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the Euler function and the sum of divisors function

Luis Elesban Santos Cruz and Florian Luca





# Power values of the product of the Euler function and the sum of divisors function

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We find examples of positive integers  $n$  such that  $\phi(n^3)\sigma(n^3)$  is a perfect square.

## 1. Introduction

The Euler function  $\phi(n)$  counts the number of positive integers  $m \leq n$  which are coprime to  $n$ , the sum of divisors function  $\sigma(n)$  is equal to the sum of the positive proper divisors of  $n$ , and both of these functions have fascinated mathematicians for centuries. A lot of effort has been spent trying to find positive integers  $n$  such that  $\phi(n)$  and  $\sigma(n)$  have nice arithmetic properties.

It is easy to make  $\phi(n)$  a square. Just take  $n = 2^{2k+1}$  for some  $k \geq 0$ . Exactly half of all integers  $m \leq 2^{2k+1}$  are odd, and hence, coprime to  $n$ . Thus,  $\phi(2^{2k+1}) = 2^{2k}$  is a perfect square. The situation for the sum of divisors function is harder. A nice presentation of this problem is in [Beukers et al. 2012]. Following that reference, we look at the factorizations

$$\begin{aligned} \sigma(2) &= 3, & \sigma(11) &= 2^2 \times 3, \\ \sigma(3) &= 2^2, & \sigma(13) &= 2 \times 7, \\ \sigma(5) &= 2 \times 3, & \sigma(17) &= 2 \times 3^2, \\ \sigma(7) &= 2^3, & \sigma(19) &= 2^2 \times 5. \end{aligned}$$

There are many ways to multiply together some of the above numbers to get a perfect square. First let us notice that 13 and 19 are useless because  $\sigma(13) = 2 \times 7$  and  $\sigma(19) = 2^2 \times 5$ , and neither 7 nor 5 ever appear again on the right-hand side of the above equations. Throw out 13 and 19 and group squares on the right-hand sides in the following way, where  $\square$  represents a perfect square:

$$\sigma(2) = 3, \quad \sigma(3) = \square, \quad \sigma(5) = 2 \times 3, \quad \sigma(7) = 2\square, \quad \sigma(11) = 3\square, \quad \sigma(17) = 2\square.$$

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Note that all six inputs are prime numbers and all outputs have prime factorizations consisting of only 2 and 3. Let the primes 2, 3, 5, 7, 11, 17 correspond to the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$  in the six-dimensional vector space  $\mathbb{F}_2^6$ , where  $\mathbf{v}_i$  has  $i$ -th component equal to 1 and all others equal to 0 for  $i = 1, \dots, 6$ . In  $\mathbb{F}_2^2$  we let  $\mathbf{w}_1$  and  $\mathbf{w}_2$  be the vectors  $(1, 0)^\top$  and  $(0, 1)^\top$  and think of them as corresponding to the primes 2 and 3 respectively. We define a linear map from  $\mathbb{F}_2^6 \mapsto \mathbb{F}_2^2$  whose matrix is

$$T = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

This matrix has rank 2, so it has  $2^4 = 16$  vectors in its nullspace, and any of these vectors gives us a solution. For example, the vector  $(1, 1, 1, 1, 0, 0)^\top$ , which is in  $\text{Null}(T)$ , gives us the solution  $n = 2 \times 3 \times 5 \times 7$ , having  $\sigma(n) = 2^6 \times 3^2$ .

In [Beukers et al. 2012], the equation  $\sigma(n^k) = m^l$  in positive integers  $n$  and  $m$  was studied for some exponents  $k > 1$  and  $l > 1$ . On page 377, they conjecture that  $\sigma(n^k) = m^l$  has only finitely many solutions if  $k > 3$  and  $l > 1$  are given. Here, we propose the following counterconjecture.

**Conjecture 1.** *For every  $k > 1$  and  $l > 1$ , there are infinitely many  $n$  such that  $\sigma(n^k) = m^l$  for some positive integer  $m$ .*

To give some evidence, we propose a different conjecture. Let  $P(n)$  denote the largest prime factor of the integer  $n$ , with the convention that  $P(0) = P(\pm 1) = 1$ .

**Conjecture 2.** *Let  $f(x) \in \mathbb{Z}[x]$  be a polynomial such that  $f(0) \neq 0$ . For every  $\varepsilon > 0$ , there exists  $c := c(\varepsilon)$  and  $x_0 := x_0(\varepsilon)$  such that*

$$\#\{p \leq x : P(f(p)) < x^\varepsilon\} > cx / \log x \quad \text{for all } x > x_0. \quad (1)$$

The substance of the above conjecture is the following. It is well known that the numbers  $n$  such that  $P(n) < n^\varepsilon$  form a positive-density subset of  $\mathbb{N}$ . It is conjectured that the primes  $p$  such that  $P(p-1) < p^\varepsilon$  form a positive-density subset of all primes. This is not known for small values of  $\varepsilon > 0$ . So, we venture even further and replace  $p-1$  by any fixed polynomial  $f(p)$  such that  $f(0) \neq 0$  (in order to make sure that  $p$  does not show up as a natural divisor of  $f(p)$ ) and conjecture that, in fact, the set of primes  $p$  such that  $P(f(p)) < p^\varepsilon$  is of positive density. This is known if all roots of  $f(x)$  are rational, with some  $\varepsilon < 1$  (like  $\varepsilon = 1 - 1/2d$ , where  $d$  is the degree of  $f(x)$ ), but it is not known for any  $\varepsilon < 1$  once  $f(x)$  has an irreducible factor of degree at least 2. The quantity  $x / \log x$  in the right-hand side of (1) arises from the prime number theorem, which asserts that, asymptotically, the function  $\pi(x) = \#\{p \leq x\}$  equals  $x / \log x$  as  $x \rightarrow \infty$ .

