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Completions of reduced local rings with prescribed minimal prime ideals

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Let T be a complete local ring of Krull dimension at least one, and let $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m$ each be countable sets of prime ideals of T. We find necessary and sufficient conditions for T to be the completion of a reduced local ring A such that A has exactly m minimal prime ideals $Q_1, Q_2, \dots, Q_m$, and such that, for every $i = 1, 2, \dots, m$, the set of maximal elements of $\{P \in \operatorname{Spec}(T) \mid P \cap A = Q_i\}$ is the set $\mathcal{C}_i$.

1. Introduction and preliminaries

As rings in general have a poorly understood structure but complete local rings are fully characterized, the relationship between local rings and their completions is an important area of study. Instead of beginning with a ring and examining its completion, we work backwards. In other words, we ask the question: when is a complete local ring T the completion of a local subring A if some restriction is placed on A? Certain restrictions have produced answers to this question. Notably, Lech [1986] gives necessary and sufficient conditions for T to be the completion of a local integral domain, and Heitmann [1993] does the same when A is required to be a local unique factorization domain.

Charters and Loepp [2004] address the question when A is a local integral domain whose generic formal fiber has finitely many maximal elements. For this paper, we define the generic formal fiber of an integral domain A to be the set $\{P \in \operatorname{Spec}(T) \mid P \cap A = (0)\}$, where T is the completion of A with respect to its maximal ideal. Charters and Loepp show that for any complete local ring T with maximal ideal M and collection G of prime ideals of T, where G has finitely many maximal elements, there exists a local integral domain A whose completion is T and whose generic formal fiber is precisely G if and only if G contains only the zero ideal and T is a field, or the following conditions are true:

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(1) $M \notin G$, and $\mathrm{Ass}(T) \subseteq G$.

(2) If $P \in \mathrm{Spec}(T)$ and $Q \in G$ with $P \subseteq Q$, then $P \in G$.

(3) If $Q \in G$, then the intersection of Q with the prime subring of T is $\langle 0 \rangle$.

The techniques employed in [Charters and Loepp 2004] also apply when A is required to be an excellent ring. In particular, the authors show that their main theorem holds for A excellent if two conditions are added to the three listed above: (1) T is equidimensional, and (2) for any P that is maximal in G, T_P is a regular local ring.

Suppose A is a local ring, T is the completion of A with respect to its maximal ideal, and Q is a prime ideal of A. We define the formal fiber of A at Q to be the set $\{P \in \mathrm{Spec}(T) \mid P \cap A = Q\}$. If the number of maximal elements of this set is finite, then we say that the formal fiber of A at Q is semilocal. In [Arnosti et al. 2012], the authors generalize the work of Charters and Loepp to the case where A is reduced with semilocal formal fibers at each of its minimal prime ideals, showing that such a ring A exists if T contains the rationals. Furthermore, they allow for control over the minimal prime ideals of A as follows: let $\mathcal{C}$ be a finite collection of prime ideals of T, and let $\mathcal{P} = \{\mathcal{C}_1, \dots, \mathcal{C}_m\}$ be a partition of $\mathcal{C}$ that simultaneously partitions all of the associated prime ideals of T (called a *feasible partition*). As A is constructed so that for any $\mathcal{C}_i$ and $P, P' \in \mathcal{C}_i$, we have $P \cap A = P' \cap A$, it is sensible to write $\mathcal{C}_i \cap A$ to denote $P \cap A$ for any $P \in \mathcal{C}_i$. Then the set of minimal prime ideals of A is precisely $\{\mathcal{C}_1 \cap A, \dots, \mathcal{C}_m \cap A\}$. Moreover, for each i, the formal fiber of A at $\mathcal{C}_i \cap A$ has maximal elements exactly the elements of $\mathcal{C}_i$. Defining Q_i as the intersection of all minimal prime ideals contained within any $P \in \mathcal{C}_i$, Arnosti et al. also show that A can be made excellent whenever

(1) T is reduced;

(2) for each Q_i and each $P \in \mathcal{C}_i$, $(T/Q_i)_{\overline{P}}$ is a regular local ring;

(3) for each Q_i, we have T/Q_i is equidimensional.

The method employed in [Arnosti et al. 2012] resembles that of [Loepp 2003] and [Heitmann 1994]. To ensure that T is the completion of A, Arnosti et al. construct A step by step so that it satisfies the conditions on R stated below:

Proposition 1.1 [Heitmann 1994, Proposition 1]. *Let T be a complete local ring with maximal ideal M. Suppose R is a quasilocal subring of T, $R \cap M$ is the maximal ideal of R, the map $R \to T/M^2$ is onto, and $IT \cap R = I$ for every finitely generated ideal I of R. Then R is Noetherian and the natural homomorphism $\widehat{R} \to T$ is an isomorphism.*

They begin with $\mathbb{Q}$, which is a subring of T by assumption, and repeatedly adjoin elements of T to build up A until it satisfies the conditions of Proposition 1.1. In order for the minimal prime ideals of A to have the desired properties, several constraints are enforced on each intermediate subring, the most important of which is that for any $\mathcal{C}_i$, it is the case that each prime ideal in $\mathcal{C}_i$, and every associated prime ideal contained within a prime ideal in $\mathcal{C}_i$, intersects identically with A, and that for each i this intersection is distinct. A subring satisfying these constraints is called an *intersection-preserving subring*, or IP subring (see Definition 1.4).

In this paper, we extend the work of Arnosti et al. by weakening several restrictions on T. First, we permit the collection $\mathcal{C}$ of prime ideals of T to be countably infinite, where previously it was required to be finite. This is a relatively simple task, as only one step in their construction needs to be fixed, and a lemma of Aiello, Loepp, and Vu [Aiello et al. 2015] appropriately extends the one lemma used in [Arnosti et al. 2012] that assumes $\mathcal{C}$ is finite. Second, we show that T need not contain the rationals. However, for the main theorem to hold in this case, T must satisfy a set of new conditions, detailed in our main theorem (Theorem 2.14), which we state here:

Theorem 2.14. *Let T be a complete local ring of dimension at least one, M be the maximal ideal of T, and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be a feasible partition. Then T is the completion of a reduced local subring A such that* $\operatorname{Min}(A) = \{\mathcal{C}_1 \cap A, \dots, \mathcal{C}_m \cap A\}$ *and the formal fiber of A at each $\mathcal{C}_i \cap A$ has countably many maximal elements, which are precisely the elements of $\mathcal{C}_i$, if and only if T has either zero or prime characteristic and at least one of the following is true*:

(1) $\operatorname{char}(T) \neq 0$.

(2) $\operatorname{char}(T) = 0$ *and* $M \cap \mathbb{Z} = \langle 0 \rangle$.

(3) $\operatorname{char}(T) = 0$ *and, for all* $P \in \mathcal{C}$, *we have* $M \cap \mathbb{Z} \not\subseteq P$.

(4) $\operatorname{char}(T) = 0$, $M \cap \mathbb{Z} = \langle p \rangle$ *for some prime integer p, and the following three conditions hold*:

   (a) *For each $P \in \mathcal{C}$ and for each $Q \in \operatorname{Ass}(T)$ with $Q \subseteq P$, we have $p \in Q$ whenever $p \in P$.*

   (b) *For each subcollection $\mathcal{C}_i$ and for any $P, P' \in \mathcal{C}_i$, we have $p \in P$ if and only if $p \in P'$.*

   (c) *For each $Q \in \operatorname{Ass}(T)$, if $p \in Q$, then* $\operatorname{Ann}_T(p) \not\subseteq Q$.

Furthermore, when one of the above four conditions is true, if J is an ideal of T such that $J \not\subseteq P$ for every $P \in \mathcal{C}$, A can be constructed so that the natural map $A \to T/J$ is onto.

