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Cylindrical liquid bridges

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We consider a cylindrical liquid bridge under capillary effects, spanning two horizontal plates and further bounded by a pair of parallel vertical planes. We explicitly formulate the volume-constrained problem and describe a numerical procedure for approximating the solution. Finally, a problem of finding the minimum spanning volume is considered.

1. Introduction

We consider a fluid trapped between two horizontal plates P_0, P_h, and further bounded by two parallel vertical planes Π_0, Π_d. Define the distance between P_0 and P_h to be h, and that between Π_0 and Π_d to be d. We orient a coordinate system (x, y, z) so that P_0 is given by $z \equiv 0$ and P_h is given by $z \equiv h$, while Π_0 is given by $y \equiv 0$ and Π_d is given by $y \equiv d$. We assume that the fluid is connected and any wetted portions of the plates are simply connected. The fluid then has a free interface Λ bounding a volume in the x-direction, and we denote the enclosed volume by $\mathcal{V}$. For an example, see Figure 1, where we have not drawn Π_0 or Π_d.

We consider dominant energies due to surface tension, wetting energy and gravitational potential energy. This gives the energy functional

$$\mathcal{E}[\Lambda] = \sigma \mathcal{A}[\Lambda] - \sigma\beta\mathcal{W}[\Lambda] + \int_{\mathcal{V}} \rho g z \, dz, \tag{1}$$

where σ is the (constant) surface tension, β is the wetting coefficient, taken to be constant on each plate, ρ is the uniform fluid density, and g is the gravitational constant. Further, $\mathcal{A}$ is the area functional for the free-surface, and $\mathcal{W}$ is the area functional for the wetted portions of P_0, P_h, Π_0 and Π_d.

It is well known that the first variation for this functional implies

$$2H = \kappa u - \lambda, \tag{2}$$

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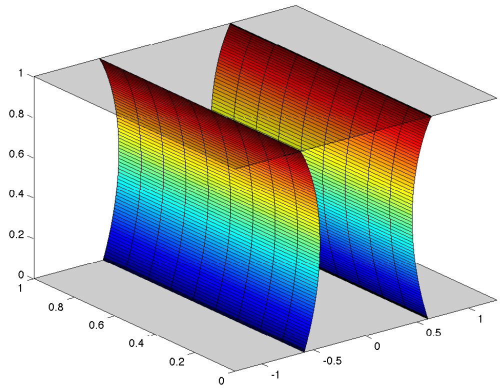

Figure 1. A cylindrical bridge.

where H is the mean curvature of Λ, u is the height of the interface, the capillary constant is $\kappa = \rho g/\sigma$, and we have included a Lagrange multiplier λ. It may also be derived that $\beta = \cos\gamma$ for a contact angle γ measured within the fluid. The standard reference is a manuscript by Finn [1986]. In what follows we do not assume that the interface is a graph over a base domain, though we do restrict our attention to the physical case where the interface is embedded. See Theorem 2.1 for details on how we interpret (2).

We make the assumption that $\beta = 0$ on Π_0 and Π_d. This implies a contact angle of $\pi/2$ along the intersection of Λ with those planes. As we shall see in Section 2, this also implies that the free-surface is generated by curves in the plane Π_0 and is extended as a right cylinder. See Figure 2 for an example of the generating curves, where $\delta/2$ denotes the value of the horizontal displacement of the fluid interface on P_0. On the plates P_0 and P_h, we allow the constant β to differ at heights 0 and h and to be any number in $[-1, 1]$. This corresponds to contact angles along the intersection of Λ with those plates, which we will denote by γ_0 and γ_h respectively.

In Section 3, we derive a formula for the enclosed volume in terms of the solution to a version of the differential equation (2) when the fluid remains connected. Then we give an algorithm for computing the interface Λ with a volume constraint in

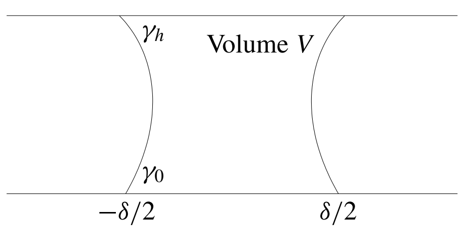

Figure 2. A liquid bridge.

Section 4. We use this algorithm to give a numerical approximation of Λ for different parameters h, γ_0, γ_h, and volume $\mathcal{V}$. Next, in Section 5, we present a collection of examples. Finally, we explore the minimum spanning volume for $(\gamma_0, \gamma_h) \in [0, \pi/2] \times [0, \pi/2]$ in Section 6.

