involve
a journal of mathematics

Global regularity of chemotaxis equations with advection

Saad Khan, Jay Johnson, Elliot Cartee and Yao Yao

msp

2016 vol. 9, no. 1

INVOLVE 9:1 (2016)
dx.doi.org/10.2140/involve.2016.9.119

Global regularity of chemotaxis equations with advection

Saad Khan, Jay Johnson, Elliot Cartee and Yao Yao

(Communicated by Behrouz Emamizadeh)

We study the Patlak–Keller–Segel (PKS) equations in 2D that describe chemotaxis with an additional advection term. We show that solutions are globally regular for smooth initial data with subcritical mass as long as the flow has nonpositive divergence. For initial data with supercritical mass, numerical simulations suggest that blow-up might be prevented by imposing some strong incompressible advection term.

1. Introduction

In this paper, we study the effect of adding an advective flow term in the Patlak–Keller–Segel (PKS) equations that model chemotaxis. Chemotaxis is the means by which small organisms such as bacteria and somatic cells direct their movements towards or against the gradient of some chemical concentration.

For such organisms, which often swim in low Reynold's numbers settings, movement is a huge challenge for many reasons. The organism's small size relative to the fluid it inhabits means that it has to overcome the effects of diffusion. Furthermore, organisms at this level typically do not have any sort of neural system, and thus cannot process the large amounts of information that would be needed to purposefully move from one place to another. The available information is also limited to whatever chemicals can bind to ligands on the cell membranes. For these reasons, chemotaxis is a very simple mechanism. The organism simply measures the gradient of a relevant chemical concentration (such as glucose, cAMP, bicoid, etc.) and moves in that direction. Its motion can be modeled by combining the diffusive and chemotactic components of its trajectory into an evolution equation

$$\rho_t = \Delta\rho + \nabla\cdot(\rho\nabla c), \tag{1-1}$$

MSC2010: 35A01.

Keywords: chemotaxis, partial differential equations, PDE, symmetric decreasing rearrangement, advection, Keller–Segel.

where ρ is the density of the organism and c represents the concentration of the chemical. The PKS equation arises by assuming that the chemical is produced by the organism itself. In this case, the chemical diffuses like

$$\epsilon c_t - \Delta c = \rho.$$

In the physically relevant regime, the diffusion of the chemical is much faster than the diffusion of the organism. Hence the $\epsilon \to 0$ limit is often taken, and the equation above becomes

$$-\Delta c = \rho. \tag{1-2}$$

The PKS equation [Patlak 1953; Keller and Segel 1970; 1971] is then obtained by combining (1-1) and (1-2):

$$\rho_t = \Delta \rho + \nabla \cdot (\rho \nabla (\mathcal{N} * \rho)), \tag{1-3}$$

where $*$ denotes convolution and $\mathcal{N}(x) = 1/(2\pi) \log |x|$ is the Newtonian potential in 2D.

This equation has been studied extensively (see [Horstmann 2003] and the references therein). In particular, it is well known that solutions with different mass sizes exhibit different behaviors: For nonnegative initial data with L^1-norm greater than 8π (i.e., solutions with *supercritical mass*), solutions blow up in finite time [Patlak 1953; Perthame 2007]. On the other hand, if the L^1-norm is less than 8π (i.e., *subcritical mass*), the diffusive term dominates the dynamics of the equation, where the solutions are globally regular, and indeed the L^∞-norm goes to 0 as $t \to \infty$.

In this paper, we incorporate an extra advection term into the PKS equation, which then becomes

$$\rho_t = \Delta \rho + \nabla \cdot (\rho \nabla (\mathcal{N} * \rho)) - \nabla \cdot (\rho \vec{u}). \tag{1-4}$$

The motivation of this extra term is that the fluid medium the organisms inhabit may have its own current, which we denote by $\vec{u}$. Throughout this paper, we assume that the underlying flow $\vec{u}(x, y)$ is an a priori given velocity field and does not depend on ρ. We are particularly interested in the case where the flow $\vec{u}$ is incompressible, since this is the physically relevant case. The goal of this paper is to investigate the effect of the advection term on the behavior of the solution. In particular, we would like to answer the following questions:

(1) For smooth initial data with subcritical mass (i.e., $\|\rho_0\|_{L^1} < 8\pi$) and any incompressible flow $\vec{u}$, does the solution to (1-4) always have global regularity?

(2) For initial data with supercritical mass (i.e., $\|\rho_0\|_{L^1} > 8\pi$), is it possible to prevent a finite-time blow-up by imposing a strong incompressible advection term in (1-4)?

In Section 2, we give a positive answer to the first question. More precisely, we prove that as long as the velocity field $\vec{u}$ has nonpositive divergence (which

includes incompressible velocity fields as a special case), any solution to (1-4) with subcritical mass remains regular for all time. The main ingredients of our proof are a comparison principle and symmetric decreasing rearrangements.

