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Expected maximum vertex valence in pairs of polygonal triangulations

Timothy Chu and Sean Cleary

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Edge-flip distance between triangulations of polygons is equivalent to rotation distance between rooted binary trees. Both distances measure the extent of similarity of configurations. There are no known polynomial-time algorithms for computing edge-flip distance. The best known exact universal upper bounds on rotation distance arise from measuring the maximum total valence of a vertex in the corresponding triangulation pair obtained by a duality construction. Here we describe some properties of the distribution of maximum vertex valences of pairs of triangulations related to such upper bounds.

1. Introduction

Binary trees are widely used in a broad spectrum of computational settings. Binary search trees underlie many modern structures devoted to efficient searching, for example. Shapes of binary trees affect the performance of searches, and there have been a wide variety of approaches to ensure such efficiency. Natural dual structures to rooted ordered binary trees are triangulations of polygons with a marked edge or vertex. Rotations in binary trees correspond to edge-flip moves in such triangulations of polygons, so the rotation distance between two rooted ordered binary trees corresponds exactly to the edge-flip distance between the two corresponding triangulations of marked polygons.

Properties of rotations have been widely studied; see Knuth [1973] for background and fundamental algorithms. There is no known polynomial-time algorithm for computing rotation (or equivalently, edge-flip) distance, though there are a variety of efficient approximation algorithms [Baril and Pallo 2006; Cleary and St. John 2010; 2009]. A straightforward argument of Culik and Wood [1982] shows that for

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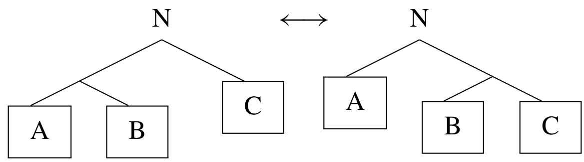

Figure 1. Rotation at a node N. Right rotation at N transforms the left tree to the right one, and left rotation at N is the inverse operation which transforms the right tree to the left one. A, B, and C represent leaves or subtrees, and the node N could be at the root or any other position in the tree.

any two trees with n internal nodes, there is always a path of length at most $2n-2$. Sleator, Tarjan and Thurston [Sleator et al. 1988] showed that the distance is never more than $2n-6$ using an argument described below based upon maximum summed vertex valence in the pair of triangulations, and furthermore that for all very large n, that bound is achieved. Recently, Pournin [2014] showed that, in fact, the upper bound is achieved for all $n \geq 11$.

A rotation move in a rooted binary tree relative to a fixed node N is a promotion of one grandchild node of N to a child node of N, a demotion of a child of N to a grandchild of N, and a switch of parent node for one grandchild of N, preserving order. This occurs in the vicinity of a single node, as pictured in Figure 1. The corresponding edge-flip move in a triangulation occurs in a single quadrilateral formed by two triangles which share an edge. The common edge between two adjacent triangles is exchanged for the opposite diagonal in that quadrilateral, as shown in Figure 2. If the edge-flip distance between two triangulations of a regular polygon is k, that means that there is a sequence of k edge flips transforming the first triangulation to the second and there is no shorter sequence accomplishing the same transformation.

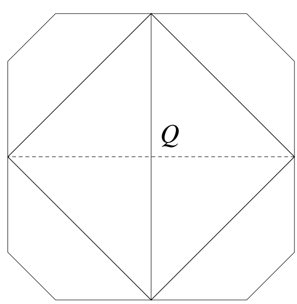

Figure 2. An edge flip across a quadrilateral Q. The four peripheral quadrilaterals denote (possibly empty) triangulated polygons whose triangulations are unchanged by the edge flip in quadrilateral Q.

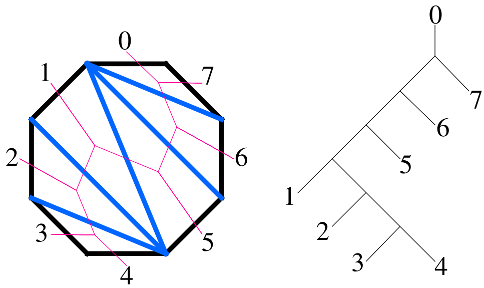

Figure 3. A triangulation of the octagon and the corresponding dual tree, with sides numbered to match the leaves. Pulling up on the edge from the marked side of the octagon (marked as leaf 0) gives the tree on the right.

2. Triangular subdivisions of polygons

Here, by a *triangulation of size n*, we mean a triangulation of $n-1$ interior edges subdividing a regular $(n+2)$-gon, where we choose to label vertices from 0 to $n+1$. Such a triangulation is dual to a tree with $n+1$ leaves and n internal nodes, with leaves labeled from 0 to n. See Figure 3.