We additionally prove that these conditions are necessary for the desired subring A of T to exist. While the work of [Arnosti et al. 2012] focuses on the excellent case, ours does not, as we suspect that it is very difficult to construct A to be excellent unless T contains the rationals. Only two lemmas in [loc. cit.] depend on T containing the rationals: Lemma 3.7, which allows A to be built up while remaining an intersection-preserving subring, and Lemma 3.11, which establishes the existence of an initial IP subring of T. Our approach is therefore to extend these two lemmas; our main result, Theorem 2.14, then follows.

We assume throughout this paper that all rings are commutative rings with unity and that local rings are Noetherian. The term *quasilocal ring*, on the other hand, denotes a ring that has one maximal ideal but is not necessarily Noetherian. The notation (R, M) indicates a quasilocal ring R with maximal ideal M.

For any complete local ring T containing the rationals and set of prime ideals $\mathcal{C} \subseteq \operatorname{Spec}(T)$, Arnosti et al. obtain a high degree of control over the minimal prime ideals of the reduced local subring A whose completion is T and over the formal fibers of $\operatorname{Min}(A)$. To achieve this control, $\mathcal{C}$ must be partitioned in such a way that the set of associated prime ideals of T is partitioned as well. The following definition formalizes this notion; we change it slightly so that $\mathcal{C}$ is allowed to be countable.

Definition 1.2 [Arnosti et al. 2012, Definition 2.2]. Let T be a complete local ring. Let $\mathcal{C} = \{P_i\}_{i=1}^{\infty}$ be a countable collection of incomparable nonmaximal prime ideals of T, and let $\mathcal{C}$ be partitioned into $m \geq 2$ subcollections $\mathcal{C}_1, \dots, \mathcal{C}_m$. We call $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^{m})$ a *feasible partition on* $\mathcal{C}$ (or simply a *feasible partition*) if, for each Q in $\operatorname{Ass}(T)$, $\mathcal{P}$ satisfies the following conditions:

(1) $Q \subseteq P_i$ for at least one $P_i \in \mathcal{C}$.

(2) There exists exactly one ℓ such that whenever $Q \subseteq P_i$, we have $P_i \in \mathcal{C}_\ell$.

Example 1.3. Let

$$T = \frac{\mathbb{Q}[[x, y, z]]}{\langle xyz \rangle},$$

and let $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_1, \mathcal{C}_2\})$, where $\mathcal{C}_1 = \{\langle x \rangle\}$ and $\mathcal{C}_2 = \{\langle y, z \rangle\}$. Then $\mathcal{P}$ is a feasible partition because each element of $\operatorname{Ass}(T) = \{\langle x \rangle, \langle y \rangle, \langle z \rangle\}$ is a subset of some $P \in \mathcal{C}$ and no element of $\operatorname{Ass}(T)$ is contained within more than one subcollection $\mathcal{C}_i$.

Central to the construction of A in [Arnosti et al. 2012] is the concept of an *intersection-preserving subring*, or IP subring. Following the approach of Heitmann [1994] and Loepp [2003], Arnosti et al. establish the existence of an IP subring of T, and then adjoin elements of T to build the ring A. Since A is constructed according to the feasible partition $\mathcal{P}$, it is essential that, for any i, the intersection of A with any $P \in \mathcal{C}_i$, or any associated prime ideal contained in any such P, is the same. Moreover, Arnosti et al. ensure that for every prime ideal P of T not contained within any prime ideal in $\mathcal{C}$, $P \cap A$ does not consist only of zerodivisors, so that only those prime ideals of T in $\mathcal{C}$ may be in the formal fiber of A at any minimal prime ideal of A. These requirements inspire their definition of an IP subring. We reproduce this definition below, with the only significant alteration again being that $\mathcal{C}$ may be countable.

Definition 1.4 [Arnosti et al. 2012, Definition 2.6]. Let (T, M) be a complete local ring, $\mathcal{C}$ a countable set of incomparable nonmaximal prime ideals of T and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ a feasible partition on $\mathcal{C}$. A quasilocal subring $(R, M \cap R)$ of T is called an *intersection-preserving subring* (IP subring) if the following conditions hold:

(1) R is infinite.

(2) For any $P \in \mathcal{C}$, we have $P \cap R = Q \cap R$ for any $Q \in \operatorname{Ass}(T)$ satisfying $Q \subseteq P$.

(3) For $P, P' \in \mathcal{C}$, we have $P, P' \in \mathcal{C}_i$ if and only if $P \cap R = P' \cap R$.

(4) For each $P \in \mathcal{C}$, we have $r \in P \cap R$ implies $\operatorname{Ann}_T(r) \not\subseteq P$.

The ring R is called *small intersection preserving* (abbreviated SIP) if, additionally, $|R| < |T|$.

We note here that in [Arnosti et al. 2012, Definition 2.6], $\operatorname{Min}(T)$ is used in part (2) instead of $\operatorname{Ass}(T)$. Using $\operatorname{Ass}(T)$ simply helps us keep track of the zerodivisors of T, and it is a minor change in the definition.

The following result, based on [Lee et al. 2001, Lemma 5], implies that an IP subring is reduced. Consequently, since the ring we construct in our main theorem is an IP subring, we know it is reduced. In [Arnosti et al. 2012, Lemma 2.8], more conditions were assumed, but were not needed in their proof. So we state the result here with only the needed conditions.

Lemma 1.5 [Arnosti et al. 2012, Lemma 2.8]. *Let T be a ring, $\mathcal{C}$ be a countable set of incomparable nonmaximal prime ideals of T, and $\mathcal{P}$ be a feasible partition on $\mathcal{C}$. Let R be a subring of T such that, for each $P \in \mathcal{C}$, if $r \in P \cap R$, then $\operatorname{Ann}_T(r) \not\subseteq P$. Then R is reduced.*

Throughout their paper, Arnosti et al. use the assumptions stated in the following remark.

Remark 1.6. Let (T, M) be a complete local ring of dimension at least one which contains the rationals. Let $\mathcal{C}$ be a finite set of incomparable nonmaximal prime ideals of T. Let $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be a feasible partition, and let R be an IP subring of T. Let $P \in \mathcal{C}$; then $P \cap R$ is a prime ideal of R, and $P \in \mathcal{C}_i$ for some i. Abusing notation, we denote $P \cap R$ by $\mathcal{C}_i \cap R$. This abuse of notation makes sense because if $P, P' \in \mathcal{C}_i$, then $P \cap R = P' \cap R$.

Our assumptions closely follow those printed above, but with two substantial changes. First, $\mathcal{C}$ is permitted to be countable, not only finite. Second, we allow T to not contain the rationals, but add conditions that are necessary for the construction of A to be possible. If T contains the rationals, then every integer is a unit, so condition (2) of the following remark holds.

Remark 1.7. Hereafter, let (T, M) be a complete local ring of dimension at least one such that T has either zero or prime characteristic. Assume that at least one of the following is true:

(1) $\text{char}(T) \neq 0$.

(2) $\text{char}(T) = 0$ and $M \cap \mathbb{Z} = \langle 0 \rangle$.

(3) $\text{char}(T) = 0$ and, for all $P \in \mathcal{C}$, we have $M \cap \mathbb{Z} \not\subseteq P$.

(4) $\text{char}(T) = 0$, $M \cap \mathbb{Z} = p\mathbb{Z}$ for some prime p, and the following three conditions hold:
   - (a) For each $P \in \mathcal{C}$ and for every $Q \in \text{Ass}(T)$ with $Q \subseteq P$, we have $p \in Q$ whenever $p \in P$.
   - (b) For each subcollection $\mathcal{C}_i$ and any $P, P' \in \mathcal{C}_i$, we have $p \in P$ if and only if $p \in P'$.
   - (c) For each $Q \in \text{Ass}(T)$, if $p \in Q$, then $\text{Ann}_T(p) \not\subseteq Q$.

