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Domino tilings of Aztec diamonds, Baxter permutations, and snow leopard permutations

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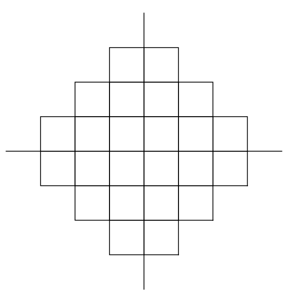
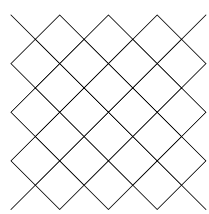
In 1992, Elkies, Kuperberg, Larsen, and Propp introduced a bijection between domino tilings of Aztec diamonds and certain pairs of alternating-sign matrices whose sizes differ by one. In this paper we first study those smaller permutations which, when viewed as matrices, are paired with the matrices for doubly alternating Baxter permutations. We call these permutations snow leopard permutations, and we use a recursive decomposition to show they are counted by the Catalan numbers. This decomposition induces a natural map from Catalan paths to snow leopard permutations; we give a simple combinatorial description of the inverse of this map. Finally, we also give a set of transpositions which generates these permutations.

1. Introduction and background

An *Aztec diamond of order n* is a two-dimensional array of unit squares with $2i$ squares in rows $i \leq n$ and $2(2n-i+1)$ squares in rows $n < i \leq 2n$, in which the squares are centered in each row. In the figure below (left) we have the Aztec diamond of order 3. We will be interested in the vertices of an Aztec diamond, which we prefer to arrange in rows and columns, so we will orient all of our Aztec diamonds as in the figure on the right.





MSC2010: 05A05, 05A15.
Keywords: domino tiling, Aztec diamond, Baxter permutation, alternating permutation, alternating-sign matrix, Catalan number.

Aztec diamonds can be tiled using 2×1 domino rectangles, which is to say they can be completely covered by disjoint dominoes whose union is the entire diamond. We call a tiling of an Aztec diamond with dominoes a TOAD for short.

In [Elkies et al. 1992], Elkies, Kuperberg, Larsen, and Propp describe how to construct, for each TOAD T of order n, a pair of matrices $SASM(T)$ and $LASM(T)$ of sizes $n \times n$ and $(n+1) \times (n+1)$, respectively. Each of these matrices is an *alternating-sign matrix* (ASM), which is a matrix with entries in $\{0, 1, -1\}$ whose nonzero entries in each row and in each column alternate in sign and sum to 1. (For an introduction to ASMs and a variety of related combinatorial objects, see [Robbins 1991; Bressoud 1999; Propp 2001].) To carry out this construction, first note that in Figure 1 the vertices that compose the tiled Aztec diamond fall naturally into two matrices: the red vertices form an $(n+1) \times (n+1)$ matrix while the blue vertices form an $n \times n$ matrix. We construct $LASM(T)$ on the red vertices by labeling each vertex of degree 4 with a 1, labeling each vertex of degree 3 with a 0, and labeling each vertex of degree 2 with a -1. We construct $SASM(T)$ on the blue vertices in the same way, except the degree 4 and degree 2 rules are reversed. Note that the TOAD T in Figure 1 has

$$LASM(T) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad SASM(T) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Following [Elkies et al. 1992] and [Canary 2010], we say an $(n+1)\times(n+1)$ ASM A and an $n \times n$ ASM B are *compatible* whenever there is a TOAD T such that $A = LASM(T)$ and $B = SASM(T)$. Elkies et al. showed that an $(n+1)\times(n+1)$ ASM with k entries equal to -1 is compatible with 2^k $n \times n$ ASMs, while an $n \times n$ ASM with j entries equal to 1 is compatible with 2^j $(n+1) \times (n+1)$ ASMs. In general, then, the compatibility relation is not one-to-one. However, each $(n+1) \times (n+1)$ ASM with no -1 entries (that is, each $(n+1) \times (n+1)$ permutation matrix) is

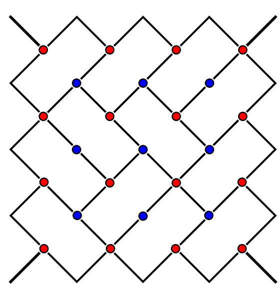


Figure 1. A domino tiling of the Aztec diamond of order 3.

compatible with exactly one $n \times n$ ASM. In this case, Canary [2010] gives an algorithm to construct the unique smaller ASM compatible with a given larger permutation matrix. (Asinowski [2014] gives a different formulation of the same algorithm, in which he first reconstructs the underlying TOAD.) To implement Canary's algorithm for an $(n+1) \times (n+1)$ permutation matrix A, first label the red vertices in a diagram for an Aztec diamond of the appropriate size with the entries of A. For each blue vertex, if the two red vertices immediately to the left, and all of the red vertices left of those, are labeled with 0, then label the blue vertex 0. Now repeat this process in each of the other three directions (up, right, and down). Canary shows that each row and column of blue vertices will now contain an odd number of unlabeled vertices, and there is a unique way to label these vertices with 1s and -1s to create an ASM.

Canary proves that the $n \times n$ ASM compatible with a given $(n+1) \times (n+1)$ permutation matrix A will also be a permutation matrix if and only if A is the matrix of a Baxter permutation. To understand the definition of a Baxter permutation, first note that we can interpret each permutation matrix A as the permutation π in one-line notation for which $A_{ij} = \delta_{j\pi(i)}$. That is, the 1 in the first row of A is in position $\pi(1)$, the 1 in the second row is in position $\pi(2)$, and in general the 1 in the jth row is in position $\pi(j)$. For example, if T is the TOAD in Figure 1, then the permutation for $LASM(T)$ is 4132 and the permutation for $SASM(T)$ is 312. We will often identify a permutation matrix with its corresponding permutation in one-line notation. With this convention, a *Baxter* permutation is a permutation that avoids 2–41–3 and 3–14–2. In other words, π is a Baxter permutation whenever there are no indices $i < j < j+1 < k$ such that $\pi(j+1) < \pi(i) < \pi(k) < \pi(j)$ (for 2–41–3) or $\pi(j) < \pi(k) < \pi(i) < \pi(j+1)$ (for 3–14–2). For example, 174962835 is not Baxter because the subsequence 4625 is an instance of 2–41–3. In contrast, 879164325 is Baxter because it contains no instances of 2–41–3 or 3–14–2. Note that the compatibility relation is still not one-to-one when we restrict it to Baxter permutations. For example, 12 is compatible with the Baxter permutations 123, 132, and 213. On the other hand, as suggested in [Asinowski et al. 2013], for every permutation π of length n which is compatible with a Baxter permutation of length $n+1$, the number of Baxter permutations of length $n+1$ compatible with π appears to be a product of Fibonacci numbers.

Baxter permutations first arose in connection with the question of whether two commuting continuous functions from the closed interval [0, 1] to itself must have a common fixed point [Baxter 1964; Boyce 1967]. Since their introduction they have been studied by many authors; some relevant references are [Chung et al. 1978; Mallows 1979; Cori et al. 1986; Dulucq and Guibert 1996; 1998; Guibert and Linusson 2000; Ouchterlony 2006; Ackerman et al. 2006; Asinowski et al. 2013].

Our work involves a particular class of Baxter permutations, which are known as doubly alternating Baxter permutations. We call a permutation π *alternating* whenever $\pi(i) < \pi(i+1)$ if i is odd and $\pi(i) > \pi(i+1)$ if i is even. That is, π is alternating whenever it begins with an ascent, and its ascents and descents alternate. A *doubly alternating* permutation is an alternating permutation whose inverse is also alternating, and we call permutations that are both doubly alternating and Baxter *doubly alternating Baxter permutations (DABPs).* Guibert and Linusson [2000] show that the Catalan number $C_n = 1/(n+1)\binom{2n}{n}$ counts both the DABPs of length $2n$ and the DABPs of length $2n+1$. The Catalan numbers are known to count many other combinatorial objects (see [Stanley 1999, Exercise 6.19] and [Stanley 2013]), including lattice paths from $(0, 0)$ to (n, n) using only north $(0, 1)$ and east $(1, 0)$ steps which do not pass below the line $y = x$; we call these paths *Catalan paths*. In addition to the explicit definition of C_n in terms of binomial coefficients, the Catalan numbers also satisfy the recurrence relation $C_n = \sum_{j=1}^{n} C_{j-1}C_{n-j}$ for $n \geq 0$, with initial condition $C_0 = 1$.

In this paper, we introduce the *snow leopard permutations (SLPs)*, which are the permutations that are compatible with the doubly alternating Baxter permutations. More formally, we write S_n to denote the set of permutations of length n, and we make the following definition.

Definition 1.1. We say a permutation $\pi \in S_n$ is a *snow leopard permutation* whenever there is a TOAD T of order n such that $LASM(T)$ is a DABP and $SASM(T) = \pi$.