The number of triangulations of size n is the n-th Catalan number, C_n, and since C_n grows exponentially at rate of $4^n n^{-3/2}$, the number of pairs of trees of size n grows on the order of $16^n n^{-3}$. Because of the rapid growth of the number of tree pairs (or equivalently, triangulation pairs), computing these quantities exhaustively via complete enumeration is not feasible beyond small n. To explore this exponentially growing space, we use sampling techniques to characterize the expected behavior of randomly selected triangulation pairs. We experiment computationally by choosing pairs of triangulations of size n uniformly at random, computing the relevant vertex sums and tabulating the results. As in [Chu and Cleary 2013], we use the linear-time random tree-generation procedure of Rémy [1985] to generate efficiently ordered trees uniformly at random, rather than considering the Yule distribution on tree pairs studied by Cleary, Passaro and Toruno [Cleary et al. 2015]. The quantities studied here are the maximum valence sums of vertices, described in the next section.

3. Vertex valence sums

Vertex valence sums play a role in the upper bounds for edge-flip distance. Given a pair of triangulations S and T, we count the number of interior edges s_i and t_i incident to each vertex i in the polygon and form the *vertex valence sum* for vertex i as $s_i + t_i$. See Figure 4. The bound of $2n-6$ for $n \geq 11$ from [Sleator et al. 1988] is obtained by the following argument. There are a total of $n-1$ edges in each triangulation, each with 2 endpoints, giving $4n-4$ total endpoints of interior edges

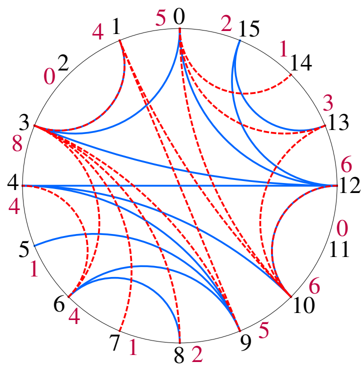

Figure 4. Two superimposed triangulations, one drawn in dashed red and one in solid blue. The vertices are numbered and the total valences are given in purple for each vertex. For example, the total valence of vertex 0 is five, with 3 red edges and 2 blue edges incident there. In this example, the summed vertex valences range from a low of 0 (at vertex 2, indicating a common peripheral triangle) to a high of 8, which occurs at vertex 3.

for the pair of triangulations. Each endpoint occurs at one of the $n+2$ vertices, so the average valence of a vertex is merely $(4n-4)/(n+2) = 4 - 12/(n+2)$. For $n \geq 11$, this gives average valence larger than 3 (approaching 4 in the limit of large n). Since the summed valences can only be integers, if the sum is more than 3, there must be a vertex j of total valence 4 or more. We consider a path of triangulations from S to T by way of a fan triangulation F_j, which is the fan from vertex j (that is, a triangulation of the polygon where every triangle has an edge incident on vertex j). Transforming S to F_j takes exactly $n-1-s_j$ edge flips, as

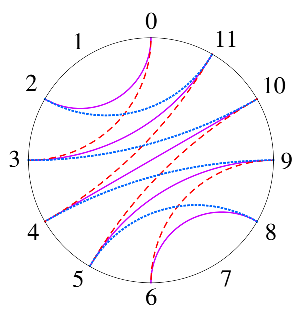

Figure 5. Two superimposed zigzag triangulations, one drawn in dotted blue and one in dashed red, with common segments in solid purple. Each red dashed edge can be directly flipped to the corresponding blue one, giving an edge-flip distance of 4 between the two triangulations, despite the vertex valence sums commonly being 4.

there is always an edge flip which increases the valence of vertex j by one and there are exactly that many edges to flip. Similarly, from T to F_j there is a path of length $n-1-t_j$ and such a path is minimal. So there is a path from S to F_j to T of length no more than $2n-2-s_j-t_j$, giving the $2n-6$ bound in the case that the maximum vertex valence s_j+t_j is exactly 4. In cases where the maximal vertex valence is higher, the upper bound is correspondingly decreased.

We note that for a triangulation pair, the vertex sums need to be quite evenly distributed around the polygon to have a chance of being maximally distant for that size. The average valence sum is between 3 and 4 for $n \geq 11$ and no maximally distant pair of triangulations can have any vertex sums of 5 or larger. Note that having a maximal vertex sum of 4 is necessary but not sufficient for being a maximally distant triangulation pair, as can be easily seen by considering a zigzag triangulation of a regular n-gon beginning at vertex 0 and a reflected zigzag triangulation beginning at vertex 2, as shown in Figure 5. The maximal vertex sum is 4 but the edge-flip distance between these two is less than $n/2$ (flipping the red edges to the blue edges in the figure), far less than that of the maximum possible. There are many other configurations with maximal vertex 4 which are not remotely close to the $2n-6$ upper bound as well. Nevertheless, if there is even a single vertex with summed valence 5 or more, the two triangulations cannot be at the maximal $2n-6$ distance.

4. Discussion

There is an obviously increasing relationship between triangulation size and expected maximum observed summed valence across the vertices. However, the growth rate appears slow, as the relationship appears to be either straightening or slightly convex downward in log-scaled Figure 6, where a straight line would indicate logarithmic growth. The experimental evidence suggests that the relationship is at most logarithmic. Figure 7 shows an example of a distribution of maximum summed valence for a particular size, $n = 950$, showing the shape of typical distributions that arise in these computations.

