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Some projective distance inequalities for simplices in complex projective space

Mark Fincher, Heather Olney and William Cherry

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We prove inequalities relating the absolute value of the determinant of $n+1$ linearly independent unit vectors in $\mathbb{C}^{n+1}$ and the projective distances from the vertices to the hyperplanes containing the opposite faces of the simplices in complex projective n-space whose vertices or faces are determined by the given vectors.

A basis of unit vectors in $\mathbb{C}^{n+1}$ determines the vertices (or the faces) of a simplex in n-dimensional complex projective space. For reasons originally motivated by an inequality in complex function theory proven by Cherry and Eremenko [2011], we investigated the relationship between the determinant of the vectors forming the basis and the projective distances from each vertex of the simplex to the hyperplane containing the face of the opposite side. We show that if $d_{\min}$ denotes the minimum of these projective distances and if D denotes the determinant of the basis vectors, then $d_{\min}^n \le |D| \le d_{\min}$.

Let $\boldsymbol{e}_0, \ldots, \boldsymbol{e}_n$ be a basis for $\mathbb{C}^{n+1}$. Given two vectors $\boldsymbol{a} = a_0\boldsymbol{e}_0 + \cdots + a_n\boldsymbol{e}_n$ and $\boldsymbol{b} = b_0\boldsymbol{e}_0 + \cdots + b_n\boldsymbol{e}_n$ in $\mathbb{C}^{n+1}$, we use $\boldsymbol{a}\cdot\boldsymbol{b}$ to denote the standard dot product,

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_0b_0 + \cdots + a_nb_n,$$

rather than the Hermitian inner product more typically used with complex vector spaces. Thus, in our notation,

$$|\boldsymbol{a}|^2 = \boldsymbol{a}\cdot\bar{\boldsymbol{a}},$$

where the bar denotes complex conjugation, as usual.

For $k = 1, \ldots, n+1$, we let $\Lambda^k\mathbb{C}^{n+1}$ denote the k-th exterior power of the vector space $\mathbb{C}^{n+1}$, and we recall that

$$\boldsymbol{e}_0\wedge\boldsymbol{e}_1\wedge\cdots\wedge\boldsymbol{e}_{k-1}, \quad \ldots, \quad \boldsymbol{e}_{i_1}\wedge\boldsymbol{e}_{i_2}\wedge\cdots\wedge\boldsymbol{e}_{i_k}, \quad \ldots, \quad \boldsymbol{e}_{n+1-k}\wedge\boldsymbol{e}_{n+2-k}\wedge\cdots\wedge\boldsymbol{e}_n,$$

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where $0 \leq i_1 < i_2 < \cdots < i_k \leq n$ form a basis for $\Lambda^k\mathbb{C}^{n+1}$. By declaring this basis to be orthonormal in $\Lambda^k\mathbb{C}^{n+1}$, the norm and dot product on $\mathbb{C}^{n+1}$ extend to a norm and inner product on $\Lambda^k\mathbb{C}^{n+1}$. For a detailed introduction to exterior algebras and wedge products, see [Bowen and Wang 1976].

Proposition 1. *Let $1 \leq k \leq n+1$ be an integer, and let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$ be vectors in $\mathbb{C}^{n+1}$. Then,*

$$(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_k) \cdot (\boldsymbol{w}_1 \wedge \cdots \wedge \boldsymbol{w}_k) = \det(\boldsymbol{v}_i \cdot \boldsymbol{w}_j)_{1 \leq i,j \leq k}.$$

Remark. The matrix of dot products on the right is called a *Gramian* matrix.

Proof. This is Exercise 39.3 in [Bowen and Wang 1976]. □

Corollary 2. *Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ be k vectors in $\mathbb{C}^{n+1}$. Then,*

$$|\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_k|^2 = \det(\boldsymbol{v}_i \cdot \bar{\boldsymbol{v}}_j)_{1 \leq i,j \leq k}.$$

Corollary 3. *Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ be k vectors in $\mathbb{C}^{n+1}$. Then,*

$$|\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_k| \leq |\boldsymbol{v}_1| \cdots |\boldsymbol{v}_k|.$$

Equality holds if and only if one of the vectors is the zero vector or if $\boldsymbol{v}_i \cdot \bar{\boldsymbol{v}}_j = 0$ for all $i \neq j$.