In Section 2, we characterize these permutations recursively, and we use this recursive characterization to show that in this case the compatibility relation is one-to-one. This implies that the snow leopard permutations of length $2n$ are also counted by C_n, as are the snow leopard permutations of length $2n+1$. Matching our recursive description of the snow leopard permutations with the first-return decomposition of a Catalan path gives us a recursively defined bijection from Catalan paths from $(0, 0)$ to (n, n) to snow leopard permutations of length $2n$. In Section 3 we give a simple combinatorial description of the inverse of this map. Finally, in Section 4 we describe how to generate all of the snow leopard permutations from the decreasing permutation with a specific set of transpositions.

2. Recursive decompositions of DABPs, TOADs, and snow leopard permutations

In this section we describe how to construct snow leopard permutations recursively, and we use our recursive decomposition to show that there are C_n snow leopard permutations of length $2n$, as well as C_n snow leopard permutations of length $2n+1$. Our snow leopard permutation decomposition is induced by similar decompositions of the associated TOADs and DABPs, so we first describe how to decompose these objects. We begin with a recursive decomposition of a DABP, for which it will be helpful to use several common operations on permutations.

Permutation tools. Throughout we write S_n to denote the set of all permutations of length n, and for any permutation π, we write $|\pi|$ to denote the length of π. The following four operations on permutations will be especially useful for us.

Definition 2.1. For any permutation $\pi \in S_n$, we write π^c to denote the *complement* of π, which is the permutation in S_n with

$$\pi^c(j) = n + 1 - \pi(j)$$

for all j, $1 \leq j \leq n$, and we write π^r to denote the *reverse* of π, which is the permutation in S_n with

$$\pi^r(j) = \pi(n + 1 - j)$$

for all j, $1 \leq j \leq n$. For any permutations $\pi \in S_n$ and $\sigma \in S_k$, we write $\pi \oplus \sigma$ to denote the permutation in S_{n+k} with

$$(\pi \oplus \sigma)(j) = \begin{cases} \pi(j) & \text{if } 1 \leq j \leq n, \\ n + \sigma(j - n) & \text{if } n < j \leq n + k \end{cases}$$

for all j, $1 \leq j \leq n$, and we write $\pi \ominus \sigma$ to denote the permutation in S_{n+k} with

$$(\pi \ominus \sigma)(j) = \begin{cases} k + \pi(j) & \text{if } 1 \leq j \leq n, \\ \sigma(j - n) & \text{if } n < j \leq n + k \end{cases}$$

for all j, $1 \leq j \leq n$.

Note that on matrices the complement is a reflection over a vertical line, while the reverse is a reflection over a horizontal line. In addition, one can also show that for any permutations π and σ, we have $(\pi \oplus \sigma)^{-1} = \pi^{-1} \oplus \sigma^{-1}$, $(\pi^r)^{-1} = (\pi^{-1})^c$, and $(\pi^c)^{-1} = (\pi^{-1})^r$. We sometimes write i to denote the inverse map on S_n; with this notation, our last two equations are equivalent to $i \circ r = c \circ i$ and $i \circ c = r \circ i$, respectively.

Example 2.2. If $\pi = 32154$ and $\sigma = 3124$ then $\pi^c = 34512$, $\sigma^r = 4213$, $\pi \oplus \sigma = 321548679$, and $\pi \ominus \sigma = 765983124$.

In some situations our permutations will naturally have length 0 or -1. To incorporate these cases into our results, we use the following notation.

Definition 2.3. We write $\varnothing$ to denote the empty permutation, which is the unique permutation of length 0, and we write @ to denote the *antipermutation*, which is the unique permutation of length -1. We have $@^c = @^r = @^{-1} = @$, and $1 \oplus @ = @ \oplus 1 = 1 \ominus @ = @ \ominus 1 = \varnothing$.

As we show next, the set of Baxter permutations is closed under $\oplus$, $\ominus$, taking complements, and taking the reverse of a permutation.

Lemma 2.4. *The following are equivalent for any permutation π.*

(i) π *is Baxter.*

(ii) π^c *is Baxter.*

(iii) π^r *is Baxter.*

(iv) π^{-1} *is Baxter.*

Proof. (i) $\Rightarrow$ (ii) If π^c contains a subsequence of type 2–41–3, then the corresponding subsequence of π will have type 3–14–2. Similarly, if π^c contains a subsequence of type 3–14–2 then the corresponding subsequence of π will have type 2–41–3. If π is Baxter then π avoids 2–41–3 and 3–14–2, so π^c avoids 3–14–2 and 2–41–3, which means π^c is Baxter.

(ii) $\Rightarrow$ (i) This is immediate from (i) $\Rightarrow$ (ii), since $(\pi^c)^c = \pi$.

(i) $\Leftrightarrow$ (iii) This is similar to the proof of (i) $\Leftrightarrow$ (ii).

(i) $\Leftrightarrow$ (iv) Since $(\pi^{-1})^{-1} = \pi$, it's sufficient to show that if π contains a subsequence of type 2–41–3 or a subsequence of type 3–14–2 then π^{-1} does, as well. With this in mind, suppose $abcd$ is a subsequence of π of type 2–41–3 for which $d-a$ is minimal. If $d = a+1$ then the corresponding subsequence in π^{-1} has type 3–14–2. Otherwise, $a+1$ is either to the left of b or to the right of c, since b and c are adjacent. If $a+1$ is to the left of b, then we can replace a with $a+1$, so $d-a$ was not minimal, which is a contradiction. On the other hand, if $a+1$ is to the right of c then we can replace d with $a+1$, so $d-a$ was not minimal in this case, either.

The proof that if π contains a subsequence of type 3–14–2 then π^{-1} contains a subsequence of type 2–41–3 or 3–14–2 is similar. □

Lemma 2.5. *The following are equivalent for permutations π and σ.*

(i) π *and* σ *are Baxter.*

(ii) $\pi \oplus \sigma$ *is Baxter.*

(iii) $\pi \ominus \sigma$ *is Baxter.*

Proof. (i) $\Rightarrow$ (ii) Suppose to the contrary that π and σ are Baxter permutations but $\pi \oplus \sigma$ is not Baxter. Call the first $|\pi|$ entries of $\pi \oplus \sigma$ the front of $\pi \oplus \sigma$, and call the last $|\sigma|$ entries the back. Note that every entry in the front is less than every entry in the back.

If $\pi \oplus \sigma$ contains a subsequence α of type 2–41–3, then α cannot be entirely contained in the front or in the back, since π and σ are Baxter. Therefore $\alpha(1)$ is in the front and $\alpha(4)$ is in the back. Now $\alpha(2)$ must be in the back, since it is greater than $\alpha(4)$, so $\alpha(3)$ must also be in the back. But this contradicts the fact that $\alpha(1) > \alpha(3)$.

If $\pi \oplus \sigma$ contains a subsequence α of type 3–14–2, then α cannot be entirely contained in the front or in the back, since π and σ are Baxter. But this contradicts the fact that $\alpha(1) > \alpha(4)$.

(ii) $\Rightarrow$ (i) If π or σ contains a subsequence of type 2–41–3 or 3–14–2 then so does $\pi \oplus \sigma$, and the result follows.

(i) $\Leftrightarrow$ (iii) This is similar to the proof of (i) $\Leftrightarrow$ (ii). □

Note that if π is alternating then π^c is not alternating in general, and π^r is alternating if and only if π has odd length. Similarly, if π and σ are alternating, then $\pi \oplus \sigma$ is not alternating in general, while $\pi \ominus \sigma$ is alternating if and only if π has even length. As a result, the set of DABPs is not closed under $\oplus$, $\ominus$, complements, or reverses.

The DABP decompositions. As we will see, snow leopard permutations inherit their recursive structure from DABPs, so our first goal is to describe how to decompose DABPs into smaller DABPs. Several of these results are not new, so we will refer to the work of others, especially [Dulucq and Guibert 1998] and [Ouchterlony 2006], as needed.

Lemma 2.6 [Ouchterlony 2006, Lemma 4.1(i)]. *If π is a DABP of odd length then* $\pi(1) = 1$.

Ouchterlony uses Lemma 2.6 to conclude that π is a DABP of length $2n+1$ if and only if $\pi = 1 \oplus (\sigma^r)^{-1}$ for some DABP σ of length $2n$ [Ouchterlony 2006, Corollary 4.2(i)], and that this correspondence is a bijection between the set of DABPs of length $2n+1$ and the set of DABPs of length $2n$. However, as we show next, more is true.

