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Local well-posedness of a nonlocal Burgers' equation

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In this paper, we explore a nonlocal inviscid Burgers' equation. Fixing a parameter h, we prove existence and uniqueness of the local solution of the equation $u_t + \big(u(x+h,t) \pm u(x-h,t)\big)u_x = 0$ with given periodic initial condition $u(x,0) = u_0(x)$. We also explore the blow-up properties of the solutions to this Cauchy problem, and show that there exist initial data that lead to finite-time-blow-up solutions and others to globally regular solutions. This contrasts with the classical inviscid Burgers' equation, for which all nonconstant smooth periodic initial data lead to finite-time blow-up. Finally, we present results of simulations to illustrate our findings.

1. Introduction

Burgers' equation is a common equation that arises naturally in the study of fluid mechanics, traffic, and other fields. It is a relatively simple partial differential equation that has been extensively studied. In finite time, solutions to the inviscid Burgers' equation are known to develop shock waves and rarefactions for smooth initial data. It also serves as a basic example of conservation laws. Many different closed forms, series approximations, and numerical solutions are known for particular sets of boundary conditions.

The more general form of dissipative Burgers' equation is

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \gamma \Delta u, \tag{1-1}$$

where $u(x,t)$ represents the velocity at point $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$, $\gamma \in \mathbb{R}^+$, and the term on the right-hand side is the viscosity term which induces diffusion properties. For the inviscid one-dimensional case, Burgers' equation reduces to

$$\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = 0. \tag{1-2}$$

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The equation that we will be studying is

$$\frac{\partial u}{\partial t}(x,t) + \big(u(x+h,t) \pm u(x-h,t)\big)\frac{\partial u}{\partial x}(x,t) = 0, \tag{1-3}$$

with $h \geq 0$. As we can see in the equation, which is a generalized form of the usual one-dimensional Burgers' equation, it includes nonlocal factors. Unlike the local Burgers' equation, analytical solutions are extremely hard to discover for this kind of nonlocal equation. Also, the existence of solutions cannot be easily derived from the method of characteristics. If we look at the characteristics, which are defined by $dx/dt = u(x+h,t) \pm u(x-h,t)$, they are hard to analyze due to the nonlocality.

In Section 2, we prove the following two theorems, illustrating respectively the existence and uniqueness of classical local solutions for periodic initial data $u(x,0) = u_0(x)$. First we introduce the norm which in the following part of the paper will facilitate our proof

Define the Sobolev norm as follows:

Definition 1.1 (Sobolev norm). Let $u(x,t) \in C^\infty(\mathbb{T})$ for some $m \in \mathbb{Z}^+$. Then the Sobolev norm is defined as

$$\begin{aligned}\|u(\cdot,t)\|^2_{H^m([0,L])} &= \int_0^L u(x,t)\big((-\partial_{xx})^m u(x,t)\big)\,\mathrm{d}x \\ &= \int_0^L |\partial_x^m u(x,t)|^2\,\mathrm{d}x.\end{aligned}$$

Remark 1.2. Without loss of generality we can assume that the functions defined on torus have period L. The Sobolev space $H^m([0,L])$ is the closure of $C^\infty([0,L])$ with respect to this norm. Observe that we will work with what is usually called the homogeneous Sobolev space $\dot{H}^m$.

Theorem 1.3 (local existence). *Suppose $u_0 \in C^\infty(\mathbb{T})$. Then there exists a classical local solution $u(x,t)$ to (2-1) for $0 \leq t \leq T(u_0)$ for some $T(u_0) > 0$.*

Theorem 1.4 (uniqueness). *The solution $u(x,t)$ to (2-1) which is in $C^1([0,T], H^r)$ for large enough r is unique.*

We resort to functional analysis skills in Sobolev spaces. Basically, we use the original equation to generate a recursive sequence of functions and prove that in appropriately chosen Sobolev spaces, the sequence admits a unique limit that converges to a classical local solution, which turns out to be regular by the topological structure of the Sobolev spaces. In Section 3 we look at blow-up and non-blow-up of solutions in finite time, presenting examples of both cases and contrasting with the local Burgers' equation. Interestingly, owing to the nonlocality factors introduced, the blow-up behaviors of (1-3) vary greatly from the local Burgers' equation (1-2). Finally, we use graphics to show simulations run on our equation in Section 4 to illustrate our results.

2. Existence and uniqueness of solution

Let us now consider the following nonlocal variation of Burgers' equation:

$$u_t + \big(u(x+h,t) \pm u(x-h,t)\big)u_x = 0. \tag{2-1}$$

We will prove Theorem 1.3 by justifying Proposition 2.2 and Lemma 2.7 below. To do this, we construct a sequence of functions $u_n(x,t)$ and show that $u_n(x,t)$ will be uniformly bounded in $C([0,T], H^m)$ with large m, while du_n/dt are also uniformly controlled. Thus, by a well-known compactness criterion, there exists a limit which we show solves the equation.

Remark 2.1. Throughout the rest of the paper, we will denote any universal constant by C, which does not depend on $u(x,t)$ and may vary from line to line.

