold. Hutchings [2011] gives a combinatorial method to compute these capacities for toric domains over convex moment regions that do not touch the axes of the moment space $\mathbb{R}^2_{\geq 0}$ (that is, the torus action is free). This method is presented in Section 3. In [Hutchings 2014, Remark 4.15] and [Choi et al. 2014, §1.2], it was conjectured that Hutchings' formula should remain true in most, and probably all, cases where $\mu(U)$ does touch one or both axes. Theorem 1.2 shows that this is true for the ECH capacities of a large class of toric domains by showing that it is true for all symplectic capacities.

Given $a, b \in \mathbb{R}^+$, define the ellipsoid

$$E(a,b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} \leq 1 \right\}, \tag{1}$$

the ball

$$B(a) := E(a,a),$$

and the polydisk

$$P(a,b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \pi |z_1|^2 \leq a, , \pi |z_2|^2 \leq b \right\}, \tag{2}$$

where each inherits the standard symplectic form from $\mathbb{C}^2$.

In Section 3, we use Theorem 1.2 to compute ECH capacities of a class of intersections of ellipsoids. We also study symplectic embeddings of domains from this class, proving the following proposition:

Proposition 1.3. *Let $a > b$ and $c > d$. Let R be the radius of the smallest ball containing $E(a,b) \cap E(c,d)$, and let $\rho = \inf\{r \mid E(a,b) \cap E(c,d) \hookrightarrow B(r)\}$. If $2a, 2d \geq R$, then $\rho = R$.*

It is known that ECH capacities provide sharp obstructions to symplectically embedding ellipsoids into ellipsoids (proved by McDuff [2011]) and ellipsoids into polydisks [Frenkel and Müller 2012]. Recall that by Gromov's nonsqueezing theorem [1985], a ball symplectically embeds into a cylinder in $\mathbb{R}^{2n}$ if and only if the radius of the cylinder exceeds that of the ball. This is an illustration of symplectic rigidity and is easily recovered from the ECH capacities on these domains. The computation of ECH capacities of the ellipsoid intersections above shows that they give sharp obstructions to symplectically embedding those ellipsoid intersections into balls. Since the balls have much larger volume than the ellipsoid intersections, Proposition 1.3 is another example of symplectic rigidity.

In Proposition 1.3, the ECH capacities give a sharp obstruction. Recent work of Hind and Lisi [2014] shows that neither ECH capacities nor Ekeland–Hofer capacities give sharp obstructions to symplectic embeddings of arbitrary toric domains; in particular the ECH and Ekeland–Hofer obstructions to symplectically embedding a product of polydisks into a ball are not always sharp. The torus action on polydisks and balls is not free, so we might ask whether the situation is any different if we consider only toric domains for which the action is free. However, the case of free torus action is not different in this way, as the following corollary of Theorem 1.2 shows:

Corollary 1.4. *Let* $P^*(1,2) = \mu^{-1}(\mu(P(1,2)) + (1,1))$ *be a toric domain, let* $a < 3$ *and let* $B^*(a) = \mu^{-1}(\mu(B^4(a)) + (1,1))$. *There is no symplectic embedding* $P^*(1,2) \hookrightarrow B^*(a)$.

This shows that neither ECH nor Ekeland–Hofer capacities are sharp even when we consider only toric domains with a free action because the obstruction given by both of these sequences of capacities is $a \geq 2$ (see [Hind and Lisi 2014]). This corollary is proved in Section 3B.

2. Proof of main theorem

In this section, we prove Theorem 1.2 by constructing symplectomorphisms as the products of area preserving maps. It will be convenient to have the following standard lemma, which shows that translations in the moment space induce symplectomorphisms on toric domains whose moment regions do not touch any coordinate plane.

Lemma 2.1. *Suppose* $U \subset (\mathbb{R}^{2n}, \omega_{\text{std}})$ *is a toric domain with free torus action such that* $\mu(U) = A$, *and* B *is any translate of* A *such that the torus action on* μ^{-1} *is also free. Then* U *and* $V = \mu^{-1}(B)$ *are symplectomorphic. In particular, they have the same symplectic capacity for any capacity.*

Proof. We can parametrize U by $g : A \times \mathbb{T}^n \to U$ defined by

$$g(\rho_1, \dots, \rho_n, e^{i\theta_1}, \dots, e^{i\theta_n}) = \left(\sqrt{\frac{\rho_1}{\pi}} e^{i\theta_1}, \dots, \sqrt{\frac{\rho_n}{\pi}} e^{i\theta_n}\right).$$