Proof. If any of the vectors $\boldsymbol{v}_j$ are the zero vector, then the inequality is obvious. So, assume that none of the $\boldsymbol{v}_j$ are zero. Let

$$\boldsymbol{u}_j = \frac{\boldsymbol{v}_j}{|\boldsymbol{v}_j|}$$

be unit vectors in the directions of the $\boldsymbol{v}_j$. Then, clearly,

$$|\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_k| = \Big| |\boldsymbol{v}_1|\boldsymbol{u}_1 \wedge \cdots \wedge |\boldsymbol{v}_k|\boldsymbol{u}_k \Big| = |\boldsymbol{v}_1| \cdots |\boldsymbol{v}_k| |\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_k|.$$

Thus, it suffices to show that $|\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_k| \leq 1$. To this end, by Corollary 2,

$$|\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_k|^2 = \det(\boldsymbol{u}_i \cdot \bar{\boldsymbol{u}}_j). \tag{1}$$

The matrix $(\boldsymbol{u}_i \cdot \bar{\boldsymbol{u}}_j)$ is a $k \times k$ Hermitian matrix with nonnegative eigenvalues $\lambda_1, \ldots, \lambda_k$. Thus, by the geometric-arithmetic mean inequality,

$$\det(\boldsymbol{u}_i \cdot \bar{\boldsymbol{u}}_j) = \lambda_1 \cdots \lambda_k \leq \left(\frac{\lambda_1 + \cdots + \lambda_k}{k}\right)^k = 1,$$

where the equality on the right follows from the fact that

$$\lambda_1 + \cdots + \lambda_k = \operatorname{Trace}(\boldsymbol{u}_i \cdot \bar{\boldsymbol{u}}_j) = k,$$

since $\boldsymbol{u}_i \cdot \bar{\boldsymbol{u}}_i = 1$.

Equality holds in the arithmetic-geometric mean inequality if and only if all the eigenvalues are equal, and hence all equal to one. This is the case if and only if $(\boldsymbol{u}_i \cdot \bar{\boldsymbol{u}}_j)$ is the $k \times k$ identity matrix, which happens if and only if $\boldsymbol{v}_i \cdot \bar{\boldsymbol{v}}_j = 0$ for all $i \neq j$. □

We will be most interested in the n-th exterior power of $\mathbb{C}^{n+1}$, where

$$\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n, \quad \ldots, \quad \boldsymbol{e}_0 \wedge \cdots \wedge \boldsymbol{e}_{j-1} \wedge \boldsymbol{e}_{j+1} \wedge \cdots \wedge \boldsymbol{e}_n, \quad \ldots, \quad \boldsymbol{e}_0 \wedge \cdots \wedge \boldsymbol{e}_{n-1}$$

form a basis of $\Lambda^n \mathbb{C}^{n+1}$. Let L denote the isometric isomorphism from $\Lambda^n \mathbb{C}^{n+1}$ to $\mathbb{C}^{n+1}$ defined on the basis vectors as follows:

$$\begin{aligned} L(\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n) &= \boldsymbol{e}_0, \\ &\vdots \\ L(\boldsymbol{e}_0 \wedge \cdots \wedge \boldsymbol{e}_{j-1} \wedge \boldsymbol{e}_{j+1} \wedge \cdots \wedge \boldsymbol{e}_n) &= (-1)^j \boldsymbol{e}_j, \\ &\vdots \\ L(\boldsymbol{e}_0 \wedge \cdots \wedge \boldsymbol{e}_{n-1}) &= (-1)^n \boldsymbol{e}_n. \end{aligned}$$

Observe that if $n = 2$ and $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors in $\mathbb{C}^3$, then $L(\boldsymbol{a} \wedge \boldsymbol{b}) = \boldsymbol{a} \times \boldsymbol{b}$, where the product on the right is the ordinary cross product in $\mathbb{C}^3$.

We will use $L(\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n)$ as a generalized cross product.

Proposition 4. *Let $\boldsymbol{a}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ be $n+1$ vectors in $\mathbb{C}^{n+1}$. Then,*

$$\det(\boldsymbol{a}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n) = \boldsymbol{a} \cdot L(\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n).$$

Proof. If we compute the determinant of the $(n+1) \times (n+1)$ matrix whose rows are $\boldsymbol{a}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$, then the expression on the right is nothing other than the computation of the determinant by expansion of minors along the first row. □

Corollary 5. *The vector $L(\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n)$ is orthogonal to each of the $\boldsymbol{b}_j$.*

We define an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$ by declaring that two nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{C}^{n+1}$ are equivalent if there exists a nonzero complex scalar c such that $\boldsymbol{v} = c\boldsymbol{w}$. The set of all such equivalence classes is denoted by $\mathbb{CP}^n$ and is called the *complex projective space* of dimension n. A point in $\mathbb{CP}^n$ is an equivalence class of vectors in $\mathbb{C}^{n+1}$ and by the definition of the equivalence relation, we can always represent a point in $\mathbb{CP}^n$ by a unit vector in $\mathbb{C}^{n+1}$. The set of equivalence classes associated with the vectors in a $k+1$ dimensional subspace of $\mathbb{C}^{n+1}$ is a k-dimensional subspace of $\mathbb{CP}^n$. When $k = n-1$, such a subspace is called a hyperplane in $\mathbb{CP}^n$. We say that $n+1$ points in $\mathbb{CP}^n$ are in *general position* if they are not all contained in any one hyperplane. This is equivalent to the vectors representing the points being linearly independent in $\mathbb{C}^{n+1}$. Similarly, we say that $n+1$ hyperplanes in $\mathbb{CP}^n$ are in *general position* if there is no point in

$\mathbb{CP}^n$ contained in all the hyperplanes. Note that a nonzero vector $\boldsymbol{v}$ in $\mathbb{C}^{n+1}$ can be thought of as representing a hyperplane where the points in the hyperplane are represented by the vectors $\boldsymbol{x}$ in $\mathbb{C}^{n+1}$ such that $\boldsymbol{v} \cdot \boldsymbol{x} = 0$.