As an attempt to address the second question, we perform some numerical simulations in Section 3. Using a stochastic particle simulation based on [Haškovec and Schmeiser 2009], we experiment with a variety of flows on solutions with supercritical mass. Our numerical results suggest that the answer might be positive. Namely, some incompressible flows, such as the shear flow and the strain flow, seem to be good candidates for preventing solutions with supercritical mass from blowing up, as long as the flow strength is sufficiently large. A rigorous proof of this phenomenon remains a very interesting open question.

2. Global regularity for solutions with subcritical mass

In this section, we consider velocity fields $\vec{u}$ with nonpositive divergence, and our goal is to prove global regularity for (1-4) with subcritical initial data.

Radially symmetric case. We first deal with the PKS equation without advection (1-3), and we prove some results for radially symmetric solutions. Although these results are known (e.g., see [Perthame 2007]), we sketch their proofs below for the sake of completeness, since some of these techniques will be useful for the PKS equation with advection as well.

For convenience, we define the mass function, which will be used throughout this section, as follows.

Definition 2.1. For a function $f \in L^1(\mathbb{R}^2)$, we say that $M_f(r) := \int_{B_0(r)} f\,dx$ is the *mass function* associated with f.

Making use of the mass function, we identify all the radially symmetric steady state solutions for the PKS equation (without advection) in the next theorem.

Theorem 2.2. *Consider the PKS equation* (*without advection*)

$$\rho_t = \Delta\rho + \nabla\cdot(\rho\nabla(\mathcal{N} * \rho)). \tag{2-1}$$

All nonzero radially symmetric steady state solutions of (2-1) *are of the form*

$$\rho_\lambda(r) = \frac{8\lambda}{(\lambda + r^2)^2} \tag{2-2}$$

for some $\lambda > 0$.

Proof. When $\rho(x, t)$ satisfies the PKS equation, one can check by direct computation that the mass function $M_\rho(r, t)$ (which we denote by $M(r, t)$ for simplicity) solves the equation

$$M_t = M_{rr} - \frac{1}{r}M_r + \frac{MM_r}{2\pi r}, \tag{2-3}$$

and hence the mass function of a radial steady state solution satisfies

$$M_{rr} - \frac{1}{r}M_r + \frac{MM_r}{2\pi r} = 0. \tag{2-4}$$

Although this ODE is nonlinear, we can multiply it by r, then rewrite rM_{rr} as $(rM_r)_r - M_r$ and MM_r as $\frac{1}{2}(M^2)_r$ to obtain

$$(rM_r)_r - 2M_r + \frac{1}{4\pi}(M^2)_r = 0, \tag{2-5}$$

which yields

$$rM_r - 2M + \frac{M^2}{4\pi} = c \tag{2-6}$$

for some constant c. Since $M(r)$ is the mass contained in a disk of radius r, we have $M(0) = 0$ by definition, which implies that $c = 0$. Hence the equation above becomes separable and can be written as

$$\frac{M_r}{M(2 - M/(4\pi))} = \frac{1}{r}, \tag{2-7}$$

and by integrating it, we obtain the following family of steady state solutions labeled by a parameter $\lambda > 0$:

$$M_\lambda(r) = \frac{8\pi r^2}{\lambda + r^2}. \tag{2-8}$$

Lastly, observe that $\rho_\lambda(r) = M'_\lambda(r)/(2\pi r)$, which gives (2-2). □

Now we want to show that all radially symmetric solutions with subcritical mass are controlled by (2-2) for some λ. Even though there is no comparison principle for ρ in (2-1), a comparison principle does hold for the mass function M for (2-3), which we describe below.

Theorem 2.3. *Assume $\rho_1(x,t)$ and $\rho_2(x,t)$ are two radially symmetric solutions to* (2-1), *where they satisfy $M_2(r,0) \geq M_1(r,0)$ for all $r \geq 0$. (Here M_1 and M_2 are the mass functions for ρ_1 and ρ_2 respectively.) Then for every t, we have $M_2(r,t) \geq M_1(r,t)$.*

The proof can be found in [Kim and Yao 2012] and will be omitted here.

Making use of this comparison principle, we can now show global regularity for radially decreasing initial data. Here we say ρ_0 is *radially decreasing* if it is radially symmetric, and $\rho_0(r)$ is nonincreasing in r.