Then we can pull back the standard symplectic form to $A \times \mathbb{T}^n$. A simple calculation shows that for the first term,

$$g^*(dx_1 \wedge dy_1) = \frac{1}{2\pi} d\rho_1 \wedge d\theta_1,$$

and thus

$$g^*\omega_{std} = \frac{1}{2\pi}\sum_{i=1}^{n} d\rho_i \wedge d\theta_i.$$

Clearly translation in moment space does not affect this last form, so conjugating a translation by this parametrization yields the desired symplectomorphism. □

Another important fact that can be seen from the proof of Lemma 2.1 is that for a toric domain U with free torus action and moment region A, the symplectic volume of U is equal to the volume of A:

$$\begin{aligned}\mathrm{vol}(U, \omega_{\mathrm{std}}) &= \frac{1}{n!}\int_U \omega_{\mathrm{std}}^n = \frac{1}{n!}\int_{A\times\mathbb{T}^n} (g^*\omega_{\mathrm{std}})^n \\ &= \frac{1}{(2\pi)^n}\int_{A\times\mathbb{T}^n} d\rho_1\wedge\cdots\wedge d\rho_n\wedge d\theta_1\wedge\cdots\wedge d\theta_n \\ &= \int_A d\rho_1\wedge\cdots\wedge d\rho_n = \mathrm{vol}(A).\end{aligned}$$

So a symplectic embedding of toric domains $U \hookrightarrow V$ may be possible only if $\mathrm{vol}(\mu(U)) \le \mathrm{vol}(\mu(V))$.

We will also use the following version of the "Traynor trick" (cf. Proposition 5.2 of [Traynor 1995]):

Lemma 2.2. *Given $\varepsilon > 0$, there exists an area preserving diffeomorphism*

$$\Psi : B^2(1) \to SD^2(1+\varepsilon) = B^2(1+\epsilon) - \{x+iy \mid y = 0, x \ge 0\}$$

from the disk to the slit-disk such that

$$\delta < |\Psi(z)|^2 < |z|^2 + \varepsilon$$

for some $\delta > 0$.

Proof. The left inequality follows from continuity (given such a map). For existence and the right inequality, define a family of loops which avoid the slit as in Figure 2, and apply Schlenk [2005, Lemma 3.1]. □

With these tools we can prove Theorem 1.2.

Proof of Theorem 1.2.

Our technique is to find upper and lower bounds on $c(A)$ by producing symplectic embeddings and applying (C1) and (C2). We show that these bounds agree with each other and with $c(A + (1, 1, \ldots, 1))$.

For what follows, we define the scaling of $\mathbb{R}^n$ by $\lambda > 0$ from $p \in \mathbb{R}^n$ to be the map $q \mapsto \lambda(q - p) + p$. Since $\lambda(q - p) + p = \lambda q + (1-\lambda)p$, any scaling by λ from p is equivalent to a scaling from the origin by λ followed by translation by $(1-\lambda)p$. So

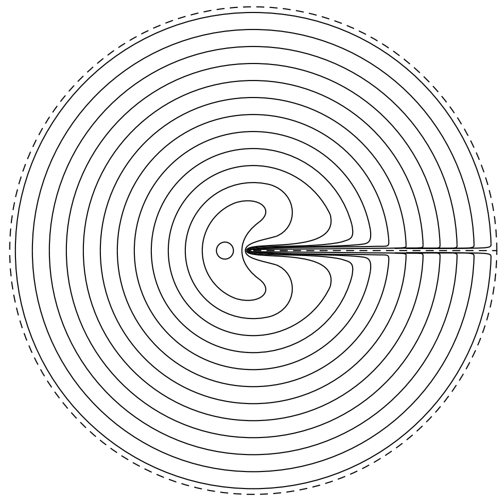

Figure 2. A family of loops defining a symplectomorphism $B^2(1) \to SD(1+\varepsilon)$.

with Lemma 2.1 we may apply conformality of capacities, axiom (C2), on moment regions scaled from points other than the origin. The reason for the requirement that rays from the star-center be transverse to the boundary will become clear in Step 2 with the scaling argument.