If $\boldsymbol{v}$ and $\boldsymbol{w}$ are two unit vectors in $\mathbb{C}^{n+1}$ representing points in $\mathbb{CP}^n$, then the *Fubini–Study distance* between the two points is defined to be $|\boldsymbol{v} \wedge \boldsymbol{w}|$. Now let $\boldsymbol{u}$ and $\boldsymbol{v}$ be unit vectors in $\mathbb{C}^{n+1}$. We think of $\boldsymbol{u}$ as representing a point in $\mathbb{CP}^n$ and $\boldsymbol{v}$ as representing a hyperplane in $\mathbb{CP}^n$. Then, the Fubini–Study distance from the point represented by $\boldsymbol{u}$ to the hyperplane represented by $\boldsymbol{v}$ is defined by

$$\begin{aligned}
&\text{distance from the point } \boldsymbol{u} \text{ to the hyperplane } \boldsymbol{v} \\
&\qquad = \min\{\text{distance from } \boldsymbol{u} \text{ to } \boldsymbol{x} : \boldsymbol{v} \cdot \boldsymbol{x} = 0 \text{ and } |\boldsymbol{x}| = 1\} \\
&\qquad = \min\{|\boldsymbol{u} \wedge \boldsymbol{x}| : \boldsymbol{v} \cdot \boldsymbol{x} = 0 \text{ and } |\boldsymbol{x}| = 1\}.
\end{aligned}$$

Second perhaps only to hyperbolic geometry, projective geometry, which arose out of the study of perspective in classical painting, is among the most ubiquitous of the non-Euclidean geometries encountered in modern mathematics. See, for instance, [Richter-Gebert 2011] for a recent accessible introduction.

Our first result is a convenient formula for the distance from a vertex of a projective simplex to the hyperplane determined by the opposite face in the simplex.

Proposition 6. *Let $\boldsymbol{a}, \boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ be $n+1$ linearly independent unit vectors in $\mathbb{C}^{n+1}$ representing $n+1$ points in general position in $\mathbb{CP}^n$. Then, the Fubini–Study distance d from the point $\boldsymbol{a}$ to the hyperplane in $\mathbb{CP}^n$ spanned by $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ is given by*

$$d = \frac{|\det(\boldsymbol{a}, \boldsymbol{b}_1, \dots, \boldsymbol{b}_n)|}{|\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n|}.$$

Proof. Without loss of generality, by making an orthogonal change of coordinates, we may choose our standard basis vectors $\boldsymbol{e}_0, \dots, \boldsymbol{e}_n$ in $\mathbb{C}^{n+1}$ so that $\boldsymbol{e}_0 \cdot \boldsymbol{b}_j = 0$ for $j = 1, \dots, n$. Let $\boldsymbol{u}$ be a unit vector in the span of $\{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$. Then,

$$\boldsymbol{u} = u_1 \boldsymbol{e}_1 + \cdots + u_n \boldsymbol{e}_n, \quad \text{with } |u_1|^2 + \cdots + |u_n|^2 = 1.$$

Let $\boldsymbol{a} = a_0 \boldsymbol{e}_0 + \cdots + a_n \boldsymbol{e}_n$. Then, the Fubini–Study distance from the point in $\mathbb{CP}^n$ represented by $\boldsymbol{a}$ to the point in $\mathbb{CP}^n$ represented by $\boldsymbol{u}$ is given by $|\boldsymbol{a} \wedge \boldsymbol{u}|$. Note that

$$\boldsymbol{a} \wedge \boldsymbol{u} = a_0 u_1 \boldsymbol{e}_0 \wedge \boldsymbol{e}_1 + \cdots + a_0 u_n \boldsymbol{e}_0 \wedge \boldsymbol{e}_n + \sum_{1 \le i < j \le n} (a_i u_j - a_j u_i) \boldsymbol{e}_i \wedge \boldsymbol{e}_j. \tag{2}$$

Hence,

$$|\boldsymbol{a} \wedge \boldsymbol{u}|^2 \ge |a_0 u_1|^2 + \cdots + |a_0 u_n|^2 = |a_0|^2 (|u_1|^2 + \cdots + |u_n|^2) = |a_0|^2. \tag{3}$$