A *combing triangulation with respect to v* is a triangulation where every edge is incident with the vertex v. Using the argument of [Sleator et al. 1988], an upper bound on rotation distance from S to T comes from the path which first flips successively edges in S to be incident with v, a vertex of maximum summed valence, to obtain the combing triangulation for v. Then the path goes from that combing triangulation to T, successively flipping to edges in T which are not incident on v. At each step of the resulting path, there is at least one edge in S which can be flipped to be incident to v or one edge incident to v which can be flipped to an edge in T. The resulting length of the path is the number of edges in S and T which are not incident on v. Combinatorial arguments give that there is always a vertex of

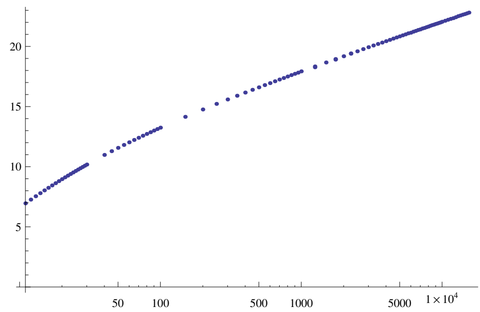

Figure 6. Maximum valence increases with triangulation size. Here we show average maximum summed vertex valence vertically, against the size of triangulations plotted horizontally on a logarithmic scale. For each size, the number of triangulation pairs sample to estimate the average maximum vertex valence ranges from 100,000 to 10 million depending upon size.

summed degree 4, giving the universal (for $n \geq 11$) bound of $2n-6$. For larger summed valence k for a particular pair of trees, the same arguments show that an upper bound for distance for that pair is $2n-k-2$. The experimental data in Table 1 shows that though it is common for randomly selected tree pairs to have higher summed valence than the minimum of 4, it is often not markedly higher than 4. In the case of randomly selected tree pairs, we know from the asymptotic analysis of Cleary, Rechnitzer and Wong [Cleary et al. 2013] that for large n, two randomly selected triangulations of size n are likely to have about $(16/\pi - 5)n \cong 0.093n$ common edges. Thus, the upper bounds for rotation distance arising from common edges, edges which are a single rotation from a common edge, and from common components of small size are generally much stronger than those upper bounds arising from the path through a fan on a vertex of maximum summed valence.

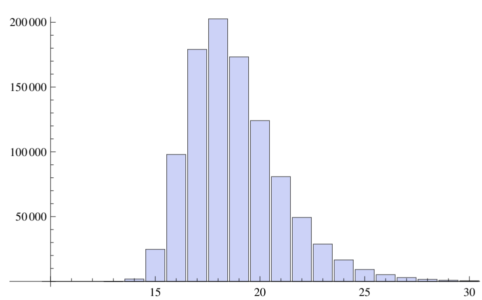

Figure 7. An example of the distribution of maximum vertex valence for one million triangulation pairs, for size 950, with average 17.8 and standard deviation 2.26.

triangulation size	average max vertex sum	σ max vertex sum
15	8.03203	1.55756
20	8.96568	1.70762
30	10.1791	1.87928
40	10.9805	1.97628
50	11.5674	2.03815
75	12.5759	2.12531
100	13.2461	2.16791
200	14.7589	2.23289
500	16.5983	2.26295
1000	17.9277	2.27465
2000	19.184	2.24797
3000	19.9369	2.2556
5000	20.8352	2.24551
7500	21.5392	2.23559
10000	22.0527	2.2108
12000	22.3505	2.20521
15000	22.7512	2.20811

Table 1. Observed averages and standard deviations of maximum vertex sums via experiments involving 10 million ($n \leq 30$), 1 million ($n < 1000$) or 100,000 (for $n \geq 10000$) runs depending upon the triangulation sizes.

n	fraction with max vertex sum 4
11	0.0050032
12	0.0015352
13	0.0004462
14	0.0001232
15	0.000035
16	$8.1 \cdot 10^{-6}$
17	$2.4 \cdot 10^{-6}$
18	$5.0 \cdot 10^{-7}$
19	$1.0 \cdot 10^{-7}$
20+	none observed

Table 2. Fractions of triangulations with maximum vertex sum exactly 4. Hundreds of millions were considered for $n \geq 20$, finding none selected at random.

The work of Pournin [2014] constructs carefully very specific examples of triangulations which are at maximal distance $2n-6$ for all $n \geq 11$. One question is how common such maximally distant pairs are. The experimental evidence in Table 2 shows that examples with this extremal behavior are quite rare. In the language of associahedra, used in [Pournin 2014], pairs of triangulations at maximal $2n-6$ distance correspond to antipodal points, which have many long geodesics between them, with typically many of those passing through the distinguished "fan" triangulations. But from the analysis here, it appears quite rare that a pair of triangulations will have a geodesic which passes through a fan point, indicating further the rarity of these extremal antipodal pairs.

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