Proposition 2.2. *Define a recursive sequence of functions $\{u_n\}$ as*

$$\partial_t u_n + \mathcal{L}u_{n-1}\,\partial_x u_n = 0, \quad u_n(x,0) = u_0(x) \in C^\infty(\mathbb{T}), \tag{2-2}$$

where $u_n = u_n(x,t)$ for $n \ge 1$, $\mathcal{L}u_n = u_n(x+h,t) \pm u_n(x-h,t)$ is a shorthand notation, and $u_0(x,t) = u_0(x)$ is smooth. Then for all sufficiently large $m \in \mathbb{Z}^+$, there exists $T(\|u_0\|_{H^m})$ such that $\|u_n(\cdot,t)\|_{C([0,T],H^m)} < C_1(T)$ and $\|du_n/dt\|_{C([0,T],H^{m-1})} \le C_2(T)$ for all $0<t<T$. Moreover, there exists a subsequence n_j such that $u_{n_j}(x,t)$ converges to $u(x,t)$ in $C([0,T],H^r)$ for any $r<m$.

Remark 2.3. We should notice that (2-2) has unique solution in $C^\infty(\mathbb{T})$ for every n. To see this we apply an inductive argument to the method of characteristics. Since $u_0 \in C^\infty(\mathbb{T})$, we inductively assume that $u_{n-1} \in C^\infty(\mathbb{T})$. In this case, denote $\mathcal{L}u_{n-1}$ by $f_h(x,t)$. The characteristics system is

$$\begin{cases} \dfrac{dt}{dr}(r,s) = 1, & t(s,0) = 0, \\ \dfrac{dx}{dr}(r,s) = f_h(x,t), & x(s,0) = s, \\ \dfrac{dz}{dr}(r,s) = 0, & z(s,0) = u_0(s). \end{cases}$$

Solving the first we have $t=r$. Thus the second is nothing but $dx/dr = f_h(x,r)$. But $f_h(x,r)$ is smooth, which implies by ODE theory that we have a solution $x = g_h(r,s)$, where g_h is implicit and again smooth. Then the implicit function theorem suggests that we can write $s = k_h(x,r) = k_h(x,t)$. Solving the third, we get $u_n = u_0(s) = u_0(k_h(x,t))$. By the smoothness of both u_0 and k_h, the smoothness of u_n is obtained. The uniqueness of each u_n is guaranteed by the method of characteristics. For more details about the method of characteristics, see [Evans 1998]. Next, since u_0 has period L, an inductive argument will also show that u_n has period L for all n.

Then we move on to prove Proposition 2.2; notice that the above remark will justify the integration by parts in the following proof.

Proof. Let us multiply (2-2) by $\partial_x^{2m} u_n$ and integrate with respect to x from 0 to L:

$$\int_0^L \partial_t u_n \, \partial_x^{2m} u_n \, \mathrm{d}x = -\int_0^L \partial_x^{2m} u_n \, \mathcal{L} u_{n-1} \, \partial_x u_n \, \mathrm{d}x.$$

We can then integrate by parts m times and pull out the partial derivative with respect to time from the left-hand side:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u_n(\cdot, t)\|_{H^m}^2 = -\int_0^L \partial_x^{2m} u_n \, \mathcal{L} u_{n-1} \, \partial_x u_n \, \mathrm{d}x \leq \left| \int_0^L \partial_x^{2m} u_n \, \mathcal{L} u_{n-1} \, \partial_x u_n \, \mathrm{d}x \right|.$$

Integrating by parts m times on the right-hand side and noting that all of the boundary terms vanish due to periodicity, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \|u_n(\cdot, t)\|_{H^m}^2 &\leq \left| \int_0^L \partial_x^m (\mathcal{L} u_{n-1} \, \partial_x u_n) \partial_x^m u_n \, \mathrm{d}x \right| \\ &\leq \left| \int_0^L \sum_{l=0}^m \binom{m}{l} \partial_x^l (\mathcal{L} u_{n-1}) \, \partial_x^{m-l+1} u_n \, \partial_x^m u_n \, \mathrm{d}x \right| \\ &\leq \sum_{l=0}^m \binom{m}{l} \left| \int_0^L \partial_x^l (\mathcal{L} u_{n-1}) \partial_x^{m-l+1} u_n \, \partial_x^m u_n \, \mathrm{d}x \right|. \end{aligned} \tag{2-3}$$

Lemma 2.4. *For all $0 \leq l \leq m$ and $m > 3/2$,*

$$\left| \int_0^L \partial_x^l (\mathcal{L} u_{n-1}) \partial_x^{m-l+1} u_n \, \partial_x^m u_n \, \mathrm{d}x \right| \leq C \|u_{n-1}\|_{H^m} \|u_n\|_{H^m}^2.$$

Proof. For the $l = 0$ case, we can reduce this to the $l = 1$ case using integration by parts:

$$\left| \int_0^L \mathcal{L} u_{n-1} \partial_x^{m+1} u_n \, \partial_x^m u_n \, \mathrm{d}x \right| = C \left| \int_0^L \partial_x (\mathcal{L} u_{n-1}) (\partial_x^m u_n)^2 \, \mathrm{d}x \right|.$$

When $l = 1$, it is not hard to see that

$$\begin{aligned}\left|\int_0^L \partial_x(\mathcal{L}u_{n-1})(\partial_x^m u_n)^2 \,\mathrm{d}x\right| &\le \|\partial_x(\mathcal{L}u_{n-1})\|_{L^\infty}\cdot\left|\int_0^L (\partial_x^m u_n)^2\,\mathrm{d}x\right| \\ &\le \|\partial_x(\mathcal{L}u_{n-1})\|_{L^\infty}\cdot\int_0^L |\partial_x^m u_n|^2\,\mathrm{d}x \\ &= \|\partial_x(\mathcal{L}u_{n-1})\|_{L^\infty}\cdot\|u_n\|_{H^m}^2.\end{aligned}$$