Let us see how Conjecture 1 would follow from Conjecture 2. Let  $k \geq 2$ ,  $f(x) = (x^{k+1} - 1)/(x - 1)$  and suppose first that  $l = 2$ . Let  $x$  be large, put  $\varepsilon = 1/2$

and let  $p_1, \dots, p_t$  be such that  $P(f(p_i)) < x^{1/2}$ . Let  $s = \pi(x^{1/2})$ . Then we can write

$$f(p_i) = w_i \square, \quad i = 1, \dots, t,$$

where the  $w_i$  are square-free numbers with  $P(w_i) \leq x^{1/2}$ . As before, we can identify the  $w_i$  with vectors in  $\mathbb{F}_2^s$  obtained by putting 1 or 0 in the  $j$ -th component according to whether the  $j$ -th prime divides  $w_i$  or not. In this way, we get a linear application from  $\mathbb{F}_2^t$  to  $\mathbb{F}_2^s$  whose nullspace has dimension at least  $t - s$ , where

$$t - s > c \frac{x}{\log x} - \pi(x^{1/2}) > c \frac{x}{\log x} - x^{1/2},$$

and this last function certainly tends to infinity with  $x$ . This is when  $l = 2$ . Assume now that  $l > 2$ . Then we write

$$f(p_i) = w_i u_i^l \quad \text{for all } i = 1, \dots, t,$$

where the  $w_i$  are  $l$ -th power free and  $P(w_i) \leq x^{1/2}$ . We attach to each  $w_i$  an element  $\mathbf{w}_i$  in the group  $(\mathbb{Z}/l\mathbb{Z})^s$  where in the  $j$ -th component we put the exponent of the  $j$ -th prime number in the factorization of  $w_i$ . Note that  $\mathbb{Z}/l\mathbb{Z}$  is not a field unless  $l$  is a prime, and even if  $l$  is a prime, we only can multiply *distinct* primes  $p_i$  in attempts to create  $n$  such that  $\sigma(n^k) = m^l$ . Thus, we are only allowed to take sums of distinct  $\mathbf{w}_i$  and get 0. There is a theorem (see [van Emde Boas and Kruyswijk 1967] and [Olson 1969, Theorem 1]) that says that if we have at least  $s(l - 1)$  such distinct elements  $\mathbf{w}_i$ , we can find some of them whose sum is 0. Thus, we can create at least  $\lfloor t/(s(l - 1)) \rfloor$  distinct (in fact, even disjoint) subsets of the  $\mathbf{w}_i$  for  $i = 1, \dots, t$  simply by finding some 0-sum among the first  $s(l - 1)$  of them, another 0-sum among the next  $s(l - 1)$  of them and so on. Since

$$\frac{t}{s(l - 1)} > \frac{c}{(l - 1) \log x} \sqrt{x},$$

and the right-hand side is a function that tends to infinity with  $x$ , we get Conjecture 1.

We can ask similar questions simultaneously for  $\phi(n)$  and  $\sigma(n)$ , like making them simultaneously squares, or cubes, etc. This has already been treated in [Freiberg 2012]. There it is shown that the number of  $n \leq x$  such that both  $\phi(n)$  and  $\sigma(n)$  are perfect powers of an exponent  $l$  is less than  $c_1 l x^{1/l} / (\log x)^{l+2}$ , where  $c_1 > 0$  is some positive constant. Square values of the product  $\phi(n)\sigma(n)$  have been investigated in [Broughan et al. 2013]. In the next section, we present some computational examples of  $n$  such that  $\phi(n^3)\sigma(n^3) = \square$ .

### 2. Computational examples

We wanted to find a positive integer  $n$  such that  $\phi(n^3)\sigma(n^3) = \square$ . For a prime  $p$ , we have  $\phi(p^3)\sigma(p^3) = p^2(p^4 - 1)$ . So, we wrote  $p^4 - 1 = w_p \square$ , where  $w_p$  is square-free for all  $p \leq 1000$ . Then we searched for a subset  $\mathcal{S}$  of cardinality  $t$  such

that the set of prime factors appearing in the factorizations of  $w_p$  for  $p \in S$  has cardinality  $s < t$ . We found the subset

$$\{2, 3, 5, 7, 13, 17, 23, 31, 41, 43, 47, 73, 83, 191, 239, 307, 443, 499, 829\},$$

with  $t = 21$  and  $s = 17$ . Thus, this set gives us  $2^{21-17} = 16$  solutions. We wrote down the  $\{0, 1\}$  matrix with 17 rows and 21 columns, which ends up having rank 17 over  $\mathbb{F}_2$ . The largest solution in the nullspace of this matrix is

$$n = 3 \times 7 \times 11 \times 13 \times 17 \times 23 \times 43 \times 47 \times 83 \times 239 \times 443 \times 499 \times 829,$$

for which  $\phi(n^3)\sigma(n^3) = m^2$ , where

$$m = 2^{30} \times 3^7 \times 5^{10} \times 7^2 \times 11 \times 13^4 \times 17^3 \times 23 \times 29 \times 37 \times 41 \times 53 \times 61 \times 83 \times 157.$$

Despite our efforts, we could not find an integer  $n > 1$  such that  $\sigma(n^5) = \square$ , and we leave finding such an example as a challenge to the reader.

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elesluis@gmail.com

*Departamento de Matemáticas Aplicadas, Universidad de Istmo, Ciudad Universitaria S/N, Barrio Santa Cruz, 4a Sección, Santo Domingo, 70110 Tehuantepec, Oaxaca, Mexico*

florian.luca@wits.ac.za

*School of Mathematics, University of the Witwatersrand, P.O. Box Wits 2050, Johannesburg, 2000 South Africa*

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