Now,

$$\det(\boldsymbol{a}, \boldsymbol{b}_1, \dots, \boldsymbol{b}_n) = \boldsymbol{a} \cdot L(\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n)$$

by Proposition 4. Of course, $L(\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n)$ is orthogonal to each of the $\boldsymbol{b}_j$. By our choice of basis, $\boldsymbol{e}_0$ is also orthogonal to each of the $\boldsymbol{b}_j$. Since the $\boldsymbol{b}_j$ form a set of n linearly independent vectors in an $(n+1)$-dimensional vector space, there is only one direction simultaneously orthogonal to all of the $\boldsymbol{b}_j$. Thus, $L(\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n)$ is in the span of $\boldsymbol{e}_0$, and so

$$|\boldsymbol{a} \cdot L(\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n)| = |a_0| \cdot |L(\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n)|.$$

Thus, observing that

$$|L(\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n)| = |\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n|,$$

we see from (3) that

$$\begin{aligned}|\boldsymbol{a} \wedge \boldsymbol{u}| \geq |a_0| &= \frac{|a_0| \cdot |L(\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n)|}{|\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n|} \\ &= \frac{|\boldsymbol{a} \cdot L(\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n)|}{|\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n|} \\ &= \frac{|\det(\boldsymbol{a}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)|}{|\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n|}.\end{aligned}$$

To complete the proof, we need to show that equality is obtained for some choice of $\boldsymbol{u}$. There are two cases. If $\boldsymbol{a}$ is the direction of $\boldsymbol{e}_0$, then equality holds for any choice of $\boldsymbol{u}$ since $a_1 = \cdots = a_n = 0$. Otherwise, if we choose

$$u_j = \frac{a_j}{\sqrt{|a_1|^2 + \cdots + |a_n|^2}} \quad \text{for } j = 1, \ldots, n,$$

we see that the terms in the sum on the far right of (2) are all zero, and so equality holds in (3). □

Corollary 7. *Let $\boldsymbol{a}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ and d be as in Proposition 6. Then,*

$$d \geq |\det(\boldsymbol{a}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)|.$$

Equality holds if and only if $\boldsymbol{b}_i \cdot \bar{\boldsymbol{b}}_j = 0$ for all $i \neq j$.

Example 8. When $n = 3$, let $0 < s \leq 1$ and consider the projective triangle with vertices represented by the unit vectors

$$\boldsymbol{a} = \left(\sqrt{\frac{1-s^2}{2}}, \sqrt{\frac{1-s^2}{2}}, s\right), \quad \boldsymbol{b}_1 = (1, 0, 0), \quad \text{and} \quad \boldsymbol{b}_2 = (0, 1, 0).$$

Then, $|\boldsymbol{b}_1 \wedge \boldsymbol{b}_2| = 1$, and so $d = \det(\boldsymbol{a}, \boldsymbol{b}_1, \boldsymbol{b}_2) = s$, and equality holds in Corollary 7. We remark that geometrically, these triangles are isosceles with projective side lengths

$$1, \quad \sqrt{\frac{1+s^2}{2}}, \quad \sqrt{\frac{1+s^2}{2}}.$$

Proof of Corollary 7. By Corollary 3, we have

$$|\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n| \leq 1.$$

Hence, by the formula for d in Proposition 6,

$$d = \frac{|\det(\boldsymbol{a}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)|}{|\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n|} \geq |\det(\boldsymbol{a}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)|.$$

Equality holds if and only if equality holds in Corollary 3. □

Proposition 9. *Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}$ be $n-1$ linearly independent vectors in $\mathbb{C}^{n+1}$ and let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ be n linearly independent vectors in $\mathbb{C}^{n+1}$. If we let*

$$\boldsymbol{a} = L(\boldsymbol{w}_1 \wedge \cdots \wedge \boldsymbol{w}_n) \quad \textit{and} \quad \boldsymbol{b} = L(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_{n-1} \wedge \boldsymbol{a}),$$

then

$$\boldsymbol{b} = (-1)^n \det \begin{pmatrix} \boldsymbol{w}_1 & \ldots & \boldsymbol{w}_n \\ \boldsymbol{v}_1 \cdot \boldsymbol{w}_1 & \ldots & \boldsymbol{v}_1 \cdot \boldsymbol{w}_n \\ \vdots & \vdots & \vdots \\ \boldsymbol{v}_{n-1} \cdot \boldsymbol{w}_1 & \ldots & \boldsymbol{v}_{n-1} \cdot \boldsymbol{w}_n \end{pmatrix}.$$

Remark. Note that the matrix specified in the proposition has vector entries in its first row, and hence its determinant results in a vector. This proposition is a generalization of Lagrange's formula for the vector triple product in $\mathbb{R}^3$. The proof of this proposition was inspired by a discussion Cherry had with Charles Conley, and we thank him for his interest. We suspect that Proposition 9 is reasonably well-known, but we were unable to find a reference to it in the literature.