Applying the Sobolev embedding theorem, we have that for $m > 3/2$,

$$\|\partial_x(\mathcal{L}u_{n-1})\|_{L^\infty} \le C\|\partial_x(\mathcal{L}u_{n-1})\|_{H^{m-1}} \le C\|\mathcal{L}u_{n-1}\|_{H^m},$$

$$\begin{aligned}\|\mathcal{L}u_{n-1}\|_{H^m} &= \|u_{n-1}(x+h,t)\pm u_{n-1}(x-h,t)\|_{H^m} \\ &\le 2\|u_{n-1}\|_{H^m}.\end{aligned}$$

We can conclude

$$\left|\int_0^L \partial_x(\mathcal{L}u_{n-1})(\partial_x^m u_n)^2\,\mathrm{d}x\right| \le C\cdot\|u_{n-1}\|_{H^m}\cdot\|u_n\|_{H^m}^2 \quad \text{for } m > \tfrac{3}{2}.$$

In general, by Hölder's inequality, terms on the right-hand side of (2-3), for $l \neq 1$, are estimated by

$$\begin{aligned}&\left|\int_0^L \partial_x^l(\mathcal{L}u_{n-1})\partial_x^{m-l+1}u_n\,\partial_x^m u_n\,\mathrm{d}x\right| \\ &\qquad \le \|\partial_x^l(\mathcal{L}u_{n-1})\|_{L^{\frac{2(m-1)}{l-1}}}\cdot\|\partial_x^{m-l+1}u_n\|_{L^{\frac{2(m-1)}{m-l}}}\cdot\|\partial_x^m u_n\|_{L^2}. \end{aligned}\tag{2-4}$$

Recall that Gagliardo–Nirenberg inequality (see, e.g., [Doering and Gibbon 1995]) has the form

$$\|\partial_x^s f\|_{L^{2m/s}} \le C\|f\|_{L^\infty}^{1-s/m}\|f\|_{H^m}^{s/m} \quad \text{for all } 1 \le s \le m. \tag{2-5}$$

Now by applying (2-5) and the Sobolev embedding theorem, we can conclude the following two facts:

$$\begin{aligned}\|\partial_x^{m-l+1}u_n\|_{L^{\frac{2(m-1)}{m-l}}} &= \|\partial_x^{m-l}(\partial_x u_n)\|_{L^{\frac{2(m-1)}{m-l}}} \\ &\le C\cdot\|\partial_x u_n\|_{L^\infty}^{1-\frac{m-l}{m-1}}\cdot\|\partial_x^m u_n\|_{L^2}^{\frac{m-l}{m-1}} \\ &\le C\cdot\|\partial_x u_n\|_{H^{m-1}}^{1-\frac{m-l}{m-1}}\cdot\|\partial_x u_n\|_{H^{m-1}}^{\frac{m-l}{m-1}} \\ &= C\cdot\|\partial_x u_n\|_{H^{m-1}} \\ &= C\cdot\|u_n\|_{H^m},\end{aligned}\tag{2-6}$$

$$
\begin{aligned}
\|\partial_x^l(\mathcal{L}u_{n-1})\|_{L^{\frac{2(m-1)}{l-1}}} &= \|\partial_x^{l-1}(\partial_x(\mathcal{L}u_{n-1}))\|_{L^{\frac{2(m-1)}{l-1}}} \\
&\le C\cdot\|\partial_x(\mathcal{L}u_{n-1})\|_{L^\infty}^{1-\frac{l-1}{m-1}}\cdot\|\partial_x^m(\mathcal{L}u_{n-1})\|_{L^2}^{\frac{l-1}{m-1}} \\
&\le C\cdot\|\partial_x(\mathcal{L}u_{n-1})\|_{H^{m-1}}^{1-\frac{l-1}{m-1}}\cdot\|\partial_x^m(\mathcal{L}u_{n-1})\|_{H^{m-1}}^{\frac{l-1}{m-1}} \\
&= C\cdot\|\partial_x(\mathcal{L}u_{n-1})\|_{H^{m-1}} \\
&= C\cdot\|\mathcal{L}u_{n-1}\|_{H^m} \\
&\le C\cdot\|u_{n-1}\|_{H^m}.
\end{aligned}
\tag{2-7}
$$

Plugging (2-6) and (2-7) into (2-4), we get

$$
\left|\int_0^L \partial_x^l(\mathcal{L}u_{n-1})\partial_x^{m-l+1}u_n\,\partial_x^m u_n\,\mathrm{d}x\right| \le C\|u_{n-1}\|_{H^m}\cdot\|u_n\|_{H^m}^2,
$$

with constant C which depends only on m. So we have proved the lemma. □

Now let

$$
f_0(t) = f_n(0) = \|u_0\|_{H^m}^2.
$$

Notice that

$$
\|u_n(\cdot,0)\|_{H^m} = \|u_0\|_{H^m}.
$$

Now we define $f_n(t)$ inductively by

$$
f_n'(t) = C(m)\sqrt{f_{n-1}(t)}\,f_n(t), \tag{2-8}
$$

where $C(m)$ is a constant depending only on m from proof above.

Observe that $f_1(t) \ge f_0(t) > 0$ for all $t \ge 0$ since the right-hand side of (2-8) is always positive. Then inductively, we can obtain that $f_n(t) \ge f_{n-1}(t)$ for all $t \ge 0$.