Proposition 2.7. *Suppose f is any of the functions r, c, $i \circ r$, and $i \circ c$ on permutations. For any nonnegative integer n and any $\pi \in S_{2n+1}$, π is a DABP if and only if there is a DABP $\sigma \in S_{2n}$ such that $\pi = 1 \oplus \sigma^f$. Moreover, for each f, this correspondence is a bijection between the set of DABPs π of length $2n+1$ and the set of DABPs σ of length $2n$.*

Proof. By [Ouchterlony 2006, Corollary 4.2(i)], the result holds for $f = i \circ r$. To prove the result for $f = c$, first note that σ is a DABP if and only if σ^{-1} is a DABP by Lemma 2.4. Now the result follows by replacing σ with σ^{-1} in [loc. cit., Corollary 4.2(i)] and using the fact that $i \circ r \circ i = c$.

The proofs when $f = r$ and $f = i \circ c$ are similar. □

With Proposition 2.7 in mind, we will focus our attention on DABPs of even length. In this case, Guibert and Linusson [2000] and Ouchterlony [2006] have found the following DABP decomposition.

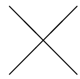


Figure 2. The TOAD of order 0.

Proposition 2.8 [Ouchterlony 2006, Corollary 4.2(ii)] and [Guibert and Linusson 2000, proof of Theorem 3]. *For any nonnegative integer n and any permutation $\pi \in S_{2n}$, π is a DABP if and only if there are DABPs π_1 and π_2 of even length such that $\pi = (1 \oplus (\pi_1^r)^{-1} \oplus 1) \ominus \pi_2$. Moreover, this correspondence is a bijection between the set of DABPs π of length $2n$ and the set of ordered pairs (π_1, π_2) of DABPs of lengths $2k$ and $2l$, where $n = k + l + 1$.*

As was the case for DABPs of odd length, more is true.

Proposition 2.9. *Suppose f is any of the functions r, c, $i \circ r$, and $i \circ c$ on permutations. For any nonnegative integer n and any permutation $\pi \in S_{2n}$, π is a DABP if and only if there are DABPs π_1 and π_2 of even length such that $\pi = (1 \oplus \pi_1^f \oplus 1) \ominus \pi_2$. Moreover, for each f, this correspondence is a bijection between the set of DABPs π of length $2n$ and the set of ordered pairs (π_1, π_2) of DABPs of lengths $2k$ and $2l$, where $n = k + l + 1$.*

Proof. This is similar to the proof of Proposition 2.7, using Proposition 2.8. □

The Aztec diamond decompositions. It is not difficult to show [Asinowski 2014; Canary 2010] that each Baxter permutation π of length $n + 1$ determines a unique TOAD $\mathcal{T}(\pi)$ of order n, and that $\mathcal{T}$ and *LASM* are inverse bijections when *LASM* is restricted to those TOADS whose *LASM* is a Baxter permutation. Computing $\mathcal{T}(\pi)$ when π has length 2 or more is routine, but some care is required when π has length 0 or 1. In particular, $\mathcal{T}(1)$ is the TOAD of order 0, which we show in Figure 2. Going a bit smaller still, we write @ to denote the TOAD $\mathcal{T}(\varnothing)$, which has order -1. Since the Aztec diamond of order -1 has no edges at all, we can't even draw it, but it will still play a role in our snow leopard decomposition.

The fact that we have the maps $\mathcal{T}$ and *LASM* means our DABP decompositions induce similar TOAD decompositions. To describe these TOAD decompositions, it's useful to introduce several ways of transforming and combining TOADs.

Definition 2.10. For any TOAD T, we write T^c to denote the *complement* of T, which is the reflection of T over a vertical line, we write T^r to denote the *reverse* of T, which is the reflection of T over a horizontal line, and we write T^{-1} to denote the *inverse* of T, which is the reflection of T over a diagonal line from upper left to lower right.

As we did for permutations, we sometimes write i to denote the inverse map on TOADs.

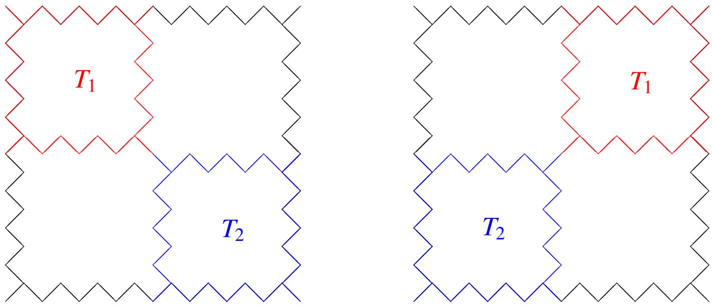


Figure 3. The construction of $T_1 \oplus T_2$ (left) and $T_1 \ominus T_2$ (right) from T_1 and T_2.

Definition 2.11. For any TOADs T_1 and T_2, we write $T_1 \oplus T_2$ to denote the TOAD we obtain by identifying the lower right vertex of T_1 with the upper left vertex of T_2, taking the smallest Aztec diamond D which contains both T_1 and T_2, and tiling the part of D outside of T_1 and T_2 with dominoes whose long sides are oriented from upper left to lower right. If T_1 has order n and T_2 has order k, then $T_1 \oplus T_2$ has order $n+k+1$.

In Figure 3 (left) we see how TOADs T_1 (in red) and T_2 (in blue) are combined to produce $T_1 \oplus T_2$. Note that the only way to tile the areas outside of T_1 and T_2 is to use dominoes whose long sides are oriented from upper left to lower right, as in the construction of $T_1 \oplus T_2$.

Definition 2.12. For any TOADs T_1 and T_2, we write $T_1 \ominus T_2$ to denote the TOAD we obtain by identifying the lower left vertex of T_1 with the upper right vertex of T_2, taking the smallest Aztec diamond D which contains both T_1 and T_2, and tiling the part of D outside of T_1 and T_2 with dominoes whose long sides are oriented from upper right to lower left. If T_1 has order n and T_2 has order k, then $T_1 \ominus T_2$ has order $n+k+1$.

In Figure 3 (right) we see how TOADs T_1 (in red) and T_2 (in blue) are combined to produce $T_1 \ominus T_2$. Note that the only way to tile the areas outside of T_1 and T_2 is to use dominoes whose long sides are oriented from upper right to lower left, as in the construction of $T_1 \ominus T_2$.

Our next result, which follows immediately from our definitions, justifies our multiple uses of the notations c, r, -1, $\oplus$, and $\ominus$.

Proposition 2.13. *For any Baxter permutations π and σ, the following hold.*

(i) $\mathcal{T}(\pi^c) = \mathcal{T}(\pi)^c$.

(ii) $\mathcal{T}(\pi^r) = \mathcal{T}(\pi)^r$.

(iii) $\mathcal{T}(\pi^{-1}) = \mathcal{T}(\pi)^{-1}$.

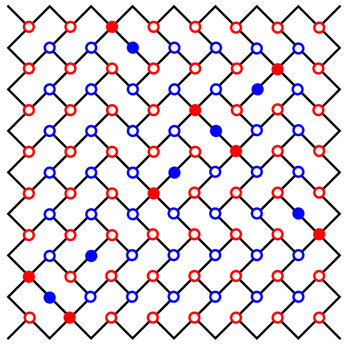


Figure 4. The DAAD corresponding to the DABP 37564812 and its compatible SLP 3654721.

(iv) $\mathcal{T}(\pi \oplus \sigma) = \mathcal{T}(\pi) \oplus \mathcal{T}(\sigma)$.

(v) $\mathcal{T}(\pi \ominus \sigma) = \mathcal{T}(\pi) \ominus \mathcal{T}(\sigma)$.

We now turn our attention to those TOADs which come from DABPs.

Definition 2.14. We call a TOAD T a *doubly alternating Aztec diamond (DAAD)* whenever $LASM(T)$ is a DABP. Note that a TOAD T is a DAAD if and only if there is a DABP π such that $\mathcal{T}(\pi) = T$. Indeed, $\pi = LASM(T)$.

In Figure 4 we have a DAAD with its DABP and its corresponding snow leopard permutation.

We saw in Proposition 2.7 that it's easy to construct DABPs of odd length from DABPs of even length. As we see next, this means it's easy to construct DAADs of even order from DAADs of odd order.

Proposition 2.15. *Suppose f is any of the functions r, c, $i \circ r$, and $i \circ c$ on DAADs. For any nonnegative integer n and any TOAD T of order $2n$, T is a DAAD if and only if there is a DAAD D of order $2n-1$ such that $T = \mathcal{T}(1) \oplus D^f$. Moreover, for each f this correspondence is a bijection between the set of DAADs of order $2n$ and the set of DAADs of order $2n-1$.*

Proof. ($\Rightarrow$) Since T is a DAAD of order $2n$, there is a DABP π of length $2n+1$ with $\mathcal{T}(\pi) = T$. By Proposition 2.7, there is a DABP σ of length $2n$ such that $\pi = 1 \oplus \sigma^f$. If we apply $\mathcal{T}$ to our expression for π and use Proposition 2.13 to simplify the result, we find $T = \mathcal{T}(1) \oplus \mathcal{T}(\sigma)^f$. Now the result follows, since $D = \mathcal{T}(\sigma)$ is a DAAD of order $2n-1$.