As far as we have been able to determine, this is the first exploration of liquid bridges of this type. Three dimensional liquid bridge problems have been studied by Athanassenas [1992]; Concus, Finn and McCuan [Concus et al. 2001]; Finn and Vogel [1992]; and Vogel [1982; 1987; 1989; 2005; 2006; 2013]. In recent work there is a trend to study the lower dimensional versions of certain related fluid mechanics problems. We point to papers by Bhatnagar and Finn [2006], as well as by McCuan and Treinen [2013; ≥ 2015], and Wente [2006] for examples of this approach. In particular, we mention a paper by McCuan [2013] as a model for the present approach.

2. Symmetries

There are two types of symmetries in the fluid configurations. The first is the cylindrical symmetry that allows us to restrict our attention to the generating curves in the Π_0 plane. The second is a reflective symmetry about the plane $x = 0$.

An Alexandrov moving plane argument has been successful in establishing symmetry properties for similar fluid configurations. See Wente [1980], Treinen [2012], and McCuan [2013]. The following is a direct consequence of first using those methods with a moving plane parallel to Π_0, then a second argument using those methods with a moving plane parallel to $x = 0$ can be used to show symmetry about $x = 0$. The details are left to the interested reader.

Theorem 2.1. *The interface Λ is right-cylindrically symmetric with generating curves restricted to the plane Π_0. The generating curves satisfy*

$$\frac{dx}{ds} = \cos\psi, \tag{3}$$

$$\frac{du}{ds} = \sin\psi, \tag{4}$$

$$\frac{d\psi}{ds} = \kappa u - \lambda, \tag{5}$$

and it suffices to compute one generating curve where $x \geq 0$.

Note that then the distance d is not important to our consideration, and hence we can view our problem in this reduced dimensional setting, or as extending infinitely in a horizontal direction. With this perspective, we normalize so that $d = 1$ so that we are considering volume per unit distance in the y-direction. The solution may be extended infinitely in both y-directions and be seen as an infinitely long liquid bridge between two horizontal plates generated by the curves in Π_0. The solution may also interpreted as a lower dimensional problem, where the interfaces are reduced to curves in the plane Π_0, spanning a *volume* that is more properly seen as an area in Π_0. This last interpretation is the easiest way to visualize the results of our computations, and so is our default for figures, even while we continue to use the terminology of *volume* and *area*, and we use them in the sense of per unit distance d.

3. Computing the fluid volume

Consider solutions to (3)–(5) with the boundary conditions

$$\sin\psi(0) = \cos\gamma_0 \quad \text{at } s = 0, \text{ where } u(0) = 0, \tag{6}$$

$$\sin\psi(\ell) = \cos\gamma_h \quad \text{at } s = \ell, \text{ where } u(\ell) = h. \tag{7}$$

Solutions to this two-point boundary value problem will determine a value of $x(0)$, which we denote by $\delta/2$. We will later use this as a parameter in the process of constructing approximate solutions, but it is immediately useful in determining a volume formula as follows.

Theorem 3.1. *The volume enclosed by the upper plate, lower plate, and the fluid-air interface given by area per unit distance in the y-direction satisfies*

$$\mathcal{V} = (h-\lambda)\Big(x(\ell) - \frac{\delta}{2}\Big) + \sin\gamma_0 - \sin\gamma_h, \tag{8}$$

where the solutions x, u, and ψ are parametrized by arc length s, with $s = 0$ at height $u = 0$ and $s = \ell$ at height $u = h$.

Proof. We find the volume of the enclosed fluid by computing the right half of the volume. The geometric idea is to start with a rectangle with height h and width $\delta/2$, and then add to it the additional volume outside this region. The first configuration is illustrated in Figure 3. This configuration contains a vertical point given by $(\bar{x}, \bar{u})$, and this partitions the volume outside of the rectangle into two regions. The lower

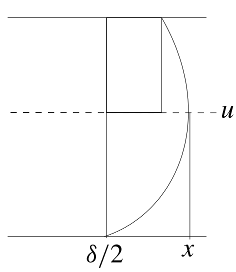

Figure 3. The configuration used in the volume computation, with only the portion $x > 0$ shown. Here $x = \bar{x}$ and $u = \bar{u}$, and the plate heights are 0 and h.