Also, given

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(\cdot,t)\|_{H^m}^2 \le C(m)\|u_{n-1}(\cdot,t)\|_{H^m}\cdot\|u_n(\cdot,t)\|_{H^m}^2,
$$

it follows that

$$
f_n(t) \ge \|u_n(\cdot,t)\|_{H^m}^2.
$$

Thus

$$
f_n'(t) = C(m)\sqrt{f_{n-1}(t)}\,f_n(t) \le C(m)\,f_n^{3/2}(t).
$$

Because $f_n(t) \ne 0$, we can divide by $f_n^{3/2}(t)$ to get

$$
\frac{f_n'(t)}{f_n^{3/2}(t)} \le C(m).
$$

We can then integrate from 0 to t, giving

$$\int_0^t \frac{f_n'(s)}{f_n^{3/2}(s)}\,\mathrm{d}s \le \int_0^t C(m)\,\mathrm{d}t,$$

$$-2f_n^{-1/2}(t) + 2(\|u_0\|_{H^m})^{-1/2} \le C(m)t,$$

$$f_n^{1/2}(t) \le \frac{1}{\|u_0\|_{H^m}^{-1/2} - C(m)t/2}.$$

If we let $T := \bigl(C(m)\sqrt{\|u_0\|_{H^m}}\bigr)^{-1}$, we can conclude that for any $0 \le t \le T$, $\{f_n(t)\}$ will be uniformly bounded by some constant $C_1(T)$. But we know that $f_n(t) \ge \|u_n(\cdot,t)\|_{H^m}^2$. Therefore

$$\sup_{t\in[0,T]} \|u_n(\cdot,t)\|_{H^m(\mathbb{R})} \le C_1(T).$$

Since u_n satisfies (2-2), and H^s in dimension one is an algebra for every $s > 1/2$, this bound also implies

$$\|\partial_t u_n(\cdot,t)\|_{H^{m-1}[0,L]} \le C_2(T),$$

if $m > 3/2$. Now standard arguments (see, e.g., [Majda and Bertozzi 2002]) yield existence of a subsequence u_{n_j} converging to a function $u(x,t)$ in $L^\infty([0,T], H^r)$ for any $r < m$. Namely, recall the following compactness criterion.

Proposition 2.5. *Define a Banach space*

$$Y = \bigl\{v \in L^{\alpha_0}([0,T], H^m),\ \partial_t v \in L^{\alpha_1}([0,T], H^s)\bigr\},$$

where $s \le m$, and $1 \le \alpha_{0,1} \le \infty$. Define the norm on the space Y by

$$\|v\|_Y = \|v\|_{L^{\alpha_0}([0,T],H^m)} + \|\partial_t v\|_{L^{\alpha_1}([0,T],H^s)}.$$

Then Y imbeds compactly into any $L^{\alpha_0}([0,T], H^r)$ with $r < s$.

Remark 2.6. This criterion can be found, for example, in [Temam 1977, page 184] (see also [Constantin and Foias 1988]).

It follows that for any $r < m$, we can find u_{n_j} converging to some u strongly in $L^\infty([0,T], H^r)$. This concludes the proof of Proposition 2.2. □

Lemma 2.7. *The function $u(x,t)$ from Proposition 2.2 is a classical solution of (2-1) and belongs to $C([0,T], H^r)$ for any $r < m$.*

Remark 2.8. Since so far u has been defined only up to sets of measure zero in time, what we mean is that it can be fixed, if necessary, on a set of times of measure zero so that the claim of the lemma holds.

Proof. Pick m large enough; $m > 7/2$ is sufficient for the argument below to work. Fix any $5/2 < l < m$. We have the recursive formula for u_n in (2-2) and we proved in Proposition 2.2 that a subsequence u_{n_j} (which we will for simplicity denote u_n) converges to u in $L^\infty([0,T], H^l)$. Take some s such that $l-1 > s > 3/2$. We have

$$\begin{aligned}\|\mathcal{L}u_{n-1}\,\partial_x u_n - \mathcal{L}u\,\partial_x u\|_{H^s} &\le \|(\mathcal{L}u_{n-1}-\mathcal{L}u)\partial_x u_n\|_{H^s} + \|\mathcal{L}u(\partial_x u_n - \partial_x u)\|_{H^s}\\ &\le \|\mathcal{L}u_{n-1}-\mathcal{L}u\|_{H^s}\|\partial_x u_n\|_{H^s} + \|\mathcal{L}u\|_{H^s}\|\partial_x u_n - \partial_x u\|_{H^s}\\ &\le C\|u_{n-1}-u_n\|_{H^s}\|\partial_x u_n\|_{H^s} + C\|\mathcal{L}u\|_{H^s}\|u_n - u\|_{H^{s+1}}.\end{aligned}$$

By our choice of l and s, we have

$$\|u_{n-1}-u_n\|_{H^s} \to 0 \text{ uniformly in } t \in [0,T] \text{ as } n \to \infty,$$
$$\|u_n - u\|_{H^{s+1}} \to 0 \text{ uniformly in } t \in [0,T] \text{ as } n \to \infty.$$

Thus

$$\|\mathcal{L}u_{n-1}\,\partial_x u_n - \mathcal{L}u\,\partial_x u\|_{H^s} \to 0 \text{ uniformly in } t \in [0,T] \text{ as } n \to \infty.$$

Now, integrating (2-2) from 0 to t, we have

$$u_n(x,t) = u_n(x,0) - \int_0^t \mathcal{L}u_{n-1}\,\partial_x u_n\,\mathrm{d}s = u_0(x) - \int_0^t \mathcal{L}u_{n-1}\,\partial_x u_n\,\mathrm{d}s. \quad (2\text{-}9)$$