Proof. Let

$$\tilde{\boldsymbol{b}} = \det \begin{pmatrix} \boldsymbol{w}_1 & \ldots & \boldsymbol{w}_n \\ \boldsymbol{v}_1 \cdot \boldsymbol{w}_1 & \ldots & \boldsymbol{v}_1 \cdot \boldsymbol{w}_n \\ \vdots & \vdots & \vdots \\ \boldsymbol{v}_{n-1} \cdot \boldsymbol{w}_1 & \ldots & \boldsymbol{v}_{n-1} \cdot \boldsymbol{w}_n \end{pmatrix}.$$

We want to show that $\boldsymbol{b} = (-1)^n \tilde{\boldsymbol{b}}$, and for this, it suffices to show that for all $\boldsymbol{z}$ in $\mathbb{C}^{n+1}$, we have $\boldsymbol{z} \cdot \boldsymbol{b} = (-1)^n \boldsymbol{z} \cdot \tilde{\boldsymbol{b}}$. Clearly,

$$\boldsymbol{z} \cdot \tilde{\boldsymbol{b}} = \det \begin{pmatrix} \boldsymbol{z} \cdot \boldsymbol{w}_1 & \ldots & \boldsymbol{z} \cdot \boldsymbol{w}_n \\ \boldsymbol{v}_1 \cdot \boldsymbol{w}_1 & \ldots & \boldsymbol{v}_1 \cdot \boldsymbol{w}_n \\ \vdots & \vdots & \vdots \\ \boldsymbol{v}_{n-1} \cdot \boldsymbol{w}_1 & \ldots & \boldsymbol{v}_{n-1} \cdot \boldsymbol{w}_n \end{pmatrix}.$$

On the other hand, by Proposition 4,

$$\begin{aligned}
\boldsymbol{z}\cdot\boldsymbol{b} &= \det(\boldsymbol{z}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-1}, \boldsymbol{a})\\
&= (-1)^n \det(\boldsymbol{a}, \boldsymbol{z}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-1})\\
&= (-1)^n \boldsymbol{a}\cdot L(\boldsymbol{z}\wedge\boldsymbol{v}_1\wedge\cdots\wedge\boldsymbol{v}_{n-1})\\
&= (-1)^n L(\boldsymbol{w}_1\wedge\cdots\wedge\boldsymbol{w}_n)\cdot L(\boldsymbol{z}\wedge\boldsymbol{v}_1\wedge\cdots\wedge\boldsymbol{v}_{n-1})\\
&= (-1)^n (\boldsymbol{w}_1\wedge\cdots\wedge\boldsymbol{w}_n)\cdot(\boldsymbol{z}\wedge\boldsymbol{v}_1\wedge\cdots\wedge\boldsymbol{v}_{n-1}) \quad \text{(since } L \text{ is an isometry)}\\
&= (-1)^n (\boldsymbol{z}\wedge\boldsymbol{v}_1\wedge\cdots\wedge\boldsymbol{v}_{n-1})\cdot(\boldsymbol{w}_1\wedge\cdots\wedge\boldsymbol{w}_n)\\
&= (-1)^n \det\begin{pmatrix} \boldsymbol{z}\cdot\boldsymbol{w}_1 & \dots & \boldsymbol{z}\cdot\boldsymbol{w}_n\\ \boldsymbol{v}_1\cdot\boldsymbol{w}_1 & \dots & \boldsymbol{v}_1\cdot\boldsymbol{w}_n\\ \vdots & \vdots & \vdots\\ \boldsymbol{v}_{n-1}\cdot\boldsymbol{w}_1 & \dots & \boldsymbol{v}_{n-1}\cdot\boldsymbol{w}_n \end{pmatrix} \quad \text{(by Proposition 1).} \qquad \square
\end{aligned}$$

Proposition 10. *Let $\boldsymbol{a}, \boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ be $n+1$ linearly independent vectors in $\mathbb{C}^{n+1}$. For $j = 1, \dots, n$, let*

$$\boldsymbol{v}_j = L(\boldsymbol{a}\wedge\boldsymbol{u}_1\wedge\cdots\wedge\boldsymbol{u}_{j-1}\wedge\boldsymbol{u}_{j+1}\wedge\cdots\wedge\boldsymbol{u}_n).$$

Then, $L(\boldsymbol{v}_1\wedge\cdots\wedge\boldsymbol{v}_n) = \pm D^{n-1}\boldsymbol{a}$, where $D = \det(\boldsymbol{a}, \boldsymbol{u}_1, \dots, \boldsymbol{u}_n)$.

Remark. The unspecified sign depends only on n and can be explicitly determined from the proof. Since the sign will not matter for our purpose, we did not bother to record it here.