($\Leftarrow$) Since D is a DAAD of order $2n-1$, there is a DABP σ of length $2n$ such that $\mathcal{T}(\sigma) = D$. By Proposition 2.7, we have $\mathcal{T}(1 \oplus \sigma^f) = T$, so T is a DAAD.

The fact that this correspondence is a bijection follows from the last statement of Proposition 2.7 and the fact that $\mathcal{T}$ is a bijection. $\square$

Proposition 2.15 says that we can understand all DAADs if we understand DAADs of odd order. With this in mind, we now describe how to decompose a DAAD of odd order into a combination of two smaller DAADs of odd order.

Theorem 2.16. *Suppose f is any of the functions r, c, $i \circ r$, or $i \circ c$ on TOADs. For any TOAD T of odd order, T is a DAAD if and only if there are DAADs T_1 and T_2 of odd order such that $T = (\mathcal{T}(1) \oplus T_1^f \oplus \mathcal{T}(1)) \ominus T_2$. Moreover, for each f, this correspondence is a bijection between the set of DAADs T of order $2n-1$ and the set of ordered pairs (T_1, T_2) of DAADs of orders $2k-1$ and $2l-1$, where $n = k + l + 1$.*

Proof. ($\Rightarrow$) Since T is a DAAD or order $2n-1$, we know that $\pi = LASM(T)$ is a DABP of length $2n$ with $T = \mathcal{T}(\pi)$. By Proposition 2.9, there are DABPs π_1 and π_2 of lengths $2k$ and $2l$, respectively, such that $\pi = (1 \oplus \pi_1^f \oplus 1) \ominus \pi_2$ and $n = k + l + 1$. If we apply $\mathcal{T}$ to our expression for π and use Proposition 2.13 to simplify the result, we find

$$T = \mathcal{T}(\pi) = \big(\mathcal{T}(1) \oplus \mathcal{T}(\pi_1)^f \oplus \mathcal{T}(1)\big) \ominus \mathcal{T}(\pi_2).$$

Now the result follows, since $T_1 = \mathcal{T}(\pi_1)$ and $T_2 = \mathcal{T}(\pi_2)$ are DAADs by definition.

($\Leftarrow$) Since T_1 and T_2 are DAADs, we know that $\pi_1 = LASM(T_1)$ and $\pi_2 = LASM(T_2)$ are DABPs of lengths k and l respectively such that $\mathcal{T}(\pi_1) = T_1$ and $\mathcal{T}(\pi_2) = T_2$. Moreover, $n = k+l+1$. By Proposition 2.9, the permutation $(1 \oplus \pi_1^f \oplus 1) \ominus \pi_2$ is also a DABP, so its image under $\mathcal{T}$ is a DAAD. But if we apply $\mathcal{T}$ to $(1 \oplus \pi_1^f \oplus 1) \ominus \pi_2$ and use Proposition 2.13 to simplify the result, we find that

$$\mathcal{T}((1 \oplus \pi_1^f \oplus 1) \ominus \pi_2) = (\mathcal{T}(1) \oplus T_1^f \oplus \mathcal{T}(1)) \ominus T_2.$$

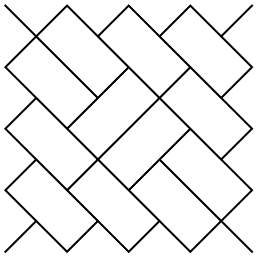
Therefore $(\mathcal{T}(1) \oplus T_1^f \oplus \mathcal{T}(1)) \ominus T_2$ is a DAAD.

The fact that this correspondence is a bijection follows from the last statement of Proposition 2.9 and the fact that $\mathcal{T}$ is a bijection. □

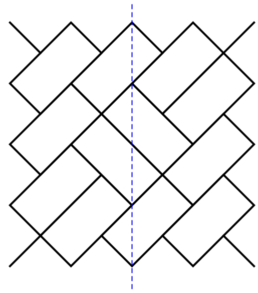
When we consider how our DAAD decomposition gives us a decomposition of the associated snow leopard permutation, we will be especially interested in pairs of dominoes that share a long side. With this in mind, we sometimes think of the process of building $\mathcal{T}(1) \oplus T \oplus \mathcal{T}(1)$ from a TOAD T in terms of adding a "hat" and pair of "shoes" to T. In Figure 5 we add a hat (in blue) and shoes (in *Wizard of Oz* ruby red) to $\mathcal{T}(1324)^c$.

When we construct $(\mathcal{T}(1) \oplus T_1 \oplus \mathcal{T}(1)) \ominus T_2$ from $\mathcal{T}(1) \oplus T_1 \oplus \mathcal{T}(1)$ and T_2, we add one more pair of dominoes which are adjacent along long sides; we call this pair the "connector". In Figure 6(d) we outline the connector in red.

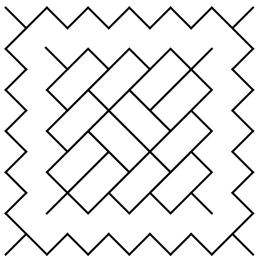
The snow leopard permutation decompositions. In the Introduction we described the function *SASM*, which maps DAADs of order n to snow leopard permutations



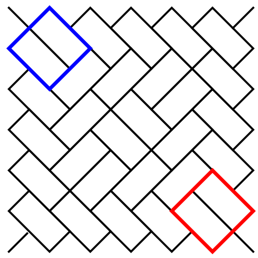
(a) $\mathcal{T}(1324)$.



(b) $\mathcal{T}(1324)^c$.



(c) Making room for the hat and shoes.



(d) A stylish blue hat and red shoes.

Figure 5. An illustration of the computation of $\mathcal{T}(1) \oplus \mathcal{T}(1324)^c \oplus \mathcal{T}(1)$, also known as the "hat and shoes" process.

of length n. In this section we use *SASM* and our DAAD decomposition to obtain our snow leopard permutation decomposition. To make this easier, we first describe a simple relationship between certain domino configurations in a DAAD T and the 1s in the matrix for $SASM(T)$.

Definition 2.17. A *block* in a TOAD T is a pair of two dominoes in T which are adjacent along a long edge, forming a 2-by-2 box.

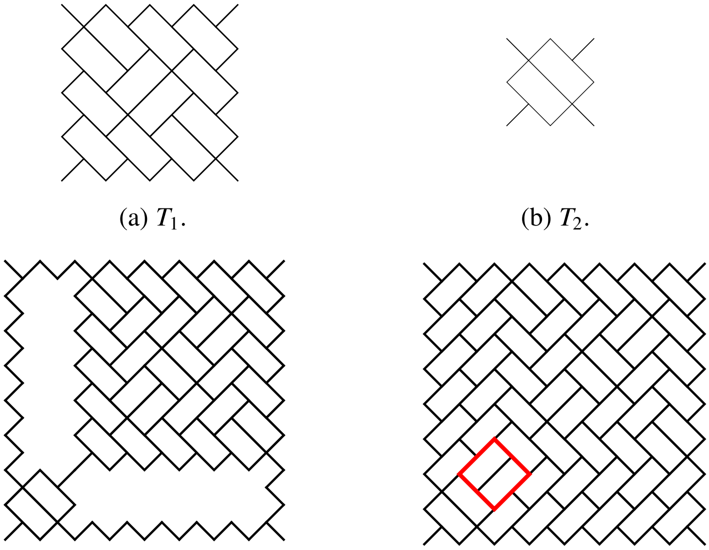
The DAAD shown in Figure 4 contains 7 blocks.

Lemma 2.18. *The vertices in a DAAD T which correspond to the* 1*s in $SASM(T)$ are exactly those vertices in the center of a block. As a result, the blocks of a DAAD are in bijection with the* 1*s in its SASM.*

Proof. Let T be a DAAD of order n that contains a block B. By Canary's algorithm, this point may correspond to a 1 in $SASM(T)$ or a -1 in $LASM(T)$. However, because $LASM(T)$ is a permutation, it cannot contain a -1. Thus, a block must correspond to a 1 in $SASM(T)$.

Conversely, a 1 in $SASM(T)$ must label a vertex of degree 2, which creates a block in T. □

Next we describe how the map *SASM* interacts with our operations on TOADs.



(a) T_1. (b) T_2.

(c) $\mathcal{T}(1) \oplus T_1^c \oplus \mathcal{T}(1)$ and T_2 in a a larger diamond.

(d) Putting the rest of the dominoes in, with the connector in red.

Figure 6. An illustration of the composition of DAADs T_1 and T_2, using the complement map. We outline the connector in red.

Proposition 2.19. *For any TOADs T_1 and T_2, the following hold*:

(i) $SASM(T_1^c) = SASM(T_1)^c$.

(ii) $SASM(T_1^r) = SASM(T_1)^r$.

(iii) $SASM(T_1^{-1}) = SASM(T_1)^{-1}$.