Step 1. The lower bound may be computed as follows. Let p be a star-center of $\operatorname{int} A$, which means that any other point in $\operatorname{int} A$ may be connected to p by a line contained in $\operatorname{int} A$. Given any $\lambda < 1$, let A_λ be the image of A under the scaling of the moment space towards p by λ. Since p is away from the coordinate planes, A_λ is bounded away from the coordinate planes and contained in A. By Lemma 2.1 and conformality, $c(A_\lambda) = \lambda c(A + (1, 1, \dots, 1))$. Then by monotonicity, $\lambda c(A + (1, 1, \dots, 1)) \le c(A)$, and since $\lambda < 1$ was arbitrary,

$$c(A + (1, 1, \dots, 1)) \le c(A).$$

Step 2. For the upper bound, we embed A into an expanded version of A, and apply an area-preserving map in each dimension in which A touches a coordinate plane P_i. We will assume that A is compact, star-shaped, and that the rays from a star-center p intersect each ∂A_j transversely.

Assume without loss of generality that A touches the first k coordinate planes and does not touch the others. Let $p = (\rho_1, \dots, \rho_n)$ be the star-center in A noted above. The projection $\tilde{p}_1 = (0, \rho_2, \dots, \rho_n)$ is also a star-center: Choose any other point $q = (x_1, \dots, x_n) \in A$. The line from $\tilde{p}_1$ to q is entirely below that from p to q in the ρ_1 coordinate. By Criterion 1.1, any perpendicular dropped from a point in A to P_1 remains in A. Hence the line from $\tilde{p}_1$ to q is also in A, so $\tilde{p}_1$ is a star-center. Repeating in the first k coordinates, we find that $\tilde{p}_k = (0, \ldots, \rho_{k+1}, \ldots, \rho_n)$ is a star-center; call this point $\tilde{p}$. A simple geometric argument making use of Criterion 1.1 shows that the rays from $\tilde{p}$ must also be transverse to each ∂A_j; we omit that here.

The next step will be to expand A to A_λ by a finite factor of λ. In order to prevent A_λ from colliding with coordinate planes, first translate A away from the coordinate planes P_{k+1} through P_n by some large amount. Note that this is possible because by assumption $p_i > 0$ for $i > k$, and furthermore translation in the moment spaces induces a symplectomorphism. So we shall instead compute the capacity of this translate, and relabel it A. Now let A_λ be the scaling of A from $\tilde{p}$ by a small $\lambda > 1$.

We show that $A \subset \operatorname{int} A_\lambda$. Consider any point $q = (x_1, \ldots, x_n) \in A$. If $q \in \operatorname{int} A$ then $q \in \operatorname{int} A_\lambda$, so suppose $q \in \partial A$. Write $q_{1/\lambda}$ for the point mapped to q under the scaling; $q_{1/\lambda}$ will be between $\tilde{p}$ and q. Now since the ray from $\tilde{p}$ to q is transverse to ∂A, it follows that $q_{1/\lambda}$ must be in $\operatorname{int} A$, so we can find an open ball U around $q_{1/\lambda}$. That ball maps under the scaling to U_λ, which is an open ball around q in A_λ. Thus $q \in \operatorname{int} A_\lambda$, and $A \subset \operatorname{int} A_\lambda$.

Let $\operatorname{ext} A_\lambda$ denote the exterior of A_λ in $\mathbb{R}^n_{\geq 0}$. Both A and A_λ are compact, so there is some d so that $0 < d < d_\lambda = \frac{1}{2}\operatorname{dist}(A, \operatorname{ext} A_\lambda)$. Now A is bounded, so let a be the maximum of the ρ_1 coordinate of A, and choose $\varepsilon > 0$ so that $\varepsilon < d$. Then by Lemma 2.2, there exists $\Psi_a : B^2(a) \to SD^2(a+\varepsilon)$ such that

$$\delta < |\Psi_a(z)|^2 < |z|^2 + \varepsilon \tag{3}$$

for $\delta > 0$. Let $F_\varepsilon = \Psi_a \times \mathrm{id} \times \cdots \times \mathrm{id}$.