Let $\mathcal{C}$ be a countable set of incomparable nonmaximal prime ideals of T, and let $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be a feasible partition. Suppose R is an IP subring of T. If $P \in \mathcal{C}_i$, then we denote $P \cap R$ by $\mathcal{C}_i \cap R$, a reasonable abuse of notation as we have guaranteed $P \cap R = P' \cap R$ for any $P, P' \in \mathcal{C}_i$.

2. Results

In [Arnosti et al. 2012, Definition 2.2], the collection $\mathcal{C}$ of nonmaximal prime ideals of T is required to be finite. Only Lemma 3.5 of their paper, however, uses the fact that $\mathcal{C}$ is finite and not merely countable. We therefore make a small modification to this lemma using the result below. This result generalizes [Arnosti et al. 2012, Lemma 3.4] (and, in fact, generalizes the prime avoidance theorem for complete local rings).

Lemma 2.1 [Aiello et al. 2015, Lemma 2.7]. *Let (T, M) be a complete local ring such that $\dim(T) \geq 1$, let $\mathcal{C}$ be a countable set of incomparable nonmaximal prime ideals of T, and let D be a subset of T such that $|D| < |T|$. Let I be an ideal of T such that $I \not\subseteq P$ for all $P \in \mathcal{C}$. Then $I \not\subseteq \bigcup\{r + P \mid r \in D, P \in \mathcal{C}\}$.*

The following result (both the statement and the proof) is taken almost exactly from [Arnosti et al. 2012, Lemma 3.5]. The statement of the result, however, has two minor changes. First, we state and prove, with the use of Lemma 2.1, that $\mathcal{C}$ can be countable instead of just finite. Second, for [Arnosti et al. 2012, Lemma 3.5], it is assumed that T contains the rationals. As that is not a necessary assumption in the proof, we need not assume T contains the rationals for our Lemma 2.2.

Lemma 2.2. *Let (T, M) and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark 1.7. Let R be an infinite subring of T such that $|R| < |T|$. Let J be an ideal of T such that*

$J \not\subseteq P$ for every $P \in \mathcal{C}$. Let $t, q \in T$. Then there exists an element $t' \in J$ such that for every $P \in \mathcal{C}$ with $q \notin P$, we have that $t + qt' + P \in T/P$ is transcendental over $R/(P \cap R)$. If, in addition, $Q \in \operatorname{Ass}(T)$, $P \in \mathcal{C}$ with $Q \subseteq P$, $q \notin P$, and $P \cap R = Q \cap R$, then $t + qt' + Q \in T/Q$ is transcendental over $R/(Q \cap R)$.

Proof. Let $\mathcal{G} = \{P \in \mathcal{C} \mid q \notin P\}$. Then $\mathcal{G}$ is a countable set of incomparable nonmaximal prime ideals of T. Suppose that $t + qt' + P = t + qs' + P$ for some $P \in \mathcal{G}$ and some $t', s' \in T$. Then $(t + qt') - (t + qs') = q(t' - s') \in P$. But $q \notin P$, so $(t' - s') \in P$. These steps are reversible, so $t + qt' + P = t + qs' + P$ if and only if $t' + P = s' + P$.

For each $P \in \mathcal{G}$, let $D_{(P)}$ be a full set of coset representatives of the cosets $t' + P$ that make $t + qt' + P \in T/P$ algebraic over $R/(P \cap R)$. Let $D = \bigcup_{P \in \mathcal{G}} D_{(P)}$. Then $|D_{(P)}| = |R/(P \cap R)| \le |R| < |T|$ for every $P \in \mathcal{G}$, and noting that D is the countable union of sets with cardinality no greater than $|R|$, we have $|D| < |T|$. Now use Lemma 2.1 with $I = J$ and $\mathcal{C} = \mathcal{G}$ to conclude that there exists an element $t' \in J$ such that $t + qt' + P \in T/P$ is transcendental over $R/(P \cap R)$ for every $P \in \mathcal{G}$. Then we have that, for every $P \in \mathcal{C}$ with $q \notin P$, $t + qt' + P \in T/P$ is transcendental over $R/(P \cap R)$. Now suppose $Q \in \operatorname{Ass}(T)$, $P \in \mathcal{C}$ with $Q \subseteq P$, $q \notin P$, and $P \cap R = Q \cap R$. Then, $t + qt' + P \in T/P$ is transcendental over $R/(P \cap R)$. Since $P \cap R = Q \cap R$, we have $t + qt' + Q \in T/Q$ is transcendental over $R/(Q \cap R)$ as well. □

We now work to show that T need not contain the rationals. In [Arnosti et al. 2012], only Lemma 3.7, which allows the step-by-step construction of the IP subring A whose completion is T, and Lemma 3.11, which proves the existence of an initial IP subring of T, rely on T containing the rationals. Therefore, only these lemmas need to be modified in order for their results to hold when $\mathbb{Q} \not\subseteq T$. We handle these modifications in our Lemmas 2.8 and 2.13.

The following three technical lemmas are presented without proof.

Lemma 2.3 [Arnosti et al. 2012, Lemma 3.1]. *Let (T, M) and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark 1.7. Let B be a well-ordered index set and let R_β, $\beta \in B$ be a family of SIP subrings such that if $\beta, \gamma \in B$ such that $\beta < \gamma$, then $R_\beta \subseteq R_\gamma$. Then $R = \bigcup_{\beta \in B} R_\beta$ is an IP subring. Moreover, if there exists some $\lambda < |T|$ such that $|R_\beta| \le \lambda$ for all β, and $|B| < |T|$, then $|R| \le \max\{\lambda, |B|\}$, and R is an SIP subring.*

Lemma 2.4 [Arnosti et al. 2012, Lemma 3.2]. *Let (T, M) and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark 1.6. Let R be a subring of T satisfying all conditions for an IP subring except that it need not be quasilocal. Then $R_{(M \cap R)}$ is an IP subring of T with $|R_{(M \cap R)}| = |R|$. Additionally, if $|R| < |T|$, then $R_{(M \cap R)}$ is an SIP subring of T.*

Lemma 2.5 [Arnosti et al. 2012, Lemma 3.3]. *Let R be a subring of a complete local ring T. Let P_1, P_2 be prime ideals of T such that $P_1 \cap R = P_2 \cap R$. Suppose that, for $i = 1, 2$, we have that $u + P_i \in T/P_i$ is transcendental over $R/(P_i \cap R)$.*

Then $P_1 \cap R[u] = P_2 \cap R[u]$. *Furthermore, if* $\operatorname{Ann}_T(p) \not\subseteq P_1$ *for all* $p \in P_1 \cap R$, *then* $\operatorname{Ann}_T(p) \not\subseteq P_1$ *for all* $p \in P_1 \cap R[u]$.

To apply Proposition 1.1, we need to construct a subring A of T such that the map $A \to T/M^2$ is onto. The following lemma is a starting point for that. In addition, we will use it in the proof of Lemma 2.8. The statement and proof are taken almost exactly from [Arnosti et al. 2012, Corollary 3.6], but the proof is short and so we include it here.

Lemma 2.6 [Arnosti et al. 2012, Corollary 3.6]. *Let* (T, M) *and* $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ *be as in Remark 1.7, and let* J *be an ideal of* T *such that* $J \not\subseteq P$ *for every* $P \in \mathcal{C}$. *Let* R *be an SIP subring of* T *and* $t + J \in T/J$. *Then there exists an SIP subring* S *of* T *such that* $R \subseteq S \subset T$, $t + J$ *is in the image of the map* $S \to T/J$, *and* $|S| = |R|$. *Moreover, if* $t \in J$, *then* $S \cap J$ *contains a non-zerodivisor of* T.

Proof. Apply Lemma 2.2 with $q = 1$. Then $q \notin P$ for every $P \in \operatorname{Spec}(T)$, and so it is possible to choose $t' \in J$ such that $t + t' + P \in T/P$ is transcendental over $R/(R \cap P)$ for every $P \in \mathcal{C} \cup \operatorname{Ass}(T)$. Consider the ring $S = R[t + t']_{(M \cap R[t+t'])}$. By Lemma 2.5, $R[t + t']$ satisfies conditions (2), (3), and (4) of Definition 1.4. Further, $t + t' \in S$ and $(t + t') + J = t + J$, and so $t + J$ is in the image of the map $S \to T/J$.