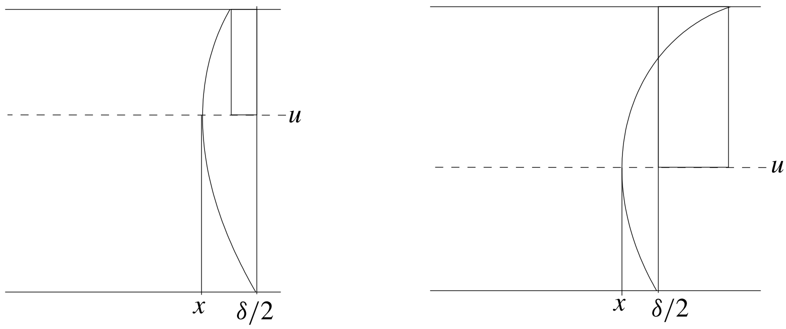

Figure 4. Two volume configurations. Here $x = \bar{x}$ and $u = \bar{u}$, and plate heights are 0 and h.

region is bounded by $\delta/2$ on the left, $u = \bar{u}$ above, and the fluid interface on the right. The upper region is bounded by $x = x(\ell)$ on the left, $u = \bar{u}$ below, and the fluid interface on the right. Added to this upper region is a second, smaller, rectangle of height $h - \bar{u}$ and width $x(\ell) - \delta/2$. So, we calculate using equations (3)–(5) and integration by parts as follows:

$$\mathcal{V} = \int_{\delta/2}^{\bar{x}} (\bar{u} - u)\,dx + \int_{x(\ell)}^{\bar{x}} (u - \bar{u})\,dx + \Big(x(\ell) - \frac{\delta}{2}\Big)(h - \bar{u}) \tag{9}$$

$$= \bar{u}\Big(\bar{x} - \frac{\delta}{2} + x(\ell) - \bar{x}\Big) + \Big(x(\ell) - \frac{\delta}{2}\Big)(h - \bar{u}) + \int_{x(\ell)}^{\bar{x}} u\,dx - \int_{\delta/2}^{\bar{x}} u\,dx \tag{10}$$

$$= h\Big(x(\ell) - \frac{\delta}{2}\Big) + \bar{u}(0) + \int_{x(\ell)}^{\bar{x}} u\,dx - \int_{\delta/2}^{\bar{x}} u\,dx \tag{11}$$

$$= h\Big(x(\ell) - \frac{\delta}{2}\Big) + \int_{x(\ell)}^{\bar{x}} \Big(\frac{d\psi}{ds} + \lambda\Big)\,dx - \int_{\delta/2}^{\bar{x}} \Big(\frac{d\psi}{ds} + \lambda\Big)\,dx \tag{12}$$

$$= (h - \lambda)\Big(x(\ell) - \frac{\delta}{2}\Big) + \int_{x(\ell)}^{\bar{x}} \frac{d\psi}{ds}\,dx - \int_{\delta/2}^{\bar{x}} \frac{d\psi}{ds}\,dx \tag{13}$$

$$= (h - \lambda)\Big(x(\ell) - \frac{\delta}{2}\Big) + \int_{\gamma_h - \pi}^{\pi/2} \cos\psi\,d\psi - \int_{-\gamma_0}^{\pi/2} \cos\psi\,d\psi \tag{14}$$

$$= (h - \lambda)\Big(x(\ell) - \frac{\delta}{2}\Big) + \sin\gamma_0 - \sin\gamma_h. \tag{15}$$

There are multiple possible configurations; however, it suffices to adapt the above calculation to these remaining cases:

- $x(s) < \delta/2$ for $0 < s < \ell$ and $x(\ell) < \delta/2$. See Figure 4 (left).
- $x(s) < \delta/2$ for some initial $s > 0$, and then $x(s)$ increases and $x(\ell) > \delta/2$. See Figure 4 (right).

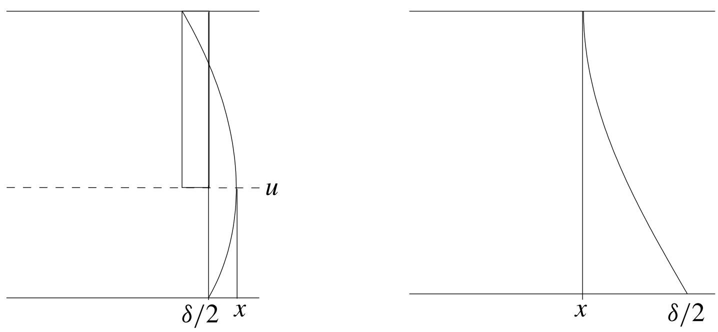

Figure 5. Two remaining volume configurations. Here $x = \bar{x}$ and $u = \bar{u}$, and plate heights are 0 and h.