Set $B = \mu \circ F_\varepsilon(\mu^{-1}(A))$. Then we claim $B \subset \operatorname{int} A_\lambda$. Consider a point $(z_1, \ldots, z_n) \in \mu^{-1}(A)$, and let

$$(\rho_1, \ldots, \rho_n) \equiv \mu(z_1, \ldots, z_n) \in A.$$

By the inequality above, $\mu \circ F_\varepsilon((z_1, \ldots, z_n)) = (\tilde{\rho}_1, \ldots, \tilde{\rho}_n)$, where $\tilde{\rho}_1 < \rho_1 + \varepsilon$ and $\tilde{\rho}_i = \rho_i$ for $i > 1$. Thus every point in $\mu^{-1}(A)$ is carried by F_ε to a point less than d away from A, so $B \subset \operatorname{int} A_\lambda$; moreover, $\operatorname{dist}(B, \operatorname{ext} A_\lambda) > d_\lambda$. Then let $\delta = \frac{1}{2}\min\{\delta, d_\lambda\}$ and $\gamma = \lambda\delta$ (using $\lambda < 2$). Set $A'_\lambda = A_\lambda + (\gamma, 0, \ldots, 0)$. The lower bound on the left of equation (3), together with the distance from B to outside A_λ, show that in fact $B \subset A'_\lambda$. So by Lemma 2.1, $c(B) \leq c(A'_\lambda) = \lambda c(A + (\delta, 0, \ldots, 0))$. Now $\lambda > 1$ was arbitrary, so $c(B) \leq c(A + (\delta, 0, \ldots, 0))$. Since A and B are symplectomorphic,

$$c(A) \leq c(A + (\delta, 0, \ldots, 0)).$$

Repeating the same process in the dimensions up to k and translating up by δ in the other coordinates shows that for some $\delta > 0$, $c(A) \leq c(A + (\delta, \delta, \ldots, \delta))$.

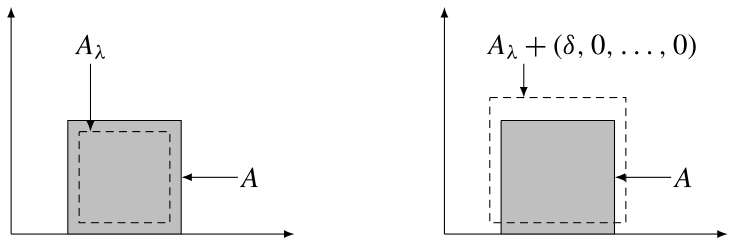

Figure 3. Illustration of the conformality argument for the lower bound (left) and the upper bound (right).

Combining with the lower bound, and using Lemma 2.1,

$$c(A) = c(A + (1, 1, \dots, 1)). \qquad \square$$

Remark 2.3. It is worth noting that we may like to consider regions A for which ∂A is not completely smooth. The ellipsoid intersections below are one example. The notion of transversality must then be generalized slightly with the goal that $A \subset \operatorname{int} A_\lambda$. If ∂A is the gluing of multiple hypersurfaces, it is sufficient that the rays from the star-center be transverse to each of the hypersurfaces at the points where they are glued together.

3. Applications

3A. ***ECH capacities.*** The remainder of this paper focuses on 4-dimensional toric domains, with accompanying planar moment regions. Using Michael Hutchings' theory of embedded contact homology (ECH), one can associate real numbers

$$0 = c_0(M) \le c_1(M) \le c_2(M) \le \cdots$$

called *ECH capacities* to any 4-dimensional Liouville domain M, such that each c_i is a symplectic capacity for 4-manifolds. For precise definitions of ECH capacities and Liouville domains, see [Hutchings 2011].

We briefly describe the computation of ECH capacities, as given by Theorem 4.14 of [Hutchings 2014]. Given a convex body A in the moment space which does not touch any coordinate plane, we can define a norm ℓ_A, not necessarily symmetric, as follows. Choose an origin in A from which to draw position vectors to ∂A. Let v_i be some vector, and q_i one of the position vectors on ∂A such that the outward normal to ∂A at q_i is parallel to v_i. If v_i has angle between the normals to ∂A at two incident edges of ∂A, let q_i be the corner where the edges meet. Then set $\ell_A(v_i) = v_i \cdot q_i$. It is not hard to check that this yields a well-defined norm; see [Hutchings 2014] for details.

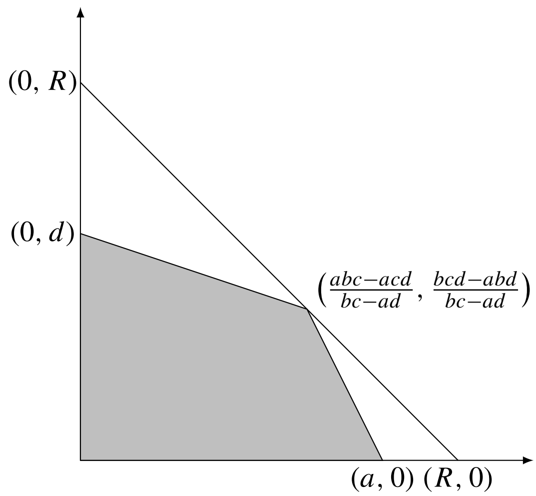

Figure 4. The image of $E(a,b) \cap E(c,d)$ under μ with suitable a, b, c, d, and the smallest ball into which it symplectically embeds.