Suppose $t \in J$ and $t + t'$ is a zerodivisor. Then $t + t' \in Q$ for some $Q \in \operatorname{Ass}(T)$. However, $Q \subseteq P$ for some $P \in \mathcal{C}$, and so $(t + t') + P = 0 + P$. Hence, $t + t' + P \in T/P$ is algebraic over $R/(R \cap P)$, a contradiction. Thus, $t + t'$ is a non-zerodivisor contained in $S \cap J$. □

Lemma 2.7 is an elementary result that will help us prove Lemma 2.8, a key lemma in this paper.

Lemma 2.7. *Let* R *be a ring and let* P *be a prime ideal of* R. *Let* $a, b \in R$, *and suppose* $a \notin P$. *If* ℓ *and* ℓ' *are units such that* $b + \ell a \in P$ *and* $b + \ell' a \in P$ *then* $\ell + P = \ell' + P$.

Proof. Suppose ℓ and ℓ' are units such that $b + \ell a \in P$ and $b + \ell' a \in P$. Then $(b + \ell a) - (b + \ell' a) = (\ell - \ell')a \in P$, and since $a \notin P$, we have $\ell - \ell' \in P$. That is, $\ell + P = \ell' + P$, completing the proof. □

The next lemma is analogous to [Arnosti et al. 2012, Lemma 3.7], which, given an SIP subring R, demonstrates the existence of an SIP subring S such that S contains R and, if I is a finitely generated ideal of R and c is an element of $IT \cap R$ then $c \in IS$. The proof is by induction on the number of generators of the ideal I of R. The only section requiring major modification for our result is the second part of the inductive step, in which it is proven that it is always possible to find generators for I satisfying the condition $(*)$ defined in the proof of the lemma. The portion of the proof of the lemma preceding the symbol $\diamondsuit$ is therefore quoted almost verbatim from the proof of [Arnosti et al. 2012, Lemma 3.7].

Lemma 2.8. *Let (T, M) and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark 1.7. Let R be an SIP subring of T. Then, for any finitely generated ideal I of R and any $c \in IT \cap R$, there exists a subring S of T with the following properties*:

(1) $R \subseteq S$.

(2) *S is an SIP subring of T.*

(3) $|S| = |R|$.

(4) $c \in IS$.

Proof. We shall proceed inductively on the number of generators of I. First suppose $I = aR$. If $a = 0$, then $S = R$ is the desired subring. Assume $a \neq 0$, and let $c = at$ for some $t \in T$. Note that, because $a \in R$, a is in some $P \in \mathcal{C}_i$ if and only if a is in every $P \in \mathcal{C}_i$. If this is the case, then, abusing notation, we shall refer to a as being contained in $\mathcal{C}_i$.

By condition (4) of Definition 1.4, $\operatorname{Ann}_T(a) \not\subseteq P$ for all $P \in \mathcal{C}$ such that $a \in P$. By Lemma 2.1 with $D = \{0\}$ and $I = \operatorname{Ann}_T(a)$, this means that $\operatorname{Ann}_T(a) \not\subseteq \bigcup_{a \in P, P \in \mathcal{C}} P$. Thus, we can choose some $q \in \operatorname{Ann}_T(a)$ such that $q \notin P$ for all $P \in \mathcal{C}$ such that $a \in P$. If $a \notin P$ for every $P \in \mathcal{C}$, we let $q = 0$. By Lemma 2.2, there exists some $t' \in T$ such that, for each $P \in \mathcal{C}$ with $a \in P$, the coset $t + qt' + P \in T/P$ is transcendental over $R/(P \cap R)$. Let $u = t + qt'$. We claim that $S = R[u]_{(R[u] \cap M)}$ is the desired subring. By Lemma 2.4, it suffices to show that $R[u]$ satisfies conditions (1), (2), (3), and (4) of being an SIP subring, and that $|R[u]| = |R|$. Condition (1) of Definition 1.4 follows immediately. We now show that condition (3) holds for $R[u]$.

For any $\mathcal{C}_i$ containing a, if $P, P' \in \mathcal{C}_i$, then $P \cap R[u] = P' \cap R[u]$ by Lemma 2.5. Next, consider any $\mathcal{C}_i$ not containing a. Let $P, P' \in \mathcal{C}_i$, and $f \in P \cap R[u]$. Then

$$f = r_n u^n + \cdots + r_1 u + r_0$$

for some $r_i \in R$. Multiplying both sides by a^n, we get

$$a^n f = r_n c^n + \cdots + a^{n-1} r_1 c + a^n r_0 \in P \cap R$$

since $au = at = c \in R$. Because R is an SIP subring, $a^n f \in P \cap R$ implies $a^n f \in P'$. However, by hypothesis, $a \notin P'$ and so f must be in P'. Consequently $f \in P' \cap R[u]$. The reverse inclusion follows by a similar argument, and so $P \cap R[u] = P' \cap R[u]$. Condition (2) of Definition 1.4 follows for $R[u]$ by a similar argument.

We will now show that condition (4) holds for $R[u]$. For each $P \in \mathcal{C}$, consider $f \in P \cap R[u]$, so that $f = r_n u^n + \cdots + r_1 u + r_0$. If $a \in P$, $u + P \in T/P$ is transcendental over $R/(P \cap R)$, so each r_i is in $P \cap R$. By assumption, for each r_i there exists a $q_i \notin P$ such that $r_i q_i = 0$. Let $q = \prod q_i \notin P$, and note that $fq = 0$. Thus, $\operatorname{Ann}_T(f) \not\subseteq P$. If $a \notin P$, recall that $a^n f \in P \cap R$. By assumption, there exists a $q \notin P$ such that $qa^n f = 0$. Note that $qa^n \notin P$, so $\operatorname{Ann}_T(f) \not\subseteq P$, and condition (4) holds. Hence, $R[u]_{(M\cap R[u])}$ is an SIP subring. Finally, observe that $|R[u]_{(M\cap R[u])}| = |R|$ and $c \in aR[u]_{(M\cap R[u])}$, as desired, so the lemma holds if I is generated by a single element.

Continuing inductively, suppose that the lemma holds when I is generated by $k-1$ elements, where $k \geq 2$. Let $I = \langle a_1, \dots, a_k\rangle R$ and $c = a_1t_1 + a_2t_2 + \cdots + a_kt_k \in R$ for some $t_i \in T$. We will first show that the lemma follows in the case where

$$\{\mathcal{C}_i \mid a_1 \in \mathcal{C}i\} = \{\mathcal{C}_j \mid a_2 \in \mathcal{C}_j\}. \tag{$*$}$$

We will then prove that it is always possible to define a generating set for I such that $(*)$ holds, completing the proof.

Assume that $(*)$ holds. Taking $a = a_1$, define q as in the principal case, and note that $a_1q = 0$. Thus, c can be rewritten as

$$c = a_1(t_1 + qt' + a_2t'') + a_2(t_2 - a_1t'') + a_3t_3 + \cdots + a_kt_k$$

for any $t', t'' \in T$. Let $u = t_1 + qt' + a_2t''$. We will choose t', t'' such that $u + P \in T/P$ is transcendental over $R/(P \cap R)$ for all $P \in \mathcal{C}$, allowing us to create an SIP subring $R[u]_{(M\cap R[u])}$.