(iv) $SASM(T_1 \oplus T_2) = SASM(T_1) \oplus 1 \oplus SASM(T_2)$.

(v) $SASM(T_1 \ominus T_2) = SASM(T_1) \ominus 1 \ominus SASM(T_2)$.

Proof. (i), (ii), (iii) These are clear from Lemma 2.18 and our construction of $SASM$, since each of c, r, and i is a reflection over a particular line.

(iv) First observe that if T_1 (resp. T_2) is the TOAD of order -1 then $T_1 \oplus T_2$ is equal to T_1 (resp. T_2). But in this case $SASM(T_1)$ (resp. $SASM(T_2)$) is the antipermutation @, and the result holds.

Now suppose T_1 and T_2 have nonnegative orders. Then in the construction of $T_1 \oplus T_2$ we create one block which is not in T_1 or T_2, where the lower right edge of T_1 meets the upper left edge of T_2. Now the result follows from Lemma 2.18.

(v) This is similar to the proof of (iv). □

We can now describe our snow leopard permutation decomposition.

Theorem 2.20. *Suppose f is any of the functions r, c, $i \circ r$, or $i \circ c$ on permutations. For any permutation π of odd length, π is a snow leopard permutation if and only if there are snow leopard permutations π_1 and π_2 of odd length such that $\pi = (1 \oplus \pi_1^f \oplus 1) \ominus 1 \ominus \pi_2$. Moreover, for each f, this correspondence is a bijection between the set of snow leopard permutations π of length $2n-1$ and the set of ordered pairs (π_1, π_2) of snow leopard permutations of lengths $2k-1$ and $2l-1$, where $n = k+l+1$.*

Proof. ($\Rightarrow$) If π is a snow leopard permutation of length $2n-1$, then by definition there is a DAAD T of order $2n-1$ such that $SASM(T) = \pi$. By Theorem 2.16, there are DAADs T_1 and T_2 of orders $2k-1$ and $2l-1$, where $n = k+l+1$, such that $T = (\mathcal{T}(1) \oplus T_1^f \oplus \mathcal{T}(1)) \ominus T_2$. Using Proposition 2.19, we find

$$\begin{aligned}
\pi &= SASM(T) \\
&= SASM\big((\mathcal{T}(1) \oplus T_1^f \oplus \mathcal{T}(1)) \ominus T_2\big) \\
&= \big(SASM(\mathcal{T}(1)) \oplus 1 \oplus SASM(T_1)^f \oplus 1 \oplus SASM(\mathcal{T}(1))\big) \ominus 1 \ominus SASM(T_2) \\
&= (1 \oplus SASM(T_1)^f \oplus 1) \ominus 1 \ominus SASM(T_2),
\end{aligned}$$

where the last step follows from the fact that $SASM(\mathcal{T}(1)) = \varnothing$. Now the result follows, since $\pi_1 = SASM(T_1)$ is a snow leopard permutation of length $2k-1$ and $\pi_2 = SASM(T_2)$ is snow leopard permutation of length $2l-1$, where $n = k+l+1$.

($\Leftarrow$) If π_1 and π_2 are snow leopard permutations of lengths $2k-1$ and $2l-1$, respectively, where $n = k+l+1$, then by definition there are DAADs T_1 and T_2 of orders $2k-1$ and $2l-1$, respectively, such that $\pi_1 = SASM(T_1)$ and $\pi_2 = SASM(T_2)$. By Theorem 2.16, we know that $(\mathcal{T}(1) \oplus T_1^f \oplus \mathcal{T}(1)) \ominus T_2$ is a DAAD of order $2n-1$. But if we apply *SASM* to this DAAD and use Proposition 2.19 as in the proof of the other direction, we find $(1 \oplus \pi_1^f \oplus 1) \ominus 1 \ominus \pi_2$ is a snow leopard permutation of length $2n-1$.

To see that the map $(\pi_1, \pi_2) \mapsto (1 \oplus \pi_1^r \oplus 1) \ominus 1 \ominus \pi_2$ is a bijection, first note that it is onto the set of snow leopard permutations by the first part of the theorem. To see it is one-to-one, suppose there are ordered pairs (π_1, π_2) and (σ_1, σ_2) of snow leopard permutations such that $(1 \oplus \pi_1^f \oplus 1) \ominus 1 \ominus \pi_2 = (1 \oplus \sigma_1^f \oplus 1) \ominus 1 \ominus \sigma_2$, and let π denote this common permutation. Then the hat (the second 1 in $1 \oplus \pi_1^f \oplus 1$ and $1 \oplus \sigma_1^f \oplus 1$) corresponds to the largest entry in π. Therefore π_1^f is a shift of the entries between the first entry of π and the largest entry of π, as is σ_1^f, so $\pi_1^f = \sigma_1^f$. But f is invertible, so $\pi_1 = \sigma_1$. Similarly, π_2 and σ_2 are both equal to the sequence of entries of π to the right of the largest entry of π, so $\pi_2 = \sigma_2$. □

It's worth noting that in small cases the permutation $(1 \oplus \pi_1^f \oplus 1) \ominus 1 \ominus \pi_2$ is not as long as it looks. For example, the antipermutation @ of length -1 is a snow leopard permutation corresponding to the TOAD of order -1. As a result, the snow leopard permutation 1 corresponds to the ordered pair $(@, @)$, since $1 = (1 \oplus @ \oplus 1) \ominus 1 \ominus @$. Similarly, for any snow leopard permutation π of odd length, $1 \oplus \pi \oplus 1$ and $1 \ominus 1 \ominus \pi$ are also snow leopard permutations of odd length, corresponding to the ordered pairs $(\pi, @)$ and $(@, \pi)$, respectively.

We can now use Theorem 2.20 to count the snow leopard permutations of each length.

Corollary 2.21. *For each* $n \geq 0$, *the number of snow leopard permutations of length* $2n-1$ *is* C_n.

Proof. For each $n \geq 0$, let a_n be the number of snow leopard permutations of length $2n-1$. There is just one snow leopard permutation of length -1, so $a_0 = 1 = C_0$ and the result holds for $n = 0$. Now fix $n \geq 1$ and suppose by induction that $a_j = C_j$ for all j, $0 \leq j \leq n-1$. By Theorem 2.20 and our induction hypothesis, we have

$$a_n = \sum_{j=0}^{n-1} a_j a_{n-1-j} = \sum_{j=0}^{n-1} C_j C_{n-1-j} = \sum_{j=1}^{n} C_{j-1} C_{n-j} = C_n. \qquad \square$$

We can also use Theorem 2.20 and Proposition 2.7 to count the snow leopard permutations of even length.

Proposition 2.22. *Suppose* f *is any of the functions* r, c, $i \circ r$, *or* $i \circ c$ *on permutations. Then for any* $n \geq 0$, *the map* $\pi \mapsto 1 \oplus \pi^f$ *is a bijection between the set of snow leopard permutations of length* $2n-1$ *and the set of snow leopard permutations of length* $2n$.

Proof. We first show that π is a snow leopard permutation of length $2n-1$ if and only if $1 \oplus \pi^f$ is a snow leopard permutation of length $2n$.

If π is a snow leopard permutation of length $2n-1$, then by definition there is a DAAD T of order $2n-1$ such that $SASM(T) = \pi$. By Proposition 2.15, the TOAD $\mathcal{T}(1) \oplus D^f$ is a DAAD of order $2n$. Now by Proposition 2.19, we have $SASM(\mathcal{T}(1) \oplus D^f) = 1 \oplus \pi^f$, since $SASM(\mathcal{T}(1)) = \varnothing$. Therefore $1 \oplus \pi^f$ is a snow leopard permutation of length $2n$.

Conversely, if $1 \oplus \pi^f$ is a snow leopard permutation of length $2n$, then by definition there is a DAAD T of order $2n$ such that $SASM(T) = 1 \oplus \pi^f$. Now by Proposition 2.15, there is a DAAD D of order $2n-1$ such that $T = \mathcal{T}(1) \oplus D^f$, and by Proposition 2.19, we have $SASM(T) = 1 \oplus SASM(D)^f$. Since π^f can be obtained from $1 \oplus \pi^f$ and f is invertible, we must have $\pi = SASM(D)$, so π is a snow leopard permutation.

Finally, it is routine to check that the map $\pi \mapsto 1 \oplus \pi^f$ is a bijection between S_{2n-1} and the set of permutations in S_{2n} whose first entry is 1, so the restriction of this map to the set of snow leopard permutations of length $2n-1$ must also be a bijection. $\square$

Corollary 2.23. *For each $n \geq 0$, the compatibility correspondence is a bijection between the set of DABPs of length n and the set of snow leopard permutations of length $n-1$.*

Proof. By definition the compatibility correspondence maps DABPs of length n onto snow leopard permutations of length $n-1$. Since each of these sets has the same number of elements, this correspondence must be a bijection. □

Theorem 2.20 also gives us useful structural information about snow leopard permutations. For instance, we have the following result concerning the parities of the entries of a snow leopard permutation.