We compute the ECH capacities according to [Hutchings 2011] as follows: for each k, $c_k(A)$ is the shortest perimeter length of an oriented lattice-polygon enclosing $k+1$ lattice points, where perimeter length is measured in the norm ℓ_A on the edge vectors of the oriented polygon.

3A1. *Embedding ellipsoid intersections into balls.* We now use Theorem 1.2 to compute the second ECH capacity of a family of ellipsoid intersections. This capacity is in turn used to prove Proposition 1.3. Throughout this section, let $a, b, c, d > 0$, $a < b$, $c > d$, and put

$$R = \frac{abc + bcd - acd - abd}{bc - ad}$$

(see Figure 4). We show that for $2a, 2d \geq R$, we have $c_2(E(a,b) \cap E(c,d)) = R$. A simple consequence is that $E(a,b) \cap E(c,d)$ symplectically embeds into a ball if and only if it embeds by inclusion (that is, Proposition 1.3). While in principle that result only requires the easier lower bound of Theorem 1.2, we illustrate the use of Theorem 1.2 to produce the actual ECH capacity, which is sufficient to prove the proposition.

A short computation, or consideration of Figure 4, shows that $B(R)$ is indeed the smallest ball into which $E(a,b) \cap E(c,d)$ embeds by inclusion. We first prove the following lemma:

Lemma 3.1. *If* $2a, 2d \geq R$, *then* $c_2(E(a,b) \cap E(c,d)) = R$.

Assuming Lemma 3.1, observe that Proposition 1.3 is immediate:

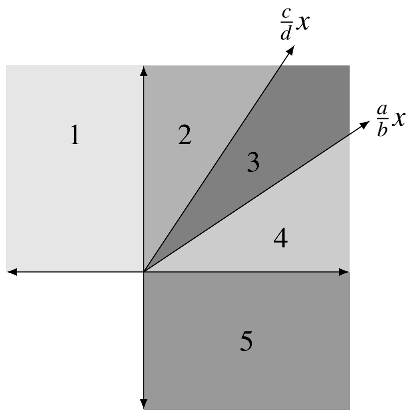

Figure 5. Calculation of $\ell_{A'}$-length by region.

Proof of Proposition 1.3. By [Hutchings 2011, Corollary 1.3], $c_2(B(r)) = r$, so we have $\rho \geq R$ by Lemma 3.1. Since $E(a, b) \cap E(c, d) \subset B(R)$, $\rho \leq R$ and the result follows. □

Proof of Lemma 3.1. Let A be the moment region of $E(a, b) \cap E(c, d)$. Since A satisfies Criterion 1.1, we know that $c_2(A) = c_2(A')$ for $A' = A + (1, 1)$.

First, we observe that the oriented lattice-polygonal path shown in Figure 6 has $\ell_{A'}$-length R when oriented clockwise, so $c_2(A) \leq R$.

Let Γ be an oriented lattice path containing three lattice points with edge vectors (α, β), (γ, δ), (ϵ, ζ) (if Γ has only two edge vectors, i.e., is just a line segment, the forthcoming argument applies *mutatis mutandis*). Suppose for a contradiction that $\ell_{A'}(\Gamma) < R$.

We first claim that $\beta, \delta, \zeta \leq 1$ and that at most one is positive. Suppose without loss of generality that $\beta \geq 2$. Depending on the region in which (α, β) lies (or its slope β/α, Figure 5), the $\ell_{A'}$-length is determined by cases:

$$\ell_{A'}((\alpha, \beta)) = \begin{cases} (\alpha, \beta) \cdot (0, d) & \text{if } \alpha \leq 0 \text{ or } \frac{\beta}{\alpha} \geq \frac{c}{d} \text{ (regions 1, 2)}, \\ (\alpha, \beta) \cdot \left(\frac{abc-acd}{bc-ad}, \frac{bcd-abd}{bc-ad}\right) & \text{if } \frac{c}{d} \leq \frac{\beta}{\alpha} \leq \frac{a}{b} \text{ (region 3)}, \\ (\alpha, \beta) \cdot (a, 0) & \text{if } 0 < \frac{\beta}{\alpha} \leq \frac{a}{b} \text{ (region 4)}. \end{cases}$$