Proof. By Proposition 9, we get that

$$L(\boldsymbol{v}_1\wedge\cdots\wedge\boldsymbol{v}_n) = (-1)^n \det\begin{pmatrix} \boldsymbol{a} & \boldsymbol{u}_1 & \dots & \boldsymbol{u}_{n-1}\\ \boldsymbol{v}_1\cdot\boldsymbol{a} & \boldsymbol{v}_1\cdot\boldsymbol{u}_1 & \dots & \boldsymbol{v}_1\cdot\boldsymbol{u}_{n-1}\\ \vdots & \vdots & \vdots & \vdots\\ \boldsymbol{v}_{n-1}\cdot\boldsymbol{a} & \boldsymbol{v}_{n-1}\cdot\boldsymbol{u}_1 & \dots & \boldsymbol{v}_{n-1}\cdot\boldsymbol{u}_{n-1} \end{pmatrix}.$$

If $i \neq j$, then

$$\boldsymbol{v}_i\cdot\boldsymbol{u}_j = L(\boldsymbol{a}\wedge\cdots\wedge\boldsymbol{u}_{i-1}\wedge\boldsymbol{u}_{i+1}\wedge\cdots\wedge\boldsymbol{u}_n)\cdot\boldsymbol{u}_j = 0$$

since $\boldsymbol{u}_j$ appears in the wedge product defining $\boldsymbol{v}_i$, and hence $\boldsymbol{v}_i$ is orthogonal to $\boldsymbol{u}_j$. Similarly, $\boldsymbol{v}_i\cdot\boldsymbol{a} = 0$. Moreover,

$$\boldsymbol{v}_j\cdot\boldsymbol{u}_j = L(\boldsymbol{a}\wedge\cdots\wedge\boldsymbol{u}_{j-1}\wedge\boldsymbol{u}_{j+1}\wedge\cdots\wedge\boldsymbol{u}_n)\cdot\boldsymbol{u}_j = (-1)^j D$$

by Proposition 4. Hence,

$$L(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n) = (-1)^n \det \begin{pmatrix} \boldsymbol{a} & \boldsymbol{u}_1 & \boldsymbol{u}_2 & \dots & \boldsymbol{u}_{n-1} \\ 0 & -D & 0 & \dots & 0 \\ 0 & 0 & D & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & (-1)^{n-1}D \end{pmatrix} = \pm D^{n-1}\boldsymbol{a}. \quad \square$$

Theorem 11. *Let $\boldsymbol{u}_0, \dots, \boldsymbol{u}_n$ be $n+1$ linearly independent unit vectors in $\mathbb{C}^{n+1}$ representing $n+1$ points in general position in $\mathbb{CP}^n$, which we think of as the vertices of a projective simplex. For each j from 0 to n, let d_j denote the Fubini–Study distance from the point represented by $\boldsymbol{u}_j$ to the hyperplane containing the opposite face of the simplex. Let $d_{\min}$ denote the minimum of the d_j. Then,*

$$d_{\min}^n \le |\det(\boldsymbol{u}_0, \dots, \boldsymbol{u}_n)|.$$

For equality to hold, at least n of the $n+1$ projective distances d_j must equal $d_{\min}$.

Proof. Let $D = \det(\boldsymbol{u}_0, \dots, \boldsymbol{u}_n)$. Note that $D \neq 0$ by the linear independence (general position) hypothesis. Without loss of generality, assume that $d_{\min} = d_n$. Then, $d_{\min}^n \le d_1 d_2 \cdots d_n$, and equality holds if and only if all of these distances are equal. By Proposition 6,

$$d_j = \frac{|D|}{|\boldsymbol{u}_0 \wedge \cdots \wedge \boldsymbol{u}_{j-1} \wedge \boldsymbol{u}_{j+1} \wedge \cdots \wedge \boldsymbol{u}_n|}.$$

Thus,

$$d_{\min}^n \le \frac{|D|^n}{\prod_{j=1}^n |\boldsymbol{u}_0 \wedge \cdots \wedge \boldsymbol{u}_{j-1} \wedge \boldsymbol{u}_{j+1} \wedge \cdots \wedge \boldsymbol{u}_n|}.$$

For j from 1 to n, let

$$\boldsymbol{v}_j = L(\boldsymbol{u}_0 \wedge \cdots \wedge \boldsymbol{u}_{j-1} \wedge \boldsymbol{u}_{j+1} \wedge \cdots \wedge \boldsymbol{u}_n),$$

and we now consider $L(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n)$. By Proposition 10,

$$L(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n) = \pm D^{n-1}\boldsymbol{u}_0.$$

Hence,

$$|L(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n)| = |D|^{n-1}$$

since $|\boldsymbol{u}_0| = 1$. We also know that

$$|L(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n)| = |\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n| \le |\boldsymbol{v}_1| \cdots |\boldsymbol{v}_n|$$

by Corollary 3. Moreover, the inequality is strict unless $\boldsymbol{v}_i \cdot \bar{\boldsymbol{v}}_j = 0$ for all $i \neq j$. Thus,

$$\begin{aligned}\prod_{j=1}^{n} |\boldsymbol{u}_0 \wedge \cdots \wedge \boldsymbol{u}_{j-1} \wedge \boldsymbol{u}_{j+1} \wedge \boldsymbol{u}_n| &= \prod_{j=1}^{n} |L(\boldsymbol{u}_0 \wedge \cdots \wedge \boldsymbol{u}_{j-1} \wedge \boldsymbol{u}_{j+1} \wedge \boldsymbol{u}_n)| \\ &= \prod_{j=1}^{n} |\boldsymbol{v}_j| \\ &\geq |L(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n)| = |D|^{n-1}.\end{aligned}$$