Use Lemma 2.2 to find t' such that, for each $P \in \mathcal{C}$ with $q \notin P$, $t_1 + qt' + P \in T/P$ is transcendental over $R/(P \cap R)$. If $q \in P$ for all $P \in \mathcal{C}$, let $t' = 0$. By our choice of q and the assumption that $(*)$ holds, each $P \in \mathcal{C}$ contains precisely one of q and a_2. Thus, if $P \in \mathcal{C}$ is such that $q \notin P$, then

$$u + P = t_1 + qt' + a_2t'' + P = t_1 + qt' + P \in T/P$$

is transcendental over $R/(P \cap R)$ regardless of the choice of t''. Now, if $P \in \mathcal{C}$ is such that $q \in P$, then $a_2 \notin P$, and so we can use Lemma 2.2 to find $t'' \in T$ such that $t_1 + a_2t'' + P$ is transcendental over $R/(P \cap R)$ for all $P \in \mathcal{C}$ satisfying $a_2 \notin P$. If $a_2 \in P$ for all $P \in \mathcal{C}$, then let $t'' = 0$. By our choice of t' and t'', we have that $u + P$ is transcendental over $R/(P \cap R)$ for all $P \in \mathcal{C}$. By Lemma 2.5, $R[u]$ satisfies condition (3) of Definition 1.4. Using an identical argument to the principal case, $R[u]$ satisfies condition (4). It clearly satisfies conditions (1) and (2), and $|R[u]| = |R|$. By Lemma 2.4, $R' = R[u]_{(M\cap R[u])}$ is an SIP subring of T with $|R'| = |R|$.

Now let $J = \langle a_2, a_3, \dots, a_k\rangle R'$ and

$$c^* = c - a_1u = a_2(t_2 - a_1t'') + a_3t_3 + \cdots + a_kt_k.$$

We have $c \in R \subseteq R'$ and $a_1u \in R'$, so $c^* \in JT \cap R'$. By our inductive hypothesis, there exists an SIP subring S of T containing R' such that $c^* \in JS$, and so $c^* = a_2s_2 + \cdots + a_ks_k$ for some $s_i \in S$. It follows that $c = a_1u + a_2s_2 + \cdots + a_ks_k \in IS$, so S is the desired SIP subring.

$\diamond$ We will now show that, given a set of generators $\langle a_1, a_2, \dots, a_k\rangle$ for I, we can reduce to the case that I satisfies $(*)$.

We first use Lemma 2.6 with $J = M$ and $t = 0$ to find an SIP subring R_0 of T such that $R \subseteq R_0 \subset T$, $|R_0| = |R|$, and $R_0 \cap M$ contains a non-zerodivisor, which we call m_0, of T. By condition (2) of Definition 1.4, if $P \in \mathcal{C}$, then $R_0 \cap P$ contains only zerodivisors of T. It follows that for every i, $m_0 \notin \mathcal{C}_i \cap R_0$.

Next, for each $P \in \mathcal{C}$, let $D_{(P)}$ be a full set of coset representatives of the cosets $t + P \in T/P$ that are algebraic over $R_0/(R_0 \cap P)$. Let $D' = \bigcup_{P \in \mathcal{C}} D_{(P)}$. Use Lemma 2.1 with $I = M$ and $D = D' \cup \{m_0\}$ to find an element m_1 of M such that, for all $P \in \mathcal{C}$, we have $m_1 + P \neq m_0 + P$ and $m_1 + P \in T/P$ is transcendental over $R_0/(R_0 \cap P)$. If $Q \in \operatorname{Ass}(T)$, then $Q \subseteq P$ for some $P \in \mathcal{C}$. Since R_0 is an SIP subring, we have $P \cap R_0 = Q \cap R_0$. It follows that $m_1 + Q \in T/Q$ is transcendental over $R_0/(R_0 \cap Q)$. Let $R_1 = R_0[m_1]_{(R_0 \cap M)}$. By Lemmas 2.4 and 2.5, R_1 satisfies conditions (2), (3), and (4) of being an IP subring. Clearly R_1 is infinite, and $|R_1| = |R_0| < |T|$. Thus, R_1 is an SIP subring of T.

Now, repeat the above procedure with R_0 replaced by R_1 and D replaced by $D' \cup \{m_0, m_1\}$ to obtain an element m_2 of M and an SIP subring R_2 of T such that $R_1 \subseteq R_2$ and, for every $P \in \mathcal{C}$, we have $m_2 + P \neq m_0 + P$ and $m_2 + P \neq m_1 + P$. Continue so that for every $n \in \{1, 2, \ldots\}$, we find $m_n \in M$ and R_n such that $R_{n-1} \subseteq R_n$, $|R_n| = |R_{n-1}|$, R_n is an SIP subring of T, and, for every $P \in \mathcal{C}$ and every $i < n$, we have $m_n + P \neq m_i + P$. Let $R' = \bigcup_{i=1}^{\infty} R_i$. Then if $P \in \mathcal{C}$, we have $m_i + P = m_j + P$ if and only if $i = j$. In addition, by Lemma 2.3, R' is an SIP subring and $|R'| = |R|$. Since $m_0 \in R' \cap M$, and $m_0 \notin P$ for all $P \in \mathcal{C}$, we have $\mathcal{C}_i \cap R' \neq M \cap R'$ for all $i = 1, 2, \ldots, m$. Also note that, for every i, in the ring $R'/(\mathcal{C}_i \cap R')$, we have $m_k + (\mathcal{C}_i \cap R') = m_j + (\mathcal{C}_i \cap R')$ if and only if $k = j$. It follows that $(1 + m_k) + (\mathcal{C}_i \cap R') = (1 + m_j) + (\mathcal{C}_i \cap R')$ if and only if $k = j$.

Now, $m_0 \in M \cap R'$ is not a zerodivisor of T and $\mathcal{C}_i \cap R'$ only contains zerodivisors of T, and so $m_0 \notin \mathcal{C}_i \cap R'$ for every i. Since m_0 is a nonunit, $m_0 + 1$ is a unit. We will consider an ideal of R' of the form $\langle m_0 a_1 + u a_2, a_1 - u a_2, a_3, \ldots, a_k \rangle$, where u is a unit we will choose later so that $(*)$ holds. This ideal is equal to $\langle (m_0 + 1) a_1, (m_0 + 1) u a_2, a_3, \ldots, a_k \rangle R'$ and therefore also equal to IR'.

For each $\mathcal{C}_i$, we know that $\mathcal{C}_i \cap R'$ is a nonmaximal prime ideal of R'. Therefore, since neither m_0 nor u is in any $\mathcal{C}_i \cap R'$, we have that for each $\mathcal{C}_i$, $m_0 a_1 \in \mathcal{C}_i \cap R'$ if and only if $a_1 \in \mathcal{C}_i \cap R'$, and $u a_2 \in \mathcal{C}_i \cap R'$ if and only if $a_2 \in \mathcal{C}_i \cap R'$. It follows that if $a_1, a_2 \in \mathcal{C}_i \cap R'$, then $m_0 a_1 + u a_2, a_1 - u a_2 \in \mathcal{C}_i \cap R'$. On the other hand, if $a_1 \in \mathcal{C}_i \cap R'$ but $a_2 \notin \mathcal{C}_i \cap R'$, then $m_0 a_1 + u a_2, a_1 - u a_2 \notin \mathcal{C}_i \cap R'$. The same holds if $a_1 \notin \mathcal{C}_i \cap R'$ but $a_2 \in \mathcal{C}_i \cap R'$.

