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# The chromatic polynomials of signed Petersen graphs

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Zaslavsky proved in 2012 that, up to switching isomorphism, there are six different signed Petersen graphs and that they can be told apart by their chromatic polynomials, by showing that the latter give distinct results when evaluated at 3. He conjectured that the six different signed Petersen graphs also have distinct zero-free chromatic polynomials, and that both types of chromatic polynomials have distinct evaluations at *any* positive integer. We developed and executed a computer program (running in SAGE) that efficiently determines the number of proper $k$-colorings for a given signed graph; our computations for the signed Petersen graphs confirm Zaslavsky's conjecture. We also computed the chromatic polynomials of all signed complete graphs with up to five vertices.

Graph coloring problems are ubiquitous in many areas within and outside of mathematics. We are interested in certain enumerative questions about coloring signed graphs. A *signed graph* $\Sigma = (\Gamma, \sigma)$ consists of a graph $\Gamma = (V, E)$ and a signature $\sigma \in \{\pm\}^E$. The underlying graph $\Gamma$ may have multiple edges and, besides the usual links and loops, also *half-edges* (with only one endpoint) and *loose edges* (no endpoints); the last are irrelevant for coloring questions, and so we assume in this paper that $\Sigma$ has no loose edges. An unsigned graph can be realized by a signed graph all of whose edges are labeled with $+$. Signed graphs originated in the social sciences and have found applications also in biology, physics, computer science, and economics; see [Zaslavsky 1998–2012] for a comprehensive bibliography.

*MSC2010:* primary 05C22; secondary 05A15, 05C15.
*Keywords:* signed graph, Petersen graph, complete graph, chromatic polynomial, zero-free chromatic polynomial.
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The *chromatic polynomial* $c_\Sigma(2k+1)$ counts the *proper k-colorings*

$$\boldsymbol{x} \in \{0, \pm 1, \dots, \pm k\}^V,$$

namely, those colorings that satisfy

$$x_v \neq \sigma_{vw}\, x_w$$

for any edge $vw \in E$ and $x_v \neq 0$ for any $v \in V$ incident with some half-edge. Zaslavsky [1982a] proved that $c_\Sigma(2k+1)$ is indeed a polynomial in $k$. It comes with a companion, the *zero-free chromatic polynomial* $c^*_\Sigma(2k)$, which counts all proper $k$-colorings $\boldsymbol{x} \in \{\pm 1, \dots, \pm k\}^V$.

The *Petersen graph* has served as a reference point for many proposed results in graph theory. Considering *signed* Petersen graphs, Zaslavsky [2012] showed that, while there are $2^{15}$ ways to assign a signature to the fifteen edges, only six of these are different up to switching isomorphism (a notion that we will make precise below), depicted in Figure 1. (In our figures we represent a positive edge with a solid line and a negative edge with a dashed line.)

Zaslavsky [2012] proved that these six signed Petersen graphs have distinct chromatic polynomials; thus they can be distinguished by this signed-graph invariant. He did not compute the chromatic polynomials but showed that they evaluate to distinct numbers at 3 [loc. cit., Table 9.2]. He conjectured that the six different signed Petersen graphs also have distinct zero-free chromatic polynomials, and that both types of chromatic polynomials have distinct evaluations at *any* positive integer [loc. cit., Conjecture 9.1]. Our first result confirms this conjecture.

**Theorem 1.** *The chromatic polynomials of the signed Petersen graphs* (denoted by $P_1, \dots, P_6$ in Figure 1) *are*