Corollary 2.24. *Snow leopard permutations preserve parity. That is, if π is a snow leopard permutation of length n, then for all j with $1 \leq j \leq n$, the entry $\pi(j)$ is even if and only if j is even.*

Proof. We first consider the case in which n is odd.

The result is vacuously true for $\pi = @$, and trivial for $\pi = 1$, so suppose by induction that $n \geq 0$ is odd and the result holds for all snow leopard permutations of odd length less than n.

In general, if σ is a permutation of odd length which preserves parity, then σ^c, $1 \oplus \sigma$, and $1 \oplus \sigma \oplus 1$ also preserve parity. Similarly, if σ is a parity-preserving permutation of odd length then $1 \ominus \sigma$ is a parity-reversing permutation. Finally, if σ_1 is a parity-preserving permutation of odd length and σ_2 is a parity-reversing permutation of even length, then $\sigma_1 \ominus \sigma_2$ is a parity-preserving permutation.

By Theorem 2.20, if π is a snow leopard permutation of odd length then there are snow leopard permutations π_1 and π_2 of odd length such that $\pi = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \pi_2$. By induction and our observations above, $1 \oplus \pi_1^c \oplus 1$ is a parity-preserving permutation of odd length and $1 \ominus \pi_2$ is a parity-reversing permutation of even length, so π preserves parity.

Now suppose π is a snow leopard permutation of even length. By Proposition 2.22, we have $\pi = 1 \oplus \sigma^c$ for some snow leopard permutation σ of odd length. By our observations above, σ^c preserves parity, so $\pi = 1 \oplus \sigma^c$ also preserves parity. □

Theorem 2.20 also gives us pattern-avoidance properties of snow leopard permutations. In particular, we can use it to show that snow leopard permutations are *anti-Baxter*, which means they avoid 2–14–3 and 3–41–2.

Corollary 2.25. *If π is a snow leopard permutation then π avoids* 2–14–3 *and* 3–41–2.

Proof. We first consider the case in which $|\pi| = n$ is odd.

The result is clear for $\pi = @$, $\pi = 1$, $\pi = 123$, and $\pi = 321$, so suppose by induction that $n \geq 0$ is odd and the result holds for all snow leopard permutations of odd length less than n. By Theorem 2.20, if π is a snow leopard permutation of odd length then there are snow leopard permutations π_1 and π_2 of odd length such that $\pi = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \pi_2$. For convenience, we call the entries of π corresponding to $1 \oplus \pi_1^c \oplus 1$ the *front* of π, and we call the remaining entries of π the *back* of π. Note that every entry in the front of π is greater than every entry in the back of π.

Now suppose π contains a subsequence $abcd$ of type 2–14–3. If a is in the front of π, then d is also in the front of π, since $d > a$. Moreover, a cannot be the first entry of the front of π and d cannot be the last, since the first and last entries are the smallest and largest entries of the front of π, and we have $b < a$ and $c > d$. Therefore our subsequence is entirely contained in the entries of π corresponding to π_1^c, and the corresponding subsequence of π_1 has type 3–41–2. This contradicts our induction hypothesis.

On the other hand, if a is not in the front of π then every entry of our subsequence is in the back of π. The first entry of the back of π is the largest, but $c > a$, so in fact our subsequence is contained in π_2, which contradicts our induction hypothesis.

The proof that π has no subsequence of type 3–41–2 is similar.

Now suppose π is a snow leopard permutation of even length. By Proposition 2.22, we have $\pi = 1 \oplus \sigma^c$ for some snow leopard permutation σ of odd length. Arguing as above, if π has a subsequence of type 2–14–3 (resp. 3–41–2) then σ has a subsequence of type 3–41–2 (resp. 2–14–3), so the result follows by induction. □

One can show that this result holds more generally: if π is a Baxter permutation of length $n+1$ and σ is a compatible permutation of length n, then σ is anti-Baxter [Asinowski et al. 2013].

3. A bijection from snow leopard permutations to Catalan paths

Like the snow leopard permutations, Catalan paths have a natural recursive decomposition. In particular, every nonempty Catalan path with $2n$ steps has the form Np_1Ep_2, where p_1 and p_2 are Catalan paths with $2k$ and $2l$ steps, respectively, and $n = k + l - 1$. In fact, this decomposition gives a bijection between the set of Catalan paths p with $2n$ steps and ordered pairs (p_1, p_2) of Catalan paths with $2k$ and $2l$ steps, where $n = k + l - 1$. Matching this decomposition with our snow leopard permutation decomposition gives us a natural bijection from the set of Catalan paths with $2n$ steps to the set of snow leopard permutations of length $2n - 1$.

Proposition 3.1. *Suppose f is any of the functions r, c, $i \circ r$, and $i \circ c$. Then for each nonnegative integer n, there is a unique bijection Γ_f from the set of Catalan paths with $2n$ steps to the set of snow leopard permutations of length $2n - 1$ such that $\Gamma_f(\varnothing) = @$ and*

$$\Gamma_f(Np_1Ep_2) = \big(1 \oplus \Gamma_f(p_1)^f \oplus 1\big) \ominus 1 \ominus \Gamma_f(p_2)$$

for any Catalan paths p_1 and p_2.

p	$\Gamma_c(p)$
$\varnothing$	@
NE	1
$NNEE$	123
$NENE$	321
$NNNEEE$	14325
$NNENEE$	12345
$NNEENE$	34521
$NENNEE$	54123
$NENENE$	54321

p	$\Gamma_c(p)$
$NNNNEEEE$	1634527
$NNNENEEE$	1654327
$NNNEENEE$	1432567
$NNNEEENE$	3654721
$NNENNEEE$	1236547
$NNENENEE$	1234567
$NNENEENE$	3456721
$NNEENENE$	5674321
$NNEENNEE$	5674123
$NENNNEEE$	7614325
$NENNENEE$	7612345
$NENNEENE$	7634521
$NENENNEE$	7654123
$NENENENE$	7654321

Table 1. Values of $\Gamma_c(p)$ for short Catalan paths p.

Proof. Since each nonempty Catalan path can be written uniquely in the form Np_1Ep_2, where p_1 and p_2 are Catalan paths, Γ_f is well-defined and unique.

To show that $\Gamma_f(p)$ is a snow leopard permutation for every Catalan path p, first note that this is true for $p = \varnothing$ and $p = NE$. Now suppose by induction that p is a Catalan path with at least 4 steps, and that the result holds for all Catalan paths with fewer steps. Then there are unique Catalan paths p_1 and p_2 such that $p = Np_1Ep_2$, and by definition we have $\Gamma_f(p) = \big(1 \oplus \Gamma_f(p_1)^f \oplus 1\big) \ominus 1 \ominus \Gamma_f(p_2)$. By induction $\Gamma_f(p_1)$ and $\Gamma_f(p_2)$ are snow leopard permutations, so by Theorem 2.20 we see that $\Gamma_f(p)$ is also a snow leopard permutation.

To show that Γ_f is onto, first note that this is true for $n = 0$ and $n = 1$, so fix $n \geq 2$ and suppose by induction that the result holds for all smaller values of n. If π is a snow leopard permutation of length $2n - 1$, then by Theorem 2.20 there are shorter snow leopard permutations π_1 and π_2 of odd length such that $\pi = (1 \oplus \pi_1^f \oplus 1) \ominus 1 \ominus \pi_2$. By induction there are Catalan paths p_1 and p_2 such that $\Gamma_f(p_1) = \pi_1$ and $\Gamma_f(p_2) = \pi_2$, and by the definition of Γ_f, we have $\Gamma_f(Np_1Ep_2) = \pi$.

Since the set of Catalan paths with $2n$ steps and the set of snow leopard permutations of length $2n - 1$ are equinumerous by Corollary 2.21, the map Γ_f must be a bijection. $\square$

Although all four maps Γ_f are bijections, we will be particularly interested in Γ_c. In Table 1 we have the values of Γ_c for all Catalan paths with 8 or fewer steps.

π	$\kappa(\pi)$
@	$\varnothing$
1	NE
123	$NNEE$
321	$NENE$
14325	$NNNEEE$
12345	$NNENEE$
34521	$NNEENE$
54123	$NENNEE$
54321	$NENENE$

π	$\kappa(\pi)$
1634527	$NNNNEEEE$
1654327	$NNNENEEE$
1432567	$NNNEENEE$
3654721	$NNNEEENE$
1236547	$NNENNEEE$
1234567	$NNENENEE$
3456721	$NNENEENE$
5674321	$NNEENENE$
5674123	$NNEENNEE$
7614325	$NENNNEEE$
7612345	$NENNENEE$
7634521	$NENNEENE$
7654123	$NENENNEE$
7654321	$NENENENE$

Table 2. Values of $\kappa(\pi)$ for short snow leopard permutations π.