Finally, consider the case where $a_1, a_2 \notin \mathcal{C}_i \cap R'$. As $m_0 a_1 \notin \mathcal{C}_i \cap R'$, by Lemma 2.7, every unit $\ell \in R'$ such that $m_0 a_1 + \ell a_2 \in \mathcal{C}_i \cap R'$ is in the same coset of $R'/(\mathcal{C}_i \cap R')$. Similarly, every unit ℓ' such that $a_1 - \ell' a_2 \in \mathcal{C}_i \cap R'$ is in the same coset of $R'/(\mathcal{C}_i \cap R')$. For each i, let ℓ_{i_+} be a representative of the coset of $R'/(\mathcal{C}_i \cap R')$ containing all units ℓ such that $m_0 a_1 + \ell a_2 \in \mathcal{C}_i \cap R'$, and let ℓ_{i_-} be a representative of the coset containing all units ℓ' such that $a_1 - \ell' a_2 \in \mathcal{C}_i \cap R'$. Since $L = \bigcup_{i=1}^{m} \{\ell_{i_+}, \ell_{i_i}\}$, is a finite set of elements of R' and $\mathcal{G} = \{\mathcal{C}_i \cap R'\}_{i=1}^{m}$ is a finite set of prime ideals of R', the set $\{\ell + P \mid \ell \in L,\, P \in \mathcal{G}\}$ is a finite set. Suppose for some $r \neq k$, $\ell \in L$, and $P \in \mathcal{G}$, we have $m_r + 1 \in \ell + P$ and $m_k + 1 \in \ell + P$. Then $m_r + P = m_k + P$, a contradiction. As the set $\{m_i + 1\}_{i=1}^{\infty}$ is infinite, there must be a positive integer r such that $m_r + 1 \notin \ell + P$ for all $\ell \in L$ and all $P \in \mathcal{G}$. So there exists a unit $u = m_r + 1 \in R'$ such that $u \notin \bigcup_{\ell \in L, P \in \mathcal{G}} \{\ell + P\}$. It follows that, for all i, we have $m_0 a_1 + u a_2 \notin \mathcal{C}_i \cap R'$ and $a_1 - u a_2 \notin \mathcal{C}_i \cap R'$.

We have now shown that, for every i, either $m_0 a_1 + u a_2$ and $a_1 - u a_2$ are both in $\mathcal{C}_i \cap R'$ or $m_0 a_1 + u a_2$ and $a_1 - u a_2$ are both not in $\mathcal{C}_i \cap R'$. Thus

$$\{\mathcal{C}_i \mid m_0 a_1 + u a_2 \in \mathcal{C}_i\} = \{\mathcal{C}_j \mid a_1 - u a_2 \in \mathcal{C}_j\},$$

and so we have $(*)$, and the previous argument applies with R replaced by R' and I replaced by IR'. Hence we can find an SIP subring S of T containing R' so that $c \in (IR')S = IS$. □

We use the following lemma to construct our ring to satisfy the hypotheses of Proposition 1.1.

Lemma 2.9 [Arnosti et al. 2012, Lemma 3.8]. *Let (T, M) and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^{m})$ be as in Remark 1.7. Let J be an ideal of T such that $J \not\subseteq P$ for all $P \in \mathcal{C}$, and let $u + J \in T/J$. Suppose R is an SIP subring of T. Then there exists an SIP subring S of T with the following properties*:

(1) $R \subseteq S$.

(2) *If $u \in J$, then $S \cap J$ contains a non-zerodivisor.*

(3) *$u + J$ is in the image of the map $S \to T/J$.*

(4) $|S| = |R|$.

(5) *For every finitely generated ideal I of S, we have $IT \cap S = I$.*

Proof. The proof follows from the proof of [Arnosti et al. 2012, Lemma 3.8], using our Lemma 2.8 where they used their Lemma 3.7. □

Lemma 2.13 demonstrates the existence of an SIP subring of T that serves as the starting point for the construction of A. Before building an SIP subring, we first find a ring in which every condition of the definition of an SIP subring is satisfied, except that each $\mathcal{C}_i \cap R$ need not be distinct. Such a ring is called *semi-SIP* and is formally defined below. Lemma 2.12 establishes a semi-SIP subring of T, making use of Lemma 2.11 in the case where T has prime characteristic.

Definition 2.10 [Arnosti et al. 2012, Definition 3.9]. Let (T, M) and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^{m})$ be as in Remark 1.7. We say that a quasilocal subring $(R, M \cap R)$ is a *semi-SIP subring of T* if the following conditions hold:

(1) R is infinite.

(2) For any $P \in \mathcal{C}$, we have $P \cap R = Q \cap R$ for any $Q \in \operatorname{Ass}(T)$ satisfying $Q \subseteq P$.

(3) For any $\mathcal{C}_i$, if $P, P' \in \mathcal{C}_i$, then $P \cap R = P' \cap R$.

(4) For each $P \in \mathcal{C}$, we have $r \in P \cap R$ implies $\operatorname{Ann}_T(r) \nsubseteq P$.

(5) $|R| < |T|$.

The following lemma is based on [Arnosti et al. 2012, Lemma 3.10]. Since the statement of that lemma requires a semi-SIP subring R of T but their proof does not use the fact that R is infinite, we weaken their requirement accordingly, and our lemma holds using their original proof. It is important to note that in their proof, the ring $R[up_i]_{(R[up_i] \cap M)}$ is infinite regardless of whether or not R is infinite.

Lemma 2.11 [Arnosti et al. 2012, Lemma 3.10]. *Let (T, M) and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark 1.7, and fix $\mathcal{C}_i$. Let R be a semi-SIP subring of T except that R need not be infinite, and let $p_i \in T$ be given such that $p_i \in Q$ for every minimal prime ideal Q contained within some $P \in \mathcal{C}_i$, but $p_i \notin P$ for any $P \in \mathcal{C}_j$, where $j \neq i$. Suppose further that $\operatorname{Ann}_T(p_i) \nsubseteq P$ for any $P \in \mathcal{C}_i$. Then there exists a unit u in T such that $R[up_i]_{(R[up_i] \cap M)}$ is a semi-SIP subring of T.*

In the proof of [Arnosti et al. 2012, Lemma 3.10], the authors define $S = R[p_i]$, and they note that, in their case, $|S| = |R| < |T|$. If R is finite, then the equality $|S| = |R|$ may not hold, but S is at most countable. Since complete local rings of positive dimension have cardinality greater than or equal to the cardinality of the real numbers (by, for example, [Charters and Loepp 2004, Lemma 2.3]), the inequality $|S| < |T|$ still holds.

Lemma 2.12. *Let (T, M) and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark 1.7. Then there exists a semi-SIP subring of T.*

Proof. By the assumptions of Remark 1.7, the characteristic of T is either 0 or some prime $\hat{p}$. In both cases, we will construct a semi-SIP subring of T.

Characteristic 0: First assume that the characteristic of T is zero. We examine three subcases, each assuming that only one of conditions (2), (3), and (4) of Remark 1.7 hold.

To begin, assume only condition (2) of Remark 1.7 is true: that is, $M \cap \mathbb{Z} = \langle 0 \rangle$. Since M contains no integers, every integer in T is a unit, so T contains the rationals. By [Arnosti et al. 2012, Lemma 3.11], there exists a semi-SIP subring of T.

Now assume that only condition (3) holds. Here $M \cap \mathbb{Z} \neq \langle 0 \rangle$, and so $M \cap \mathbb{Z} = p\mathbb{Z}$ for some prime integer p, but p is not in any of the prime ideals $P \in \mathcal{C}$. Choose any $P \in \mathcal{C}$. Since $p \notin P$, we have $P \cap \mathbb{Z} \neq p\mathbb{Z}$, and as $P \cap \mathbb{Z} \subseteq M \cap \mathbb{Z}$, we have $P \cap \mathbb{Z} = \langle 0 \rangle$. Let $R_0 = \mathbb{Z}_{(p\mathbb{Z})}$, so that R_0 is a local subring of T. Then $P \cap R_0 = \langle 0 \rangle$. Furthermore, for any $Q \in \operatorname{Ass}(T)$, we know that $Q \subseteq P$ for some

$P \in \mathcal{C}$, and so $Q \cap R_0 = \langle 0 \rangle$. Conditions (2), (3), and (4) of Definition 2.10 follow easily from these results. As R_0 is countably infinite and T has cardinality greater than or equal to the cardinality of the real numbers (by [Charters and Loepp 2004, Lemma 2.3]), $|R_0| < |T|$ and so R_0 is a semi-SIP subring of T.