Hence,

$$d_{\min}^n \leq \frac{|D|^n}{\prod_{j=1}^{n} |\boldsymbol{u}_0 \wedge \cdots \wedge \boldsymbol{u}_{j-1} \wedge \boldsymbol{u}_{j+1} \wedge \cdots \wedge \boldsymbol{u}_n|} \leq \frac{|D|^n}{|D|^{n-1}} = |D|,$$

as required, with strict inequality unless $d_1 = \cdots = d_n$ and $\boldsymbol{v}_i \cdot \bar{\boldsymbol{v}}_j = 0$ for all $i \neq j$. □

Remark. Equality of the n distances is not sufficient for equality to hold in Theorem 11, but the proof of Theorem 11 suggests the following conjecture.

Conjecture 12. *With notation as in Theorem 11, fix $0 < D \leq 1$ and consider all configurations of $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_n$ such that $D = |\det(\boldsymbol{u}_0, \ldots, \boldsymbol{u}_n)|$. Among all such configurations, the configuration with the largest $d_{\min}$ will be a regular simplex.*

Remark. When $D < 1$, equality will not hold in Theorem 11 for the regular simplex with determinant D.

We now observe that if we like, we could just as easily work with vectors defining the faces of the simplices, rather than the vertices.

Proposition 13. *Let $\boldsymbol{a}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ be $n+1$ linearly independent unit vectors in $\mathbb{C}^{n+1}$. We think of the vectors as the coefficients of linear forms defining hyperplanes in $\mathbb{CP}^n$. By linear independence, the hyperplanes are in general position and thus determine a simplex. Let d denote the distance from the hyperplane determined by $\boldsymbol{a}$ to the vertex of the simplex where the hyperplanes determined by $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ intersect. Then,*

$$d = \frac{|\det(\boldsymbol{a}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)|}{|\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n|}.$$

Remark. Observe that the distance formula here is identical to that in Proposition 6. Thus, Theorem 11 and Corollary 7 immediately translate to the following corollary.

Corollary 14. *Let $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_n$ be $n+1$ linearly independent unit vectors in $\mathbb{C}^{n+1}$ representing $n+1$ linear forms defining $n+1$ hyperplanes in general position in $\mathbb{CP}^n$, which we think of as the faces of a projective simplex. For each j from 0 to n, let d_j denote the Fubini–Study distance from the hyperplane represented by $\boldsymbol{u}_j$ to the opposite vertex of the simplex. Let $d_{\min}$ denote the minimum of the d_j. Then,*

$$d_{\min}^n \leq |\det(\boldsymbol{u}_0, \ldots, \boldsymbol{u}_n)| \leq d_{\min}.$$

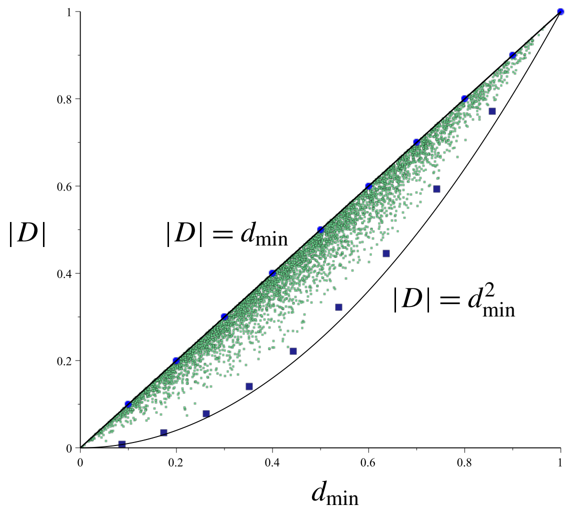

Figure 1. $|D|$ versus $d_{\min}$ in the case of dimension $n = 2$.

Remark. Figure 1 illustrates the inequalities constraining the absolute value of the determinant and the minimum distance in the case when $n = 2$, i.e., for the case of projective triangles in the projective plane. The points marked as circles along the line $|D| = d_{\min}$ illustrate isosceles triangles, as in Example 8. The points marked as squares just above the curve $|D| = d_{\min}^2$ are from equilateral triangles. The other points are triangles with randomly generated vertices.