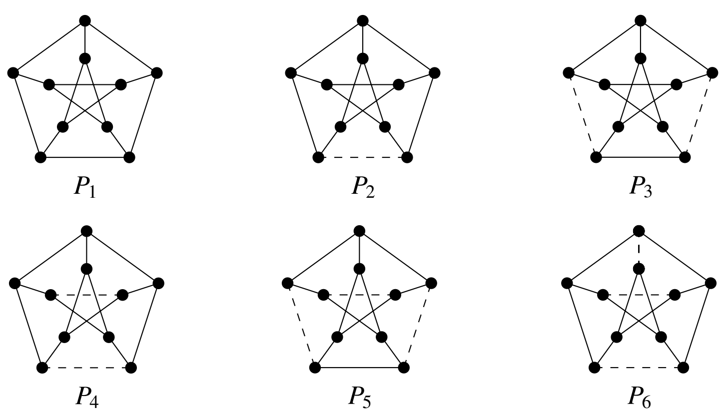



**Figure 1.** The six switching-distinct signed Petersen graphs.

$$\begin{aligned}
c_{P_1}(2k+1) &= 1024k^{10} - 2560k^9 + 3840k^8 - 4480k^7 + 3712k^6 \\
&\qquad - 1792k^5 + 160k^4 + 480k^3 - 336k^2 + 72k, \\
c_{P_2}(2k+1) &= 1024k^{10} - 2560k^9 + 3840k^8 - 4480k^7 + 3968k^6 \\
&\qquad - 2560k^5 + 1184k^4 - 352k^3 + 48k^2, \\
c_{P_3}(2k+1) &= 1024k^{10} - 2560k^9 + 3840k^8 - 4480k^7 + 4096k^6 \\
&\qquad - 2944k^5 + 1696k^4 - 760k^3 + 236k^2 - 40k, \\
c_{P_4}(2k+1) &= 1024k^{10} - 2560k^9 + 3840k^8 - 4480k^7 + 4224k^6 \\
&\qquad - 3200k^5 + 1984k^4 - 952k^3 + 308k^2 - 52k, \\
c_{P_5}(2k+1) &= 1024k^{10} - 2560k^9 + 3840k^8 - 4480k^7 + 4096k^6 \\
&\qquad - 3072k^5 + 1920k^4 - 960k^3 + 320k^2 - 48k, \\
c_{P_6}(2k+1) &= 1024k^{10} - 2560k^9 + 3840k^8 - 4480k^7 + 4480k^6 \\
&\qquad - 3712k^5 + 2560k^4 - 1320k^3 + 460k^2 - 90k.
\end{aligned}$$

*Their zero-free counterparts are*

$$\begin{aligned}
c^*_{P_1}(2k) &= 1024k^{10} - 7680k^9 + 26880k^8 - 58240k^7 + 86592k^6 \\
&\qquad - 91552k^5 + 68400k^4 - 34440k^3 + 10424k^2 - 1408k, \\
c^*_{P_2}(2k) &= 1024k^{10} - 7680k^9 + 26880k^8 - 58240k^7 + 86848k^6 \\
&\qquad - 93088k^5 + 72304k^4 - 39880k^3 + 14792k^2 - 3288k, \\
c^*_{P_3}(2k) &= 1024k^{10} - 7680k^9 + 26880k^8 - 58240k^7 + 86976k^6 \\
&\qquad - 93856k^5 + 74256k^4 - 42592k^3 + 16960k^2 - 4222k, \\
c^*_{P_4}(2k) &= 1024k^{10} - 7680k^9 + 26880k^8 - 58240k^7 + 87104k^6 \\
&\qquad - 94496k^5 + 75664k^4 - 44320k^3 + 18192k^2 - 4698k, \\
c^*_{P_5}(2k) &= 1024k^{10} - 7680k^9 + 26880k^8 - 58240k^7 + 86976k^6 \\
&\qquad - 93984k^5 + 74800k^4 - 43560k^3 + 17840k^2 - 4616k, \\
c^*_{P_6}(2k) &= 1024k^{10} - 7680k^9 + 26880k^8 - 58240k^7 + 87360k^6 \\
&\qquad - 95776k^5 + 78480k^4 - 47760k^3 + 20640k^2 - 5660k.
\end{aligned}$$

*Consequently* (*as a quick computation with a computer algebra system shows*), *none of the difference polynomials* $c_{P_m}(2k+1) - c_{P_n}(2k+1)$ *and* $c^*_{P_m}(2k) - c^*_{P_n}(2k)$, *with* $m \neq n$, *have a positive integer root.*

To compute the above polynomials, we developed and executed a computer program (running in SAGE [Stein et al. 2012]) that efficiently determines the number of proper $k$-colorings for any signed graph. This code can be downloaded from math.sfsu.edu/beck/papers/signedpetersen.sage or from the online supplement to this paper. The procedure `chrom` is the main method; it takes an incidence matrix and outputs the chromatic polynomial as an expression.

We also used our program to compute the chromatic polynomials of all signed complete graphs up to five vertices; up to switching isomorphism, there are two signed $K_3$s, three signed $K_4$s, and seven signed $K_5$s. As with the signed Petersen graphs, the chromatic polynomials distinguish these signed complete graphs:

**Theorem 2.** *The chromatic polynomials of the signed complete graphs* (denoted $K_3^{(1)}, K_3^{(2)}, \dots, K_5^{(7)}$ in Figure 2) *are*