Finally, assume that only condition (4) of Remark 1.7 holds. Again, given $M \cap \mathbb{Z} = p\mathbb{Z}$, let $R_0 = \mathbb{Z}_{(p\mathbb{Z})}$, so that R_0 is an infinite local subring of T with $|R_0| < |T|$. For any prime ideal P of T, $P \cap \mathbb{Z}$ is either $p\mathbb{Z}$, if $p \in P$, or $\langle 0 \rangle$, if $p \notin P$. Conditions (2) and (3) of Definition 2.10 follow respectively from conditions (4a) and (4b) of Remark 1.7. It remains to show that condition (4c) of Remark 1.7 implies condition (4) of Definition 2.10.

Recall that condition (4c) of Remark 1.7 ensures that for each $Q \in \operatorname{Ass}(T)$, if $p \in Q$, then $\operatorname{Ann}_T(p) \not\subseteq Q$. For contradiction, suppose that condition (4) of Definition 2.10 does not hold, so that there exists some $P \in \mathcal{C}$ and $r \in P \cap R_0$ with $\operatorname{Ann}_T(r) \subseteq P$. It must be that $p \in P$; if $p \notin P$, then $P \cap R_0 = \langle 0 \rangle$ and clearly $\operatorname{Ann}_T(0) \not\subseteq P$. As $P \cap R_0 = p\mathbb{Z}$ and $\operatorname{Ann}_T(p) \subseteq \operatorname{Ann}_T(kp)$ for any integer k, we may assume that $r = p$ and $\operatorname{Ann}_T(p) \subseteq P$. The set of zerodivisors of T_P is equal to $\bigcup\{QT_P : Q \in \operatorname{Ass}(T), Q \subseteq P\}$. Since $\operatorname{Ann}_{T_P}(p)$ consists entirely of zerodivisors, by the prime avoidance theorem, $\operatorname{Ann}_{T_P}(p) \subseteq QT_P$ for some $Q \in \operatorname{Ass}(T)$ with $Q \subseteq P$. Choose any $a \in \operatorname{Ann}_T(p)$, so that $ap = 0$. In T_P, we have $(a/1)(p/1) = 0/1$; thus $a/1 \in \operatorname{Ann}_{T_P}(p)$ and by above $a/1 \in QT_P$. It follows that $a \in Q$ and so $\operatorname{Ann}_T(p) \subseteq Q$. By condition (4c) of Remark 1.7, $p \notin Q$, but by condition (4a) of Remark 1.7, because $p \in P$, we have $p \in Q$, a contradiction. Therefore, condition (4) of Definition 2.10 holds, and R_0 is a semi-SIP subring of T.

Characteristic $\hat{p}$: We now assume that the characteristic of T is $\hat{p}$ for some prime integer $\hat{p}$. In this case, let $R_0 = \mathbb{Z}_{\hat{p}}$. Since R_0 is a field, no prime ideal of T contains a nonzero element of R_0. Therefore,

$$P \cap R_0 = \langle 0 \rangle = P' \cap R_0 = Q \cap R_0$$

for any $P, P' \in \mathcal{C}$ and $Q \in \operatorname{Ass}(T)$. Furthermore, it is trivially true that $\operatorname{Ann}_T(r) \not\subseteq P$ for any $P \in \mathcal{C}$ and $r \in P \cap R_0$, and clearly $|R_0| < |T|$. Thus R_0 satisfies every condition of being semi-SIP except for being infinite. By adjoining a carefully chosen element to R_0, we will create a ring that is semi-SIP. We will choose this element using the following method from [Arnosti et al. 2012, Lemma 3.11].

Let $\operatorname{Min}(T) = \{Q_1, \dots, Q_n\}$. For each minimal prime ideal Q_i, by Lemma 2.1 we can pick some $q_i \in Q_i$ such that $q_i \notin \bigcup\{P \in \mathcal{C} \mid Q_i \not\subseteq P\}$. Let $q = \prod_{i=1}^n q_i$. As q is nilpotent, we may choose ℓ to be the smallest positive integer such that $q^\ell = 0$. Choose any $\mathcal{C}_k$, let $\mathcal{E}_k$ be the set of minimal prime ideals of T contained within $\mathcal{C}_k$, and let $p_k = \prod_{Q_i \in \mathcal{E}_k} q_i^\ell$ and $s_k = \prod_{Q_i \notin \mathcal{E}_k} q_i^\ell$. As $p_k s_k = \prod_{i=1}^n q_i^\ell = \left(\prod_{i=1}^n q_i\right)^\ell = q^\ell = 0$, we have $\operatorname{Ann}_T(p_k) \not\subseteq P$ for every $P \in \mathcal{C}_k$.

Apply Lemma 2.11, setting $R = R_0$ and $p_i = p_k$ as chosen above, to find a unit $u \in T$ such that $R_0[up_k]_{(R_0[up_k] \cap M)}$ is the desired semi-SIP subring. □

Lemma 2.13. *Let (T, M) and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark 1.7. Then there exists an SIP subring of T. On the other hand, if* char(T) *is neither zero nor a prime p, or if none of the conditions in Remark 1.7 holds, then there does not exist an IP subring of T whose completion is T.*

Proof. By Lemma 2.12, there exists a semi-SIP subring of T, which we call R_0. Now use the proof of [Arnosti et al. 2012, Lemma 3.11], in which elements $p_1, \dots, p_m$ are adjoined to R_0 in such a way that the resulting ring is an SIP subring of T. The process is similar to the method we used in the characteristic $\hat{p}$ part of the proof of Lemma 2.12. So if (T, M) and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ are as in Remark 1.7, then there exists an SIP subring of T.

Next, we show that the characteristic of T must be either zero or prime in order to construct an IP subring. We claim that if this condition does not hold, then char(T) must be a prime power p^k, where $k > 1$. Suppose otherwise, so that R has nonzero characteristic n, where n can be written as $n = ab$ for some relatively prime integers $1 < a, b < n$. Then a and b are zerodivisors and consequently nonunits, and we have $a, b \in M$. Therefore $\langle a, b\rangle \subseteq M$. By Bezout's identity, there exist integers r and s such that $ra + sb = 1$, implying $1 \in M$, a contradiction since M consists only of nonunits. Therefore a and b cannot be relatively prime, and char(T) must be a prime power. This implies that the prime subring of T is $\mathbb{Z}_{p^k}$ for some prime p and $k > 1$. Then, however, p is nilpotent in every subring of T, so no subring of T is reduced. As every IP subring is reduced, T has no IP subring.

Second, to prove that for the construction of an IP subring it is necessary to have at least one of the four conditions in Remark 1.7, we will show that if conditions (1), (2), and (3) do not hold and any one of conditions (4a), (4b), and (4c) does not hold, then an IP subring cannot exist. As A must be an IP subring in order for us to control its minimal prime ideals, the conditions of Remark 1.7 are necessary for Theorem 2.14.

Suppose that an IP subring R of T exists when conditions (1), (2), (3), and (4a) fail. Thus char$(T) = 0$, $M \cap \mathbb{Z} = p\mathbb{Z}$ for some prime p, and there exists some $P \in \mathcal{C}$ and some $Q \in \text{Ass}(T)$ contained in P such that $p \in P$ but $p \notin Q$. By condition (2) of Definition 1.4, $P \cap R = Q \cap R$, so $p \notin R$, but this is impossible as any subring of T must contain the integers. Thus condition (4a) of Remark 1.7 is necessary in the absence of conditions (1), (2), and (3). Now suppose that there exists an IP subring R of T and conditions (1), (2), (3), and (4b) of Remark 1.7 do not hold, so $P \cap \mathbb{Z} \neq P' \cap \mathbb{Z}$ for some $\mathcal{C}_i$ and $P, P' \in \mathcal{C}_i$. As R contains the integers, $P \cap R \neq P' \cap R$, but this contradicts condition (3) of Definition 1.4, and we conclude that condition (4b) of Remark 1.7 is also necessary if we do not have conditions

(1), (2), and (3). Finally, suppose condition (4c) of Remark 1.7 fails, so for some $Q \in \operatorname{Ass}(T)$, $p \in Q$ but $\operatorname{Ann}_T(p) \subseteq Q$ (where again $M \cap \mathbb{Z} = p\mathbb{Z}$). Then, by [Lee et al. 2001, Theorem 1], T is not the completion of a reduced local subring. □

The following theorem extends the main result, Theorem 3.12, of [Arnosti et al. 2012].