We treat each case separately. In region 1, we have $(\alpha, \beta) \cdot (0, d) = \beta d \geq 2d \geq R$, a contradiction. In region 2,

$$\ell_{A'}((\alpha, \beta)) = (\alpha, \beta) \cdot \left(\frac{abc-acd}{bc-ad}, \frac{bcd-abc}{bc-ad}\right)$$

and $\alpha \geq 1$. Hence,

$$(\alpha, \beta) \cdot \left(\frac{abc-acd}{bc-ad}, \frac{bcd-abc}{bc-ad}\right) > (1, 1) \cdot \left(\frac{abc-acd}{bc-ad}, \frac{bcd-abc}{bc-ad}\right) = R.$$

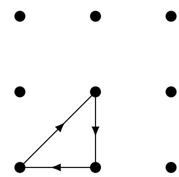

Figure 6. The minimal path for $c_2(A)$ in Lemma 3.1.

Lastly, in region 3, $\ell_{A'}((\alpha,\beta)) = (\alpha,\beta)\cdot(a,0)$ and $\alpha > \beta$, so $\ell_{A'}((\alpha,\beta)) = \alpha a > 2a \geq R$. Thus $\beta,\delta,\zeta \leq 1$.

To show that at most one of β,δ,γ is positive, assume without loss of generality that $\beta,\delta \geq 1$. Another calculation as above shows that both $\ell_{A'}((\alpha,\beta))$ and $\ell_{A'}((\gamma,\delta))$ are greater than or equal to $\min\{a,d\}$, so $\ell_{A'}(\Gamma) \geq 2\min\{a,d\} \geq R$, a contradiction.

A symmetric argument but with regions 2, 3, 4, and 5 shows that $\alpha,\gamma,\epsilon \leq 1$ and that at most one is positive. These facts imply that the maximum displacement in either coordinate is 1; that is, Γ lies in $[0,1]^2$ up to translation. We check that the shortest lattice path containing three lattice points in $[0,1]^2$ has $\ell_{A'}$-length R, so Γ cannot exist. $\square$

3B. ***Toric domains with free action.*** The proof of Corollary 1.4 simply combines the embeddings involved in the proof of Theorem 1.2 with the result that a symplectic embedding $P(1,2) \hookrightarrow B^4(a)$ is possible if and only if $a \geq 3$ [Hind and Lisi 2014, Theorem 1.1].

Proof of Corollary 1.4. Suppose to the contrary that $a < 3$ is given for which we can find an embedding $f : P^*(1,2) \hookrightarrow B^*(a)$. Let $\lambda > 1$ be close to 1 such that $\lambda^2 a < 3$. Let $P^*_\lambda(1,2) = \mu^{-1}(\mu(P(\lambda,2\lambda)) + (1,1))$ and $B^*_\lambda(a) = \mu^{-1}(\mu(B^4(\lambda a)) + (1,1))$. After scaling by λ, we can find an embedding $f_\lambda : P^*_\lambda(1,2) \hookrightarrow B^*_\lambda(a)$. This is combined with the embeddings from the proof of Theorem 1.2 as follows:

First, we can find a symplectic embedding $F : P(1,2) \hookrightarrow P^*_\lambda(1,2)$ by the same technique illustrated in that theorem since $P^*_\lambda(1,2)$ is just the translated expansion of $P(1,2)$. We also have the inclusion embedding $\iota : B^*_\lambda(a) \hookrightarrow B(\lambda^2 a)$ because of the translation law (Lemma 2.1) above. Combining these we get

$$\iota \circ f_\lambda \circ F : P(1,2) \hookrightarrow B(\lambda^2 a).$$

Since $\lambda^2 a < 3$, this violates [Hind and Lisi 2014, Theorem 1.1]. Thus no such embedding $f : P^*(1,2) \hookrightarrow B^*(a)$ exists. $\square$

By Theorem 1.2, the ECH and Ekeland–Hofer capacities of $P^*(1,2)$ and $B^*(a)$ are the same as those of $P(1,2)$ and $B(a)$, so neither of these capacities give sharp obstructions to embedding $P^*(1,2)$ into $B^*(a)$.

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