$$\begin{aligned}
c_{K_3^{(1)}}(2k+1) &= 8k^3 - 2k,\\
c_{K_3^{(2)}}(2k+1) &= 8k^3,\\
c_{K_4^{(1)}}(2k+1) &= 16k^4 - 16k^3 - 4k^2 + 4k,\\
c_{K_4^{(2)}}(2k+1) &= 16k^4 - 16k^3 + 4k^2,\\
c_{K_4^{(3)}}(2k+1) &= 16k^4 - 16k^3 + 12k^2 - 2k,\\
c_{K_5^{(1)}}(2k+1) &= 32k^5 - 80k^4 + 40k^3 + 20k^2 - 12k,\\
c_{K_5^{(2)}}(2k+1) &= 32k^5 - 80k^4 + 64k^3 - 16k^2,\\
c_{K_5^{(3)}}(2k+1) &= 32k^5 - 80k^4 + 88k^3 - 48k^2 + 10k,\\
c_{K_5^{(4)}}(2k+1) &= 32k^5 - 80k^4 + 72k^3 - 28k^2 + 4k.\\
c_{K_5^{(5)}}(2k+1) &= 32k^5 - 80k^4 + 96k^3 - 56k^2 + 12k,\\
c_{K_5^{(6)}}(2k+1) &= 32k^5 - 80k^4 + 80k^3 - 40k^2 + 8k,\\
c_{K_5^{(7)}}(2k+1) &= 32k^5 - 80k^4 + 120k^3 - 80k^2 + 20k.
\end{aligned}$$

The corresponding zero-free chromatic polynomials are

$$\begin{aligned}
c^*_{K_3^{(1)}}(2k) &= 8k^3 - 12k^2 + 4k,\\
c^*_{K_3^{(2)}}(2k) &= 8k^3 - 12k^2 + 6k,\\
c^*_{K_4^{(1)}}(2k) &= 16k^4 - 48k^3 + 44k^2 - 12k,\\
c^*_{K_4^{(2)}}(2k) &= 16k^4 - 48k^3 + 52k^2 - 24k,\\
c^*_{K_4^{(3)}}(2k) &= 16k^4 - 48k^3 + 60k^2 - 34k,\\
c^*_{K_5^{(1)}}(2k) &= 32k^5 - 160k^4 + 280k^3 - 200k^2 + 48k,\\
c^*_{K_5^{(2)}}(2k) &= 32k^5 - 160k^4 + 304k^3 - 272k^2 + 114k,\\
c^*_{K_5^{(3)}}(2k) &= 32k^5 - 160k^4 + 328k^3 - 340k^2 + 174k,\\
c^*_{K_5^{(4)}}(2k) &= 32k^5 - 160k^4 + 312k^3 - 296k^2 + 136k,\\
c^*_{K_5^{(5)}}(2k) &= 32k^5 - 160k^4 + 336k^3 - 360k^2 + 190k,\\
c^*_{K_5^{(6)}}(2k) &= 32k^5 - 160k^4 + 320k^3 - 320k^2 + 158k,\\
c^*_{K_5^{(7)}}(2k) &= 32k^5 - 160k^4 + 360k^3 - 420k^2 + 240k.
\end{aligned}$$

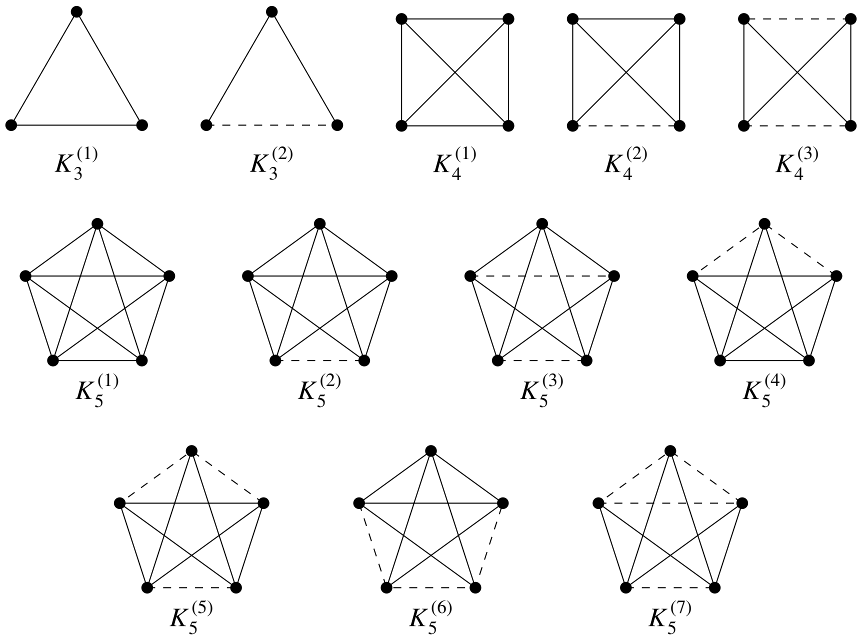


**Figure 2.** The switching classes of signed complete graphs.

We now review a few constructs on a signed graph $\Sigma = (V, E, \sigma)$ and describe our implementation. The *restriction* of $\Sigma$ to an edge set $F \subseteq E$ is the signed graph $(V, F, \sigma|_F)$. For $e \in E$, we denote by $\Sigma - e$ (the *deletion* of $e$) the restriction of $\Sigma$ to $E - \{e\}$. For $v \in V$, denote by $\Sigma - v$ the restriction of $\Sigma$ to $E - F$, where $F$ is the set of all edges incident to $v$. A component of the signed graph $\Sigma = (\Gamma, \sigma)$ is *balanced* if it contains no half-edges and each cycle has positive sign product.