Note that by our choice of l and s the H^l- or H^s-convergence implies pointwise convergence, so $u_n \to u$, $\mathcal{L}u_{n-1}\,\partial_x u_n \to \mathcal{L}u\,\partial_x u$ pointwise for almost every t. Then from (2-9), as proved in Proposition 2.2, we conclude that for almost every t,

$$u(x,t) = u_0(x) - \int_0^t \mathcal{L}u\,\partial_x u\,\mathrm{d}s.$$

This means that $u(x,t)$ is Lipschitz in time with values in H^s (up to fixing it on a measure-zero set of times). We also have that $u \in L^\infty([0,T], H^m)$ since the approximating sequence satisfies uniform bound in this space. But then for every $s < r < m$, we have

$$\|u(\cdot,t_2)-u(\cdot,t_1)\|_{H^r} \le \|u(\cdot,t_2)-u(\cdot,t_1)\|_{H^m}^{\frac{r-s}{m-s}}\|u(\cdot,t_2)-u(\cdot,t_1)\|_{H^s}^{\frac{m-r}{m-s}},$$

and so we obtain that $u \in C([0,T], H^r)$. □

We have therefore proved that there exists a solution to our equation, (2-1). We now prove uniqueness by considering two different solutions of our equation, $\theta(x,t)$ and $\varphi(x,t)$, and showing that their difference $w(x,t) = \theta(x,t) - \varphi(x,t)$ is zero for all t and x.

Next, we prove that the classical solution is also unique, which is indicated in Theorem 1.4.

Proof. Let θ and φ be solutions to (2-1) with initial data $u(x,0) = u_0(x)$. Then

$$\theta_t + \mathcal{L}\theta\,\theta_x = 0, \tag{2-10}$$

$$\varphi_t + \mathcal{L}\varphi\,\varphi_x = 0. \tag{2-11}$$

Let $w = \theta - \varphi$. Subtracting (2-11) from (2-10), we get

$$\begin{aligned}\partial_t w &= -(\mathcal{L}\theta\,\theta_x - \mathcal{L}\varphi\,\varphi_x)\\ &= -(\mathcal{L}\theta\,\theta_x - \mathcal{L}\varphi\,\varphi_x) + \mathcal{L}\theta\,\varphi_x - \mathcal{L}\theta\,\varphi_x = -\mathcal{L}\theta\, w_x - \mathcal{L}w\,\varphi_x.\end{aligned}$$

We multiply by $(-1)^r \partial_x^{2r} w$, integrate from 0 to L, and integrate the left-hand side by parts r times, giving

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (\partial_x^r w)^2\,\mathrm{d}x = (-1)^{r+1}\int_0^L \partial_x^{2r} w\,\mathcal{L}\theta\,\partial_x w\,\mathrm{d}x + (-1)^{r+1}\int_0^L \partial_x^{2r} w\,\mathcal{L}w\,\partial_x\varphi\,\mathrm{d}x,$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{H^r}^2 \le \underbrace{\left|\int_0^L \partial_x^{2r} w\,\mathcal{L}\theta\,\partial_x w\,\mathrm{d}x\right|}_{I_1} + \underbrace{\left|\int_0^L \partial_x^{2r} w\,\mathcal{L}w\,\partial_x\varphi\,\mathrm{d}x\right|}_{I_2}. \tag{2-12}$$

Integrating I_1 by parts r times gives

$$\left|\int_0^L \partial_x^{2r} w\,\mathcal{L}\theta\,\partial_x w\,\mathrm{d}x\right| \le \sum_{l=0}^{r}\binom{m}{l}\left|\int_0^L \partial_x^l(\mathcal{L}\theta)\,\partial_x^{r-l+1} w\,\partial_x^r w\,\mathrm{d}x\right|.$$

Again, when $l = 0$, we can reduce this to the $l = 1$ case using integration by parts. When $l = 1$,

$$\begin{aligned}I_1 = \left|\int_0^L \partial_x^l(\mathcal{L}\theta)\partial_x^{r-l+1} w\,\partial_x^r w\,\mathrm{d}x\right| &= \left|\int_0^L \partial_x(\mathcal{L}\theta)\,\partial_x^r w\,\partial_x^r w\,\mathrm{d}x\right|\\ &= \left|\int_0^L \partial_x(\mathcal{L}\theta)(\partial_x^r w)^2\,\mathrm{d}x\right|\\ &\le \|\partial_x(\mathcal{L}\theta)\|_{L^\infty}\cdot\int_0^L |\partial_x^r w|^2\,\mathrm{d}x\\ &\le C\cdot\|\partial_x(\mathcal{L}\theta)\|_{L^\infty}\cdot\|w\|_{H^r}^2\\ &\le C\cdot\|\theta\|_{H^r}\cdot\|w\|_{H^r}^2\end{aligned}$$

if $r-1>1/2$. When $l\neq 1$,

$$I_1=\left|\int_0^L \partial_x^l(\mathcal{L}\theta)\partial_x^{r-l+1}w\,\partial_x^r w\,\mathrm{d}x\right|\le \|\partial_x^l(\mathcal{L}\theta)\|_{L^{\frac{2(r-1)}{l-1}}}\cdot\|\partial_x^{r-l+1}w\|_{L^{\frac{2(r-1)}{r-l}}}\cdot\|\partial_x^r w\|_{L^2}$$
$$\le C\|\theta\|_{H^r}\cdot\|w\|_{H^r}^2$$

as before. We can therefore conclude that

$$I_1=\left|\int_0^L \partial_x^{2r}w\,\mathcal{L}\theta\,\partial_x w\,\mathrm{d}x\right|\le C\|\theta\|_{H^r}\cdot\|w\|_{H^r}^2.$$

The same process can be done to I_2 to determine a bound for the integral, giving the result

$$I_2=\left|\int_0^L \partial_x^{2r}w\,\mathcal{L}w\,\partial_x\varphi\,\mathrm{d}x\right|\le C\|\varphi\|_{H^r}\cdot\|w\|_{H^r}^2.$$