Proof of Proposition 13. Let

$$\boldsymbol{u} = \frac{L(\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n)}{|\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n|},$$

which is the unit vector representing the vertex of the simplex where the hyperplanes determined by $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ intersect. For $j = 1, \ldots, n$, let

$$\boldsymbol{v}_j = L(\boldsymbol{a} \wedge \boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_{j-1} \wedge \boldsymbol{b}_{j+1} \wedge \cdots \wedge \boldsymbol{b}_n).$$

Then, the vectors $\boldsymbol{v}_j$, which are not necessarily unit vectors, represent the n other vertices of the simplex. By Proposition 6 and Proposition 4,

$$d = \frac{\left|\det\left(\boldsymbol{u}, \frac{\boldsymbol{v}_1}{|\boldsymbol{v}_1|}, \ldots, \frac{\boldsymbol{v}_n}{|\boldsymbol{v}_n|}\right)\right|}{\left|\frac{\boldsymbol{v}_1}{|\boldsymbol{v}_1|} \wedge \cdots \wedge \frac{\boldsymbol{v}_n}{|\boldsymbol{v}_n|}\right|} = \frac{|\boldsymbol{u} \cdot L(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n)|}{|\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n|}.$$

By Proposition 10, $L(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n) = \pm D^{n-1}\boldsymbol{a}$, where $D = \det(\boldsymbol{a}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$. Thus,

$$\begin{aligned} d &= \frac{|u \cdot L(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n)|}{|\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n|} \\ &= \frac{|D|^{n-1}|\boldsymbol{u} \cdot \boldsymbol{a}|}{|D|^{n-1}} \quad \text{(since } \boldsymbol{a} \text{ is a unit vector)} \\ &= \frac{|L(\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n) \cdot \boldsymbol{a}|}{|\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n|} \quad \text{(by the definition of } \boldsymbol{u}\text{)} \\ &= \frac{|\det(\boldsymbol{a}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)|}{|\boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_n|} \quad \text{(by Proposition 4).} \qquad \square \end{aligned}$$

We conclude by explaining some of the initial motivation coming from complex function theory for this investigation. Let $\mathbb{D}$ denote the unit disc in the complex plane. J. Dufresnoy [1944] studied complex analytic mappings f from $\mathbb{D}$ to $\mathbb{CP}^n$ such that the image of f omits at least $2n + 1$ hyperplanes in general position in $\mathbb{CP}^n$, where here *general position* means that the linear forms defining any $n + 1$ of the hyperplanes will be linearly independent. As in [Cherry and Eremenko 2011], we let $f^\#$ denote the Fubini–Study derivative of f, which measures how much the mapping f distorts length, where length in $\mathbb{D}$ is measured with respect to the standard Euclidean metric and length in $\mathbb{CP}^n$ is measured with respect to the Fubini–Study metric. A consequence of Dufresnoy's work is that $f^\#(0)$ is bounded above by a constant depending only on the dimension n and the set of omitted hyperplanes, but Dufresnoy remarked in his 1944 paper that the constant depends on the omitted hyperplanes in a "completely unknown" way. By making a portion (see [Eremenko 1999]) of the potential-theoretic method of Eremenko and Sodin [1991] effective, Cherry and Eremenko [2011] were able to give an explicit and effective estimate on how the constant depends on the omitted hyperplanes. Cherry and Eremenko's bound was expressed in terms of the singular values of the $(n+1) \times (n+1)$ matrices formed by the coefficients of the normalized linear forms defining $n+1$ of the omitted hyperplanes. Let P be a point in $\mathbb{CP}^n$ where n of the $2n + 1$ omitted hyperplanes intersect, and let Q be a point where a different n of the $2n + 1$ omitted hyperplanes intersect. Then, the projective line connecting P with Q will intersect the $2n + 1$ omitted hyperplanes in only three points: it will intersect n of the hyperplanes at P, another n at Q and the last one at some third point R. Such a line is called a *diagonal* line for the hyperplane configuration. In the event that the hyperplane configuration is such that for some diagonal line, two of the three points P, Q, and R are very close together, it is not hard to see that one can find a complex analytic map f from $\mathbb{D}$ into the diagonal line omitting the three points such that $f^\#(0)$ is very large. One is then led to ask if this is the only way one can get a very large value of $f^\#(0)$. One would thus like to know how this minimum distance among the pairs of points in $\{P, Q, R\}$

compares to the singular values appearing in Cherry and Eremenko's bound. Rather than look initially at collections of $2n+1$ hyperplanes in $\mathbb{CP}^n$, we began with the easier situation of $n+1$ hyperplanes in $\mathbb{CP}^n$ and did some numerical experiments comparing the singular values of the matrices formed by the coefficients of the defining forms of the hyperplanes and the projective distances from the hyperplanes to the opposite vertices of the simplex whose faces are contained in the given hyperplanes. These opposite vertices would be the points determining the diagonal lines in bigger configurations of hyperplanes. Although Cherry and Eremenko's bound is expressed only in terms of some of the singular values, we realized that we could obtain prettier results for the determinant, whose absolute value is of course the square root of the product of all the singular values. We therefore decided to write this note focusing on the pure projective geometry of the simplices and leave the possible application to complex function theory to another time.

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