- $x(s) > \delta/2$ for some initial $s > 0$, and then $x(s)$ decreases and $x(\ell) < \delta/2$. See Figure 5 (left).
- There is no vertical point on the interface profile curve. There are many such configurations; see Figure 5 (right) for a typical example. The volume computation is straightforward in these cases, only requiring use of (3)–(5). □

4. Numerical solver

We use a shooting method to solve the two-point boundary value problem of (3)–(5) with boundary conditions (6) and (7). We implement this by nesting two algorithms, namely an inner implementation of an adaptive Runge–Kutta–Felberg method and an outer implementation of a multidimensional root finder.

Values for the initial and terminal contact angles γ_0, γ_h, volume $\mathcal{V}$, and height h are prescribed for the desired solution. The lower conditions for the boundary value problem are

$$r(0) = \frac{\delta}{2}, \tag{16}$$

$$u(0) = 0, \tag{17}$$

$$\psi(0) = \gamma_0, \tag{18}$$

where the tangent to the curve forms the contact angle γ_0 with the lower plate, and the upper boundary conditions are

$$u(\ell) = h, \tag{19}$$

$$\psi(\ell) = -\gamma_h, \tag{20}$$

where the ending arc length ℓ is chosen to terminate at height h with the tangent to the curve forming the angle γ_h with the upper parallel plate.

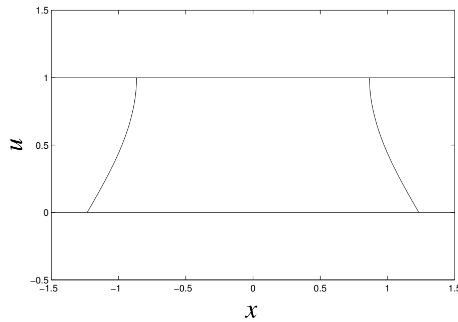

Figure 6. A liquid bridge with contact angle $\pi/2$ on the upper plate.

Again, the boundary value problem is solved using a shooting method based on an adaptive ODE solver. The solver uses the adaptive Runge–Kutta–Felberg method for 4th and 5th order, implemented by Matlab as ODE45. The absolute and relative tolerances were both set to $1e-8$. To begin to solve the problem, reasonable guesses are given for the free parameters: the distance between the generating curves δ, the ending arc length ℓ, and the Lagrange multiplier λ. These values are used to generate candidates satisfying the ODE. Then the solutions to (3)–(5) with these values of the free parameters are used to evaluate the equations

$$\mathcal{V} - V(\ell) = 0, \tag{21}$$

$$h - u(\ell) = 0, \tag{22}$$

$$\gamma_h - \psi(\ell) = 0, \tag{23}$$

which are not, in general, solved. The parameters δ, ℓ, and λ are adjusted in the multidimensional root finder implemented in Matlab as FSOLVE, which defaults to a trust region method. The tolerances for this portion of the algorithm were set to $1e-6$. We recompute the solutions to (3)–(5) with new values of the parameters δ, ℓ, and λ at each step, until (21)–(23) are satisfied to the prescribed tolerance.

5. Examples

We present some examples of note generated with the algorithm described in the previous section. In Figure 2 we saw a typical example of a configuration where $\gamma_0, \gamma_h \in \left[0, \frac{\pi}{2}\right]$. Figure 6 shows a configuration where $\gamma_h = \pi/2$, and Figure 7 shows a configuration where both $\gamma_0, \gamma_h > \pi/2$. If the volume does not span the gap between P_0 and P_h, then it will rest on the plate P_0 as a sessile drop. We see in Figure 8 a configuration where $(\gamma_0, \gamma_h) = (2.57, 1.05)$, which appears to be close to the maximum height h before the liquid bridge pinches off of the upper plate P_h and becomes a sessile drop.

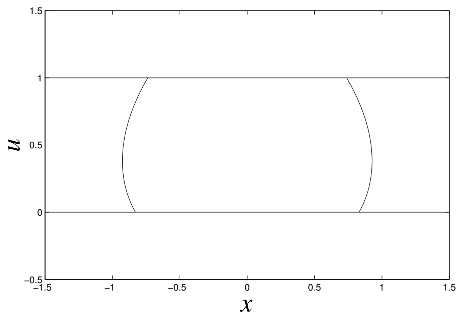

Figure 7. A liquid bridge with both γ_0 and γ_h larger than $\pi/2$.

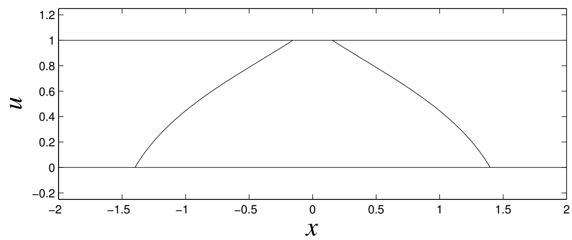

Figure 8. A liquid bridge that is visually similar to a sessile drop.