Thus, (2-12) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{H^r}^2&\le C\|\theta\|_{H^r}\cdot\|w\|_{H^r}^2+C\|\varphi\|_{H^r}\cdot\|w\|_{H^r}^2\\&=\|w\|_{H^r}^2(C\|\theta\|_{H^r}+C\|\varphi\|_{H^r}).\end{aligned}$$

Then by Grönwall's inequality, we have

$$\|w(\cdot,t)\|_{H^r}\le\|w(\cdot,0)\|_{H^r}\exp\left(\int_0^t(C\|\theta(\cdot,s)\|_{H^r}+C\|\varphi(\cdot,s)\|_{H^r})\,\mathrm{d}s\right),$$

but $\|w(\cdot,0)\|_{H^r}=0$ because θ and φ are solutions to the same Cauchy problem. Therefore, the difference $w=\theta-\varphi$ is zero a.e. Since θ and φ are sufficiently smooth, they must be equal everywhere. □

3. Blow-up and non-blow-up properties

Let us consider the following two subcases of equation (2-1), where they both have initial data $u_0(x)$ of period L:

$$u_t+\big(u(x+h,t)+u(x-h,t)\big)u_x=0, \tag{3-1}$$

$$u_t+\big(u(x+h,t)-u(x-h,t)\big)u_x=0. \tag{3-2}$$

Remark 3.1. Let us introduce the following notation: denote $u^h(x,t)$ to be the solution of an equation with spatial shift h. Looking at (3-2), it can be shown using symmetry and uniqueness that if the smooth initial condition $u_0(x)$ is even, the solution, while it remains smooth, will stay even in x. Also, $u^h(x,t)=u^{L-h}(x,t)$

for all periodic initial data. Now consider (3-1). If $u_0(x)$ is odd, the solution will stay odd in x. Also, $u^h(x,t) = u^{L-h}(x,t)$ will hold for all even initial data $u_0(x)$.

These facts are deduced from the existence and uniqueness of solutions, definitions of evenness and oddness, and periodicity applied to our equation.

We now state the existence of solutions that blow up in finite time.

Theorem 3.2 (Existence of blow-up). *There exists initial data $u_0 \in C^\infty(\mathbb{R})$ such that the solution $u(x,t)$ to (2-1) blows up in finite time.*

We prove this result in Section 3. We first derive some properties of the solution.

Lemma 3.3. *Suppose $u(x,t)$ is a periodic solution of (2-1) with period $L = 2h$. Let $u(0,0) = u(h,0) = 0$; then $u(0,t) = u(h,t) = 0$, for all $t > 0$.*

We can prove this by considering both the plus and minus cases as follows:

Proof. Let us first consider the plus sign case, (3-1). Plugging $x = 0, h$ into to the recursive formula (2-2) for the plus case, we get

$$\partial_t u_n(0,t) = -2u_{n-1}(h,t)\partial_x u_n(0,t),$$
$$\partial_t u_n(h,t) = -2u_{n-1}(0,t)\partial_x u_n(h,t).$$

Since $u(0,0) = u_0(0) = u(h,0) = u_0(h) = 0$, we easily see that $\partial_t u_1(0,t) = \partial_t u_1(h,t) = 0$; therefore u_1 is constant at $x = 0, h$. But $u_1(0,0) = u_0(0) = 0$ and $u_1(h,0) = u_0(h) = 0$, so we have $u_1(0,t) = u_1(h,t) = 0$. Then, inductively, assume $u_{n-1}(0,t) = u_{n-1}(h,t) = 0$. Then, $\partial_t u_n(0,t) = \partial_t u_n(h,t) = 0$ so they are both constant. By the same reasoning, $u_n(0,0) = u_n(h,0) = 0$; therefore they are identically zero for all time. But our solution is just the limit of a subsequence of u_n, so $u(0,t) = u(h,t) = 0$

Now let us consider the minus sign case, (3-2). Plugging $x = 0$ into (3-2), we get

$$u_t(0,t) = (u(h,t) - u(h,t))u_x(0,t) = 0,$$

because $u(-h,t) = u(h,t)$ due to the period $L = 2h$. So $u(0,t) = C$, independent of time. Therefore, if we choose $u(0,0) = 0$, then $u(0,t) = 0$ for all $t > 0$. The same may be done at $u(h,0)$ to show that if $u(h,0) = 0$, then $u(h,t) = 0$. □

Corollary 3.4. *Suppose $u_0(x) \in C^\infty(\mathbb{R})$ has period $L = kh$ for some $k \in \mathbb{Z}$ and $u_0(mh) = 0$ for all $0 \le m \le k$. Then the solution to (2-1) satisfies $u(mh,t) = 0$ for all $t \ge 0$ and $0 \le m \le k$.*

The proof is similar to that from Lemma 3.3 extended for more general integers.

Blow-up. Now we investigate the cases where $u_0(x)$ has period $L = 2h$ and $u_0(0) = u_0(h) = 0$, and derive the possibility of blow-up.

Lemma 3.5. *Consider the equation* $u_t + \big(u(x+h,t) + u(x-h,t)\big)u_x = 0$ *with* $u(x,0) = u_0(x) \in C^\infty(\mathbb{R})$, *period* $L = 2h$, *and* $u_0(0) = u_0(h) = 0$. *Assume* $u_x(0,0) < 0$ *and* $u_x(h,0) < 0$. *Then the solution* $u(x,t)$ *blows up in finite time.*

Proof. Note that in Proposition 2.2, we proved that if the initial data $u_0(x)$ has period $2h$, then $u(x,t)$ will also have period $L = 2h$. Also, in this case, by (3-1), $u(0,t) = u(h,t) = 0$.