Theorem 2.14. *Let (T, M) be a complete local ring of dimension at least one and let $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be a feasible partition. Then T is the completion of a reduced local subring A such that $\operatorname{Min}(A) = \{\mathcal{C}_1 \cap A, \dots, \mathcal{C}_m \cap A\}$ and the formal fiber of A at each $\mathcal{C}_i \cap A$ has countably many maximal elements, which are precisely the elements of $\mathcal{C}_i$, if and only if T has either zero or prime characteristic and at least one of the following is true*:

(1) $\operatorname{char}(T) \neq 0$.

(2) $\operatorname{char}(T) = 0$ *and* $M \cap \mathbb{Z} = \langle 0 \rangle$.

(3) $\operatorname{char}(T) = 0$ *and for all* $P \in \mathcal{C}$, $M \cap \mathbb{Z} \not\subseteq P$.

(4) $\operatorname{char}(T) = 0$, *we have* $M \cap \mathbb{Z} = \langle p \rangle$ *for some prime* p, *and the following three conditions hold*:

   (a) *For each* $P \in \mathcal{C}$ *and for each* $Q \in \operatorname{Ass}(T)$ *with* $Q \subseteq P$, *we have* $p \in Q$ *whenever* $p \in P$.

   (b) *For each subcollection* $\mathcal{C}_i$ *and for any* $P, P' \in \mathcal{C}_i$, *we have* $p \in P$ *if and only if* $p \in P'$.

   (c) *For each* $Q \in \operatorname{Ass}(T)$, *if* $p \in Q$, *then* $\operatorname{Ann}_T(p) \not\subseteq Q$.

Furthermore, when one of these conditions is true, if J is an ideal of T such that $J \not\subseteq P$ for every $P \in \mathcal{C}$, then the natural map $A \to T/J$ is onto.

Proof. The proof is taken almost exactly from the proof of [Arnosti et al. 2012, Theorem 3.12]. Let

$$\Omega = \{u + J \mid u \in T, J \not\subseteq P \text{ for all } P \in \mathcal{C}\}$$

equipped with a well-ordering $<$, such that every element has strictly fewer than $|\Omega|$ predecessors. Note that

$$\left|\{J \mid J \text{ is an ideal of } T \text{ with } J \not\subseteq P \text{ for every } P \in \mathcal{C}\}\right| \leq |T|.$$

For each $\alpha \in \Omega$, we let $|\alpha| = |\{\beta \in \Omega \mid \beta \leq \alpha\}|$, by abuse of notation.

Let 0 denote the first element of Ω, and let R_0 be the SIP subring of T constructed in Lemma 2.13.

For each $\lambda \in \Omega$ after the first, we define R_λ recursively as follows: assume R_β is defined for all $\beta < \lambda$ such that R_β is an SIP subring, and $|R_\beta| \leq |\beta||R_0|$ for all $\beta < \lambda$. Let $\gamma(\lambda) = u + J$ denote the least upper bound of the set of predecessors of λ. If $\gamma(\lambda) < \lambda$, we use Lemma 2.9 with $R = R_{\gamma(\lambda)}$ to find an SIP subring R_λ such that

(1) $R_{\gamma(\lambda)} \subseteq R_\lambda \subseteq T$;

(2) if $u \in J$, then $J \cap R_\lambda$ contains a non-zerodivisor:

(3) the coset $u + J$ is in the image of the map $R_\lambda \to T/J$;

(4) for all finitely generated ideals I of R_λ, $IT \cap R_\lambda = I$.

In this case,

$$|R_\lambda| = |R_{\gamma(\lambda)}| \le |\gamma(\lambda)||R_0| \\ \le |\lambda||R_0|.$$

On the other hand, if $\gamma(\lambda) = \lambda$, we let $R_\lambda = \bigcup_{\beta<\lambda} R_\beta$. We note that for all $\beta < \lambda$,

$$|R_\beta| \le |\beta||R_0| \\ \le |\lambda||R_0| \\ < |T|.$$

The last inequality holds since $|\lambda| < |\Omega| = |T|$ and $|R_0| < |T|$. By Lemma 2.3, it follows that R_λ is an SIP subring of T and $|R_\lambda| \le |\lambda||R_0|$.

Let $A' = \bigcup_{\alpha\in\Omega} R_\alpha$, and define $A = A'_{(A'\cap M)}$. Then A is an IP subring of T.

Note that $M^2 \not\subseteq P$ for every $P \in \mathcal{C}$ so, since every $\alpha = u + M^2 \in \Omega$ has a successor λ (where $\gamma(\lambda) = \alpha$), our construction guarantees that α is in the image of $A \to T/M^2$. Hence this map is onto. Next, let $I = (a_1, \dots, a_n)A$ be a finitely generated ideal of A and $c \in IT \cap A$. Then for some $\delta \in \Omega$, with $\gamma(\delta) < \delta$, $\{c, a_1, \dots, a_n\} \subset R_\delta$. It follows that $c \in IT \cap R_\delta = IR_\delta \subseteq I$. Hence $IT \cap A = I$ for all finitely generated ideals I of A. Since A is a quasilocal subring of T, Proposition 1.1 implies that A is Noetherian and $\widehat{A} = T$.

Now, since T is faithfully flat over A, the ideals $\mathcal{C}_i \cap A$ are the minimal prime ideals of A, so that $\mathrm{Min}(A)$ has m elements. Furthermore, by our construction, the natural map $A \to T/J$ is onto for any ideal J such that $J \not\subseteq P$ for all $P \in \mathcal{C}$. By construction, the formal fiber of $\mathcal{C}_i \cap A$ has maximal ideals precisely the elements of $\mathcal{C}_i$.

By Lemma 2.13, unless the conditions of the above theorem are satisfied, no IP subring of T exists whose completion is T. As A must be an IP subring in order for us to control its minimal prime ideals, if the conditions do not hold, then neither does this theorem. □

Arnosti et al. [2012] note that, for their ring A, not only are the formal fibers of the minimal prime ideals of A known, but so are the formal fibers at *all* prime ideals of A. The same holds for our A of Theorem 2.14. Specifically, let A be as in Theorem 2.14, and suppose that P is any nonminimal prime ideal of A. Then by the theorem, $A \to T/PT$ is an onto map. As T is the completion of A, T is a faithfully flat extension of A and so $PT \cap A = P$. It follows that $A/P \cong T/PT$, and so the only element in the formal fiber of A at P is PT.

To demonstrate that this result is not trivial, we examine a complete ring T for which this theorem, but not previous work, allows us to find the desired subring A.

Example 2.15. Let

$$T = \frac{\widehat{\mathbb{Z}_{\langle 5\rangle}}[[x,y,z]]}{\langle 5xy\rangle},$$

and let $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_1, \mathcal{C}_2\})$, where $\mathcal{C}_1 = \{\langle 5\rangle\}$ and $\mathcal{C}_2 = \{\langle x, y\rangle\}$. Then T is a complete local ring with dimension at least one and characteristic zero, and $\mathcal{P}$ is a feasible partition. Furthermore, T satisfies every subcondition of condition (4) of Theorem 2.14. Therefore, by that theorem, there exists a reduced local subring A of T having the desired properties. As T does not contain the rationals and is not of one of the previously characterized categories of complete rings, Theorem 2.14 is necessary to establish the existence of A.

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