6. Minimum spanning volume

Consider configurations where the contact angles γ_0 and γ_h are both less than $\pi/2$. The phenomenon explored is the minimum volume which admits a solution spanning the two plates P_0 and P_h. In Figure 9 we see that for angles $(\gamma_0, \gamma_h) = (0.99, 0)$ and a particular volume, we have a point on the interior of the fluid interface on the right that touches a corresponding point on the interior of the fluid interface on the left.

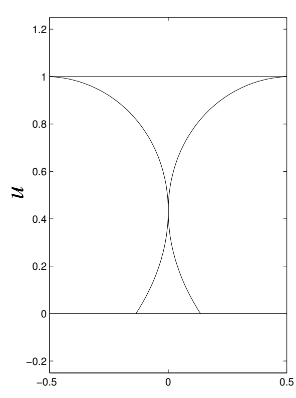

Figure 9. A liquid bridge with interfaces touching on the interior.

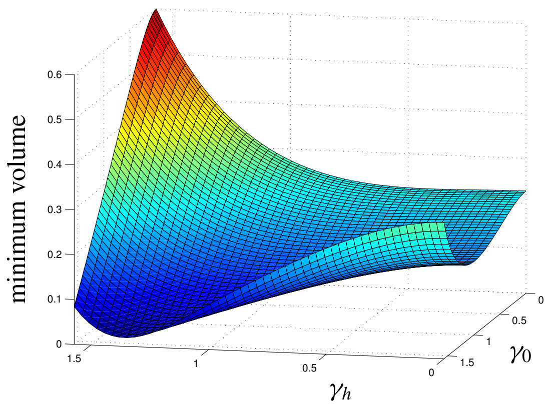

Figure 10. The minimum spanning volume over a grid of 50×50 samples in the (γ_0, γ_h)-space.

This is clearly nonphysical and represents an absolute minimum spanning volume. It is apparent that this contact between the left and right interfaces occurs on either P_0 or P_h if either $\gamma_0 > \pi/2$ or $\gamma_h > \pi/2$. Therefore, we restrict our attention to the region $0 \le \gamma_0 \le \pi/2$ and $0 \le \gamma_h \le \pi/2$. We seek a minimum volume where $x(s) = 0$ for some $s \in [0, \ell]$.

Observe the crucial fact of the system (3)–(5) that

$$\frac{d\psi}{ds} = \kappa u - \lambda$$

is independent of x, and so the x solution may be translated by a constant. We are able to use this to some degree to adjust the volume spanned. If the left and right interfaces are rigidly moved apart in the x-direction, then the spanned volume increases while still solving the boundary value problem, and conversely, if they are rigidly moved together, they will eventually touch. At this point there exists an arc length s such that $x(s) = 0$ for both the left and right portions of the configuration. We are able to use this idea in conjunction with our previous solver to obtain the minimum spanning volume at a fixed height h for a given pair of contact angles (γ_0, γ_h).

We use the following algorithm to run over a grid of 50×50 samples in the (γ_0, γ_h)-space. We solve the constrained boundary value problem similar to the method in Section 4, however, we replace the condition

$$\mathcal{V} - V(\ell) = 0$$

with

$$x(s) = 0 \quad \text{for some } s \in [0, \ell]. \tag{24}$$

The results are collected in Figure 10. Here it is worth noting that the examples from Figure 11 are generated from interesting points on the minimum spanning volume

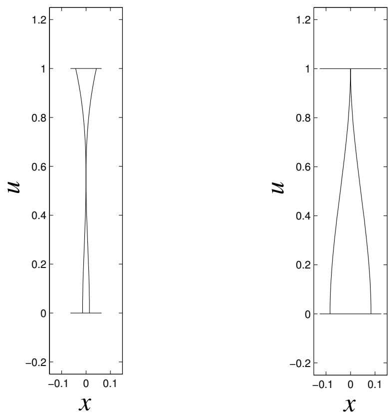

Figure 11. Left: The minimum spanning volume. Here $(\gamma_0, \gamma_h) \approx \left(\frac{\pi}{2}, 1.35\right)$. Right: A very small spanning volume, but not the minimum spanning volume. Here $(\gamma_0, \gamma_h) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

surface. The minimum spanning volume on the left is actually the minimum volume of all the contact angle pairs, and perhaps surprisingly, it is not the $(\pi/2, \pi/2)$ case (which is pictured on the right).

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