Differentiating the equation with respect to x gives

$$u_{tx}(x,t) + \big(u_x(x+h,t) + u_x(x-h,t)\big)u_x(x,t) + \big(u(x+h,t) + u(x-h,t)\big)u_{xx}(x,t) = 0. \quad (3\text{-}3)$$

Plug in $x = 0, h$ respectively and define $F_1(t) = u_x(0,t)$ and $F_2(t) = u_x(h,t)$. Noting that the last terms in both cases vanish, we get

$$F_1' + 2F_1F_2 = 0, \quad (3\text{-}4)$$

$$F_2' + 2F_1F_2 = 0. \quad (3\text{-}5)$$

It is easy to see that $F_1' - F_2' = 0$; thus $F_1 - F_2 = A$, where A is a constant. Since we assume $F_1 = F_2$, we get that $A = 0$. Plugging this into (3-4) gives

$$F_1' + 2F_1^2 = 0.$$

The solution to this differential equation is

$$F_1(t) = \frac{1}{\frac{1}{F_1(0)} + 2t}.$$

This blows up in finite time when

$$t = -\frac{1}{2F_1(0)} = -\frac{1}{2u_x(0,0)} > 0.$$

We can argue similarly for (3-5) to show that F_2 also blows up in finite time under the same conditions. □

Remark 3.6. For instance, we can take

$$u(x,0) = u_0(x) = x(x-h)(x-2h)\left(-\frac{1}{2h^2} + \frac{3}{h^3}x - \frac{3}{2h^4}x^2\right)$$

for $0 \le x \le 2h$. This satisfies our assumptions in Lemma 3.5 and thus the corresponding solution blows up in finite time.

Remark 3.7. There is an obvious case of blow-up for the plus sign equation when the period L is just h. Equation (3-1) reduces to

$$u_t + 2u \cdot u_x = 0.$$

This is the typical Burgers' equation, which is known to blow up in finite time for any nonconstant periodic initial condition $u_0(x)$ [McOwen 2003].

Lemma 3.8. *Suppose u_0 has period $L=6h$ and is even, and $u_0(kh)=0$, $u_0'(3kh)=0$ for all $k \in \mathbb{Z}$. Assume $u_x(2h,0) < 0$, $u_x(h,0) > 0$ and*

$$\frac{\ln u_x(h,0) - \ln(-u_x(2h,0))}{u_x(h,0) + u_x(2h,0)} > 0.$$

Then the solution $u(x,t)$ to the Cauchy problem,

$$u_t + \big(u(x+h,t) - u(x-h,t)\big)u_x = 0,$$
$$u(x,0) = u_0(x),$$

blows up in finite time.

Proof. By Lemma 3.3 and Corollary 3.4, we have $u(kh,t) = 0$, for all $k \in \mathbb{Z}$ and $u(x,t)$ is even if $u_0(x)$ is even. Differentiating the equation with respect to x gives

$$u_{tx}(x,t) + \big(u_x(x+h,t) - u_x(x-h,t)\big)u_x(x,t) + \big(u(x+h,t) - u(x-h,t)\big)u_{xx}(x,t) = 0.$$

Observe that $u_x(3kh,t) = 0$ for all time by an argument similar to proof of Lemma 3.3. Plugging in $x = h, 2h$ gives

$$F_1'(t) + F_1(t)F_2(t) = 0,$$
$$F_2'(t) - F_1(t)F_2(t) = 0,$$

where $F_1(t) = u_x(h,t)$ and $F_2(t) = u_x(2h,t)$. Solving this system of ordinary differential equations gives

$$F_1(t) + F_2(t) = F_1(0) + F_2(0) = A,$$
$$F_1'(t) = F_1^2(t) - AF_1(t)$$

for some constant A. Thus

$$F_1(t) = \frac{A \exp(AB)}{\exp(AB) - \exp(At)},$$

where

$$B = \frac{\ln F_1(0) - \ln(-F_2(0))}{F_1(0) + F_2(0)}.$$

This blows up in finite time if $F_2(0) = u_x(2h,0) < 0$, $F_1(0) = u_x(h,0) > 0$ and $B > 0$. □

Remark 3.9. To give an example, take $h = 4/3$. Then we can take

$$u(x,0) = u_0(x) = \frac{16(x-4)^2(x+4)^2x^2(3x-8)(3x+8)(3x+4)(3x-4)}{3375(112+153x^2)}.$$

This satisfies our assumptions in Lemma 3.8 and thus blows up in finite time. This may not be a very nicely manufactured example, but our point is that functions specified by Lemma 3.8 do exist.

Non-blow-up. We will now look for stationary solutions by taking specific initial data to (3-2) and showing that it cannot blow up in finite time. Let $u(x,t) = \sin(\pi x k/h)$, where h is fixed and $k \in \mathbb{Z}$. Noting that $u_t = 0$ and $u(x+h,t) - u(x-h,t) = 0$ (by trigonometric identities), we have that $u(x,t)$ solves the equation and never blows up.

Similarly, for (3-1), we will take $u(x,t) = \sin\big(\pi x\big(k-\frac{1}{2}\big)/h\big)$, where h is fixed and $k \in \mathbb{Z}$. Once again, noting that $u_t = 0$ and $u(x+h,t)+u(x-h,t) = 0$, we have that $u(x,t)$ solves the equation. We also know that $u(x,t) = \sin\big(\pi x\big(k-\frac{1}{2}\big)/h\big)$ never blows up. So we have found stationary solutions for both equations (3-1) and (3-2) that never blow up in finite time. So the nonlocal models are different from Burgers' equation where any nonconstant solution blows up in finite time: there exists nontrivial initial data for which solutions are globally regular for the nonlocal equation.