While it is not obvious from these data, it turns out that Γ_c^{-1} has a simple, direct description in terms of ascents and descents.

Definition 3.2. For any snow leopard permutation π of length $2n-1$, we write $\kappa(\pi)$ to denote the lattice path with $2n$ steps whose i-th step $\kappa(\pi)_i$ is given by

$$\kappa(\pi)_i = \begin{cases} N & \begin{array}{l} \pi(i) < \pi(i+1) \text{ and } i \text{ is odd} \\ \text{or} \\ \pi(i) > \pi(i+1) \text{ and } i \text{ is even,} \end{array} \\ E & \begin{array}{l} \pi(i) < \pi(i+1) \text{ and } i \text{ is even} \\ \text{or} \\ \pi(i) > \pi(i+1) \text{ and } i \text{ is odd} \end{array} \end{cases}$$

for $0 \le i \le 2n-1$. By convention, we treat the empty entries $\pi(0)$ and $\pi(2n)$ as $2n$ and 0, respectively.

Example 3.3. Consider the permutation $\pi = 789634521$, which has ascent/descent sequence $DAADDAADDD$. Thus we have $\kappa(\pi) = NNEENNEENE$.

In Table 2 we have the values of $\kappa(\pi)$ for all snow leopard permutations π of length 7 or less.

It is not immediately obvious that κ maps every snow leopard permutation to a Catalan path, so we prove this next.

Proposition 3.4. *Suppose π is a snow leopard permutation of length $2n-1$. Then $\kappa(\pi)$ is a Catalan path of length $2n$.*

Proof. It is routine to check this when π has length 3 or less, since $\kappa(@) = \varnothing$, $\kappa(1) = NE$, $\kappa(123) = NNEE$, and $\kappa(321) = NENE$. Now suppose the result holds for all snow leopard permutations of odd length less than $2n-1$, where $2n-1 \geq 5$, and that π is a snow leopard permutation of length $2n-1$. By Theorem 2.20, there are snow leopard permutations π_1 and π_2 of lengths $2k-1$ and $2l-1$, respectively, such that $n = k+l+1$ and $\pi = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \pi_2$. We now consider three cases.

Case One: If $\pi_1 = @$ then $\pi = 1 \ominus 1 \ominus \pi_2$. In this case the ascent/descent sequence for π consists of two descents, followed by the ascent/descent sequence for π_2. By the definition of κ, this means $\kappa(\pi) = NE\kappa(\pi_2)$. Since $\kappa(\pi_2)$ is a Catalan path by induction, so is $\kappa(\pi)$.

Case Two: If $\pi_2 = @$ then $\pi = 1 \oplus \pi_1^c \oplus 1$. Since the complement operation on permutations turns ascents into descents and vice versa, the ascent/descent sequence for π consists of a descent, followed by the complement of the ascent/descent sequence for π_1, followed by a descent. By the definition of κ, this means $\kappa(\pi) = N\kappa(\pi_1)E$. Since $\kappa(\pi_1)$ is a Catalan path by induction, so is $\kappa(\pi)$.

Case Three: Suppose $\pi_1 \neq @$ and $\pi_2 \neq @$. Reasoning as in the previous cases, we find that the ascent/descent sequence for π consists of a descent, followed by the complement of the ascent/descent sequence for π_1, followed by an E, followed by the ascent/descent sequence for π_2. By the definition of κ, $\kappa(\pi) = N\kappa(\pi_1)E\kappa(\pi_2)$. Since $\kappa(\pi_1)$ and $\kappa(\pi_2)$ are Catalan paths by induction, so is $\kappa(\pi)$. $\square$

The data in Tables 1 and 2, along with a close examination of the proof of Proposition 3.4, suggest that Γ_c and κ are inverses of one another; we prove this next.

Theorem 3.5. *Γ_c and κ are inverse functions.*

Proof. By Proposition 3.1 we know that Γ_c maps Catalan paths with $2n$ steps to snow leopard permutations of length $2n-1$, and by Proposition 3.4, the function κ maps snow leopard permutations of length $2n-1$ to Catalan paths with $2n$ steps. Since Γ_c is invertible, it's sufficient to show that $\Gamma_c(\kappa(\pi)) = \pi$ for every snow leopard permutation π.

The result is routine to check for $\pi = @$ and $\pi = 1$, so suppose π has length $2n-1 > 1$ and the result holds for all shorter snow leopard permutations. By Theorem 2.20, there are snow leopard permutations π_1 and π_2 such that $\pi = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \pi_2$. Reasoning as in the proof of Proposition 3.4, we see that $\kappa(\pi) = N\kappa(\pi_1)E\kappa(\pi_2)$. Now by the definition of Γ_c and our induction hypothesis, we have

$$\begin{aligned}\Gamma_c(\kappa(\pi)) &= \Gamma_c(N\kappa(\pi_1)E\kappa(\pi_2)) \\ &= N(\Gamma_c(\kappa(\pi_1)))^c E\Gamma_c(\kappa(\pi_2)) \\ &= N\pi_1^c E\pi_2 \\ &= \pi.\end{aligned}$$

$\square$

4. Using transpositions to generate snow leopard permutations

It is well known that every permutation is a product of adjacent transpositions, so the adjacent transpositions generate S_n. In this section we introduce a simple set of transpositions, and we show that the snow leopard permutations of odd length are exactly the permutations one can construct from the decreasing permutation using sequences of our transpositions. We begin with the transpositions themselves.

Definition 4.1. Suppose that π is a permutation with consecutive entries $\pi(i)$, $\pi(i+1), \ldots, \pi(j)$.

(1) If $\pi(i)$ and $\pi(j)$ are odd and either $\pi(i-1), \pi(i), \ldots, \pi(j), \pi(j+1)$ or $\pi(i-1), \pi(j), \ldots, \pi(i), \pi(j+1)$ is a decreasing sequence of consecutive integers, and σ is the permutation we obtain from π by interchanging $\pi(i)$ and $\pi(j)$, then we say π and σ are *related by* τ_1.

(2) If $\pi(i)$ and $\pi(j)$ are even and either $\pi(i-1), \pi(i), \ldots, \pi(j), \pi(j+1)$ or $\pi(i-1), \pi(j), \ldots, \pi(i), \pi(j+1)$ is an increasing sequence of consecutive integers, and σ is the permutation we obtain from π by interchanging $\pi(i)$ and $\pi(j)$, then we say π and σ are *related by* τ_2.

By convention, if $\pi(i)$ or $\pi(j)$ occurs at either end of π, then we waive any requirement for the behavior of π beyond that point.

Example 4.2. The permutations $\pi = 983654721$ and $\sigma = 983456721$ are related by τ_2, since 36547 can be replaced with 34567.

Example 4.3. The permutations $\pi = 567894321$ and $\sigma = 567894123$ are related by τ_1, since 4321 can be replaced with 4123.

In Figure 7 we have graphs showing how the snow leopard permutations of lengths 3, 5, and 7 are related to one another by τ_1 and τ_2.

Although we don't do it here, one can study the parity of the number of inversions in a snow leopard permutation of odd length to show that these graphs are always bipartite.

As we show next, snow leopard permutations are only related to other snow leopard permutations by τ_1 and τ_2. We begin with a lemma concerning snow leopard permutations which begin with a decreasing sequence of consecutive integers.

Lemma 4.4. *If π is a snow leopard permutation of odd length, and there is a permutation σ of odd length with $\pi = 1 \ominus 1 \ominus \cdots \ominus 1 \ominus \sigma$, then σ is a snow leopard permutation.*

Proof. We argue by induction on $|\pi| - |\sigma|$.

If $|\pi| = |\sigma|$ then $\pi = \sigma$, and the result is clear. If $|\pi| - |\sigma| = 2$ then $\pi = 1 \ominus 1 \ominus \sigma = (1 \oplus @ \oplus 1) \ominus 1 \ominus \sigma$ must be the snow leopard decomposition of π guaranteed by Theorem 2.20, so σ is a snow leopard permutation.

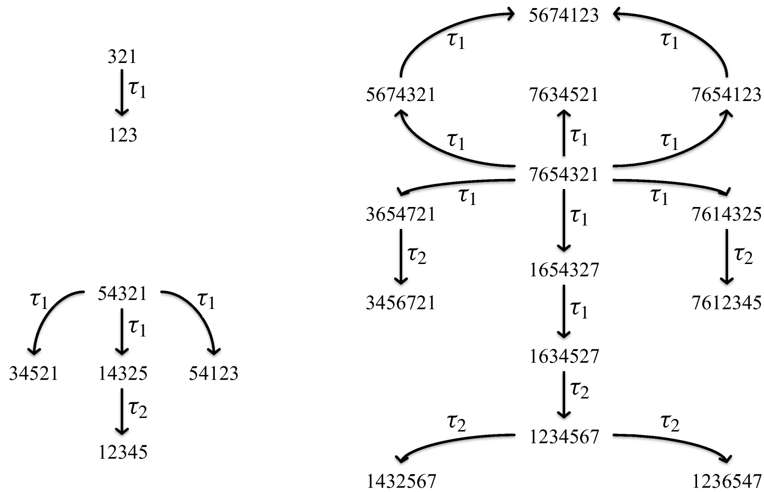


Figure 7. Graphs showing how the snow leopard permutations of lengths 3, 5, and 7 are related by τ_1 and τ_2.