*Switching* $\Sigma$ by $s \in \{\pm\}^V$ results in the new signed graph $(V, E, \sigma^s)$, where $\sigma^s_{vw} = s_v\, \sigma_{vw}\, s_w$. Switching does not alter balance, and any balanced signed graph can be obtained from switching an all-positive graph [Zaslavsky 1982b]. We also note that there is a natural bijection of proper colorings of $\Sigma$ and a switched version of it, and this bijection preserves the number of proper $k$-colorings. Thus the chromatic polynomials of $\Sigma$ are invariant under switching.

The *contraction* of $\Sigma$ by $F \subseteq E$, denoted by $\Sigma/F$, is defined as follows [Zaslavsky 1982b]: switch $\Sigma$ so that every balanced component of $F$ is all positive, coalesce all nodes of each balanced component, and discard the remaining nodes and all edges in $F$; note that this may produce half-edges. If $F = \{e\}$ for a link $e$, $\Sigma/e$ is obtained by switching $\Sigma$ so that $\sigma(e) = +$ and then contracting $e$ as in the case of unsigned graphs; that is, disregard $e$ and identify its two endpoints. If $e$ is a negative loop at $v$, then $\Sigma/e$ has vertex set $V - \{v\}$ and edge set resulting from $E$

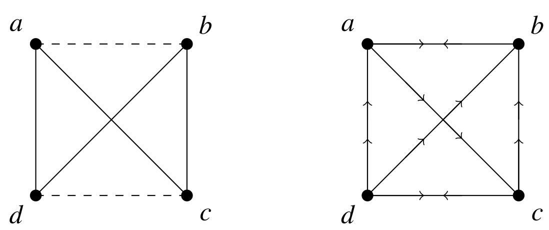
by deleting $e$ and converting all edges incident with $v$ to half-edges. The chromatic polynomial satisfies the deletion-contraction formula [Zaslavsky 1982a]

$$c_\Sigma(2k+1) = c_{\Sigma-e}(2k+1) - c_{\Sigma/e}(2k+1). \tag{1}$$

The zero-free chromatic polynomial $c^*_\Sigma(2k)$ satisfies the same identity provided that $e$ is not a half-edge or negative loop. We will use (1) repeatedly in our computations.

We encode a signed graph $\Sigma$ by its *incidence matrix* as follows: first *bidirect* $\Sigma$; i.e., give each edge an independent orientation at each endpoint (which we think of as an arrow pointing towards or away from the endpoint), such that a positive edge has one arrow pointing towards one and away from the other endpoint, and a negative edge has both arrows pointing either towards or away from the endpoints. The incidence matrix has rows indexed by vertices, columns indexed by edges, and entries equal to $\pm 1$ according to whether the edge points towards or away from the vertex (and 0 otherwise). Since half-edges and negative loops have the same effect on the chromatic polynomial of $\Sigma$, we may assume that $\Sigma$ has no half-edge. See Figure 3 for an example.

Deletion-contraction can be easily managed by incidence matrices: deletion of an edge simply means deletion of the corresponding column; contraction of a positive edge $vw$ means replacing the rows corresponding to $v$ and $w$ by their sum and then deleting the column corresponding to the edge $vw$ (it is sufficient to only consider contraction of positive edges, since we can always switch one of its endpoints if necessary, which means negating the corresponding row). Note that this process works for both links and half-edges. Note also that we will constantly look for multiple edges (with the same sign) and replace them with a single edge.




| | $ab$ | $ac$ | $ad$ | $bc$ | $bd$ | $cd$ |
|---|---|---|---|---|---|---|
| $a$ | $-1$ | $-1$ | $1$ | $0$ | $0$ | $0$ |
| $b$ | $-1$ | $0$ | $0$ | $1$ | $1$ | $0$ |
| $c$ | $0$ | $1$ | $0$ | $-1$ | $0$ | $-1$ |
| $d$ | $0$ | $0$ | $-1$ | $0$ | $-1$ | $-1$ |

**Figure 3.** $K_4^{(3)}$ with one of its bidirections and corresponding incidence matrix.

Thus we can keep track of incidence matrices as we recursively apply deletion-contraction, leading to empty signed graphs or signed graphs that only have half-edges; both have easy chromatic polynomials.

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