We can also construct a stationary solution to (3-2) by setting the period L to be h. The nonlocal terms become $u(x+h,t) = u(x-h,t) = u(x,t)$, so (3-2) reduces to $u_t = 0$. This is constant in time. Therefore $u(x,t) = u_0(x)$ for all t, so given a smooth initial condition, $u(x,t)$ will not blow up.

4. Simulations

In this section, we compare our model with the well-known "local" Burgers' equation (1-2). We used Matlab v2013 to run all simulations, with a forward-in-time, centered-in-space scheme. We illustrate many of the results of this paper in the graphics we generate.

We first look at the "local" Burgers' equation, (1-2). We know that this leads to gradient catastrophe (i.e., blow-up in gradient) in finite time for all nonconstant smooth initial data. We use $u(x,0) = \sin(\pi x)$ to generate Figure 1 (left).

As we can see, the slope of the graph in Figure 1 (left) at $x = 0$ blows up in finite time. Now, considering our equation with the plus sign,

$$u_t + \big(u(x+h,t) + u(x-h,t)\big)u_x = 0,$$

notice that there is a translation parameter h in our equation which affects the location of blow-up. As we can see in Figure 1 (right) with $h = L/8$, where L is the period of the initial data, blow-up does not occur at the origin, and two peaks

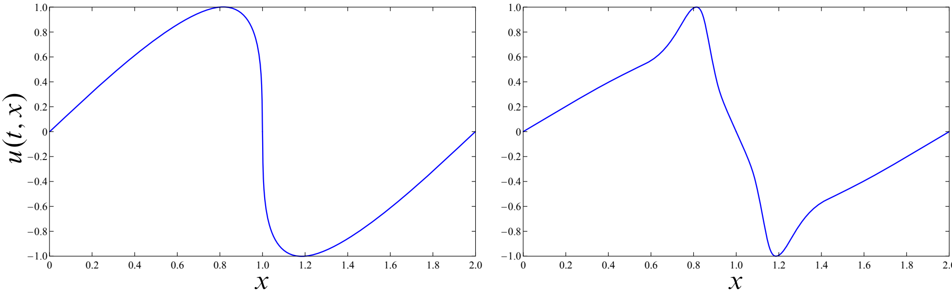

Figure 1. Local Burgers' equation with $h = 0$ (left) and nonlocal Burgers' equation with $h = L/8$ (right).

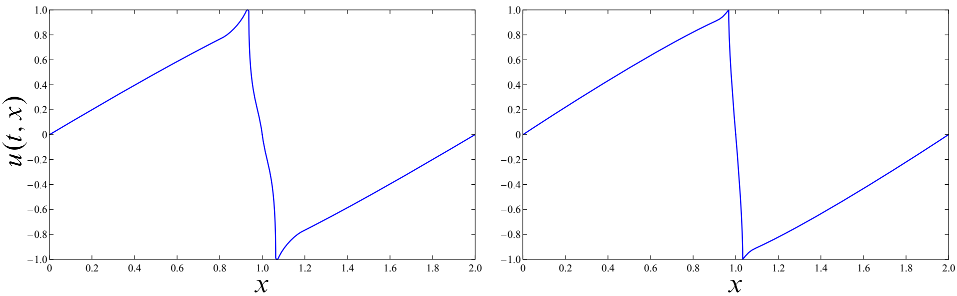

Figure 2. Nonlocal Burgers' equations with $h = L/16$ (left) and $h = L/32$ (right).

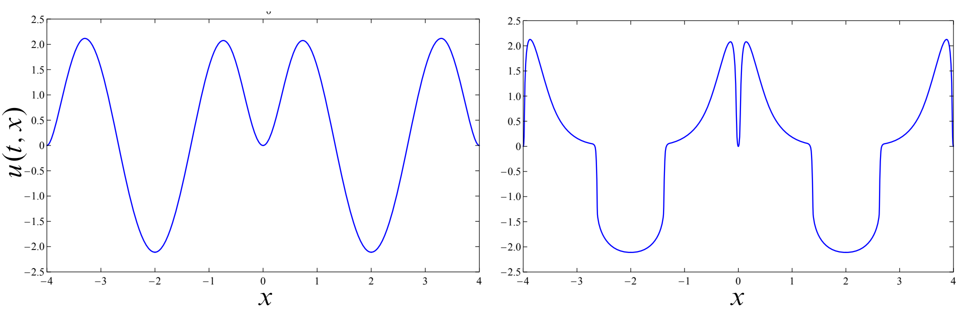

Figure 3. Nonlocal Burgers' equation minus case initial condition (left) and in finite time (right).

form instead of the usual one. We then varied the value of h to be $L/16$ and $L/32$ in Figure 2, which gives blow-up closer and closer to the origin.

Now we constructed initial data to fit Lemma 3.8 to get intuition on how it will blow up at $x = \pm L/3, \pm 2L/3$ in the minus sign case. Figure 3 (left) shows the initial data for our equation

$$u_t + \big(u(x+h,t) - u(x-h,t)\big)u_x = 0.$$

Note how $u(x, 0) = 0$ at $x = kh$, where period $L = 6h$. Now in Figure 3 (right), we see that at $x = \pm L/3, \pm 2L/3$, vertical lines form, causing blow-up in slope.

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