Now suppose $|\pi| - |\sigma| \geq 4$. By Theorem 2.20, there are snow leopard permutations π_1 and π_2 such that $\pi = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \pi_2$. But π begins with its largest element, so we must have $\pi_1 = @$ and $\pi = 1 \ominus 1 \ominus \pi_2$. Therefore π_2 has the same form as π, but with two fewer 1s, so by induction σ is a snow leopard permutation. $\square$

Theorem 4.5. *Suppose π is a snow leopard permutation of odd length and σ is a permutation.*

(i) *If π and σ are related by τ_1, then σ is a snow leopard permutation.*

(ii) *If π and σ are related by τ_2, then σ is a snow leopard permutation.*

Proof. It turns out that (i) and (ii) depend on each other, so we prove them together.

It's routine to check that (i) and (ii) hold when π and σ have lengths -1, 1, or 3, so suppose $|\pi| = |\sigma| \geq 5$; we argue by induction on $|\pi|$.

Case One: π and σ are related by τ_1. By Theorem 2.20, there are snow leopard permutations π_1 and π_2 such that $\pi = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \pi_2$.

First suppose $\pi_1 = @$, so that $\pi = 1 \ominus 1 \ominus \pi_2$. In this case, if $i \geq 3$ then our swap takes place inside π_2, so there is a permutation σ_2 which is related to π_2 by τ_1 such that $\sigma = 1 \ominus 1 \ominus \sigma_2$. By induction, σ_2 is a snow leopard permutation, so σ is also a snow leopard permutation by Theorem 2.20. On the other hand, if $i \leq 2$ then $i = 1$, since the first entry of π is odd and the second is even. In this case there is a permutation β of odd length such that $\pi = 1 \ominus 1 \ominus \cdots \ominus 1 \ominus \beta$, and β is a snow leopard permutation by Lemma 4.4. Now $\sigma = (1 \oplus \alpha^c \oplus 1) \ominus 1 \ominus \beta$, where α is an identity permutation of odd length. Since α and β are snow leopard permutations, σ is also a snow leopard permutation by Theorem 2.20.

Now suppose $\pi_1 \neq @$. In this case our decreasing sequence must be entirely contained in either π_1^c or $1 \ominus \pi_2$. Since the $1 \ominus \pi_2$ part of π begins with an even number, any decreasing sequence beginning with an odd number in this part of π must be contained in π_2. Therefore there is a permutation σ_2 which is related to π_2 by τ_1 such that $\sigma = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \sigma_2$. By induction, σ_2 is a snow leopard permutation, so σ is a snow leopard permutation by Theorem 2.20.

On the other hand, if our decreasing sequence is contained in π_1^c, then it corresponds to an increasing sequence in π_1 which begins with an even number. Therefore, there is a permutation σ_1 which is related to π_1 by τ_2, for which $\sigma = (1 \oplus \sigma_1^c \oplus 1) \ominus 1 \ominus \pi_2$. By induction, σ_1 is a snow leopard permutation, so σ is also a snow leopard permutation by Theorem 2.20.

Case Two: π and σ are related by τ_2. By Theorem 2.20, there are snow leopard permutations π_1 and π_2 such that $\pi = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \pi_2$. In addition, any increasing sequence in π must be entirely contained in the $1 \oplus \pi_1^c \oplus 1$ part of π, or in the π_2 part of π. If our increasing sequence is contained in the π_2 part of π, then there is a permutation σ_2 which is related to π_2 by τ_2, such that $\sigma = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \sigma_2$. By induction, σ_2 is a snow leopard permutation, so σ is also a snow leopard permutation by Theorem 2.20.

On the other hand, if our increasing sequence is contained in the $1 \oplus \pi_1^c \oplus 1$ part of π, then we must have $i \geq 2$ and $i \leq |\pi_1| + 1$, since this part of π begins and ends with odd numbers. That is, our increasing sequence must be entirely contained in π_1^c. Therefore, this increasing sequence corresponds to a decreasing sequence in π_1, all of whose entries have opposite parity with the corresponding entries in π. This means there is a permutation σ_1 which is related to π_1 by τ_1 such that $\sigma = (1 \oplus \sigma_1^c \oplus 1) \ominus 1 \ominus \pi_2$. By induction, σ_1 is a snow leopard permutation, so σ is also a snow leopard permutation by Theorem 2.20. □

We are interested in permutations which are connected by chains of permutations in which consecutive permutations are related by τ_1 or τ_2, so we make the following definition.

Definition 4.6. We say permutations π and σ are *τ-related* whenever there is a sequence $\alpha_1, \ldots, \alpha_n$ of permutations such that $\pi = \alpha_1$, $\sigma = \alpha_n$, and for each j, the permutations α_j and α_{j-1} are related by τ_1 or related by τ_2.

We can now show that the snow leopard permutations of odd length are exactly those permutations that are τ-related to the reverse identity.

Theorem 4.7. *A permutation π of length $2n-1$ is a snow leopard permutation if and only if it is τ-related to the decreasing permutation of length $2n-1$.*

Proof. ($\Rightarrow$) It is routine to verify this result when π has length -1, 1, or 3, so suppose $|\pi| \geq 5$; we argue by induction on $|\pi|$. By Theorem 2.20, there are snow

length	1	3	5	7	9
SLP-like permutations	1	2	7	32	175
SLPs	1	2	5	14	42

Table 3. The number of SLPs compared with the number of permutations with some properties of SLPs.

leopard permutations π_1 and π_2 of odd length such that $\pi = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \pi_2$. By induction, there is a sequence s_1 (resp. s_2) of moves of types τ_1 and τ_2 which, when applied to the decreasing permutation of the appropriate length, produces π_1 (resp. π_2). To obtain π from the decreasing permutation of length $2n-1$, first apply a move of type τ_1 to swap the entries in positions 1 and $|\pi_1|+2$. Now apply the sequence s_2 of moves to the entries to the right of position $|\pi_1|+3$. Finally, for each move in s_1 of type τ_1, apply the corresponding move of type τ_2 to the subsequence in positions 2 through $|\pi_1|+1$, and vice versa. Since we have constructed each of the pieces of π individually, the resulting permutation is π itself.

($\Leftarrow$) It is routine to check that the decreasing permutation of length $2n-1$ is a snow leopard permutation, so this part is immediate from Theorem 4.5. $\square$

Corollary 4.8. *Suppose π and σ are τ-related permutations of odd length. Then π is a snow leopard permutation if and only if σ is a snow leopard permutation.*

Proof. This is immediate from Theorem 4.7, since π and σ are snow leopard permutations if and only if they are τ-related to the decreasing permutation of length $|\pi|$, and this relationship is transitive. $\square$

5. Questions and open problems

It should be possible to build on this work in a variety of directions. For example, it may be fruitful to study the distributions of various permutation statistics on snow leopard permutations, and to look for connections between these statistics and statistics on Catalan paths, or on other Catalan objects. In addition, both κ and the compatibility relation deserve more attention. Finally, we have the following more specific questions.

(1) Can we characterize the snow leopard permutations nonrecursively?

We have given a recursive decomposition of the snow leopard permutations, so in principle we can recognize these permutations in the wild using this decomposition. Similarly, we have also characterized the snow leopard permutations as the permutations generated by a particular set of transpositions. While these points of view are useful, we would also like to have a short list of simple conditions we can check to determine whether a given permutation is an SLP. For example, we know that if π is a snow leopard permutation of odd length then π preserves parity, π

avoids 2–14–3 and 3–41–2, and $\kappa(\pi)$ is a Catalan path. These conditions rule out many permutations, but there are still permutations with all of these properties which are not SLPs. In fact, in Table 3 we see how the number of permutations with these three properties compares with the number of snow leopard permutations for small lengths.

(2) What permutations of length n are compatible with alternating Baxter permutations of length $n+1$?

Cori, Dulucq, and Viennot [Cori et al. 1986] have used bijections with binary trees to prove that the alternating Baxter permutations of lengths $2n$ and $2n+1$ are counted by the products C_n^2 and $C_n C_{n+1}$ of Catalan numbers, respectively. We conjecture that the smaller permutations which are compatible with the alternating Baxter permutations are counted by the same products of Catalan numbers. Our preliminary explorations suggest that we can extend either the work of Cori, Dulucq, and Viennot or the work of Dulucq and Guibert [1998] to prove this conjecture, but it might also be possible to extend or modify κ to give a proof.

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