Corollary 2.4. *Let $\rho_0(r)$ be a nonnegative smooth radially decreasing function, with $\|\rho_0\|_{L^1} < 8\pi$. Let $\rho(x,t)$ be the solution to* (2-1) *with initial data ρ_0. Then $\rho(x,t)$ is globally regular and bounded.*

Proof. We find a sufficiently small $\lambda > 0$ such that $M_\lambda(r) > M_{\rho_0}(r)$ for all $r \geq 0$. Then, since $M_\lambda(r)$ is a steady state solution, the comparison principle in Theorem 2.3 ensures that $M_\rho(r, t) \leq M_\lambda(r)$ for all time during the existence of ρ, which implies that $\rho(0, t) \leq \rho_\lambda(0)$ for all time. Note that the radially decreasing property of ρ is preserved by the PKS equation (see Theorem 4.2 of [Kim and Yao 2012]), so its maximum occurs at the origin for all time. Combining these two facts, we have $\|\rho(\cdot, t)\|_{L^\infty} \leq \rho_\lambda(0)$ for all time during the existence of ρ. Once we have this global L^∞-bound, one can proceed as in [Kiselev and Ryzhik 2012] to show that the solution is indeed smooth for all time. □

General, advective case with nonpositive divergence. Using the previous results, we aim to study the regularity properties of the advective chemotaxis equation (1-4) with a nonpositive divergence flow, where the initial data are not necessarily radially symmetric. To do this we will utilize symmetric decreasing rearrangements, which can map arbitrary measurable functions to symmetric decreasing functions for which the above results hold. Using this transformation and some inequalities, we can show that the symmetric case is a "supersolution" of the general case in some sense.

The symmetric decreasing rearrangement of a function is defined as follows.

Definition 2.5. For a measurable set $\Omega \subset \mathbb{R}^2$, we define its symmetric rearrangement Ω^* as $\Omega^* := B(0, r)$, where r is chosen such that $|B(0, r)| = |\Omega|$.

For a nonnegative $f \in L^1(\mathbb{R}^2)$, its *symmetric decreasing rearrangement* f^* is given by

$$f^*(x) := \int_0^\infty \chi_{\{f>t\}^*}(x)\, dt, \tag{2-9}$$

where χ denotes the characteristic function.

Below we list a couple of properties on the symmetric decreasing rearrangement.

Lemma 2.6. *Let $\rho \in L^1(\mathbb{R}^2)$ be nonnegative and let $\Omega = \{\rho > s\}$ for some $s \geq 0$. Then the following hold*:

(1) $\int_\Omega \rho\, dx = \int_{\Omega^*} \rho^*\, dx$.

(2) $\int_\Omega \Delta\rho\, dx \leq \int_{\Omega^*} \Delta\rho^*\, dx$.

(3) $\int_\Omega f\, dx \leq \int_{\Omega^*} f^*\, dx$ *for any nonnegative* $f \in L^1(\mathbb{R}^2)$.

(4) $\int_\Omega \nabla \cdot (\rho \nabla(\mathcal{N} * f))\, dx = s \int_\Omega f\, dx$ *for any nonnegative* $f \in L^1(\mathbb{R}^2)$.

Proof. The proofs for (1)–(3) can be found in [Burchard 2009]. Next we will prove (4). Apply the divergence theorem to get

$$\int_\Omega \nabla \cdot (\rho \nabla(\mathcal{N} * f))\, dx = \int_{\partial\Omega} \rho \nabla(\mathcal{N} * f) \cdot \hat{n}\, dx, \tag{2-10}$$

where $\hat{n}$ is the unit normal vector of Ω. Observe that $\partial\Omega = \{\rho = s\}$, which, when applied to the right-hand side of (2-10), gives

$$\int_{\partial\Omega} \rho\nabla(\mathcal{N} * f)\cdot\hat{n}\,dx = s\int_{\Omega} \nabla\cdot\nabla(\mathcal{N} * f)\,dx = s\int_{\Omega} f\,dx, \tag{2-11}$$

where in the last equality we used the fact that $\mathcal{N} * f$ inverts the Laplacian. We thus obtain (4) by combining the above two equations. □

Definition 2.7. Given any two functions $f, g \in L^1(\mathbb{R}^2)$, we say $f \prec g$ if $M_f(r) \leq M_g(r)$ for all $r \geq 0$.

With these tools in hand, we can show that solutions of (1-4) with subcritical mass are globally regular by showing that $\rho^*(\cdot, t) \prec \rho'$ for all time during the existence of ρ, where ρ' is some radially symmetric steady state in (2-2). This implies that ρ has an L^∞-bound that is uniform in time, which finally gives the global regularity of ρ. To do so, we proceed in stages, and construct a sequence converging to the appropriate solution. The method of this proof follows very closely the approach used in [Kim and Yao 2012].

Theorem 2.8. *Assume $\nabla\cdot\vec{u} \leq 0$. Let ρ and ρ' solve*

$$\rho_t = \Delta\rho + \nabla\cdot(\rho\nabla(\mathcal{N} * f)) + \nabla\cdot(\vec{u}\rho), \tag{2-12}$$

$$\rho'_t = \Delta\rho' + \nabla\cdot(\rho'\nabla(\mathcal{N} * f')), \tag{2-13}$$

respectively, with $\rho(\cdot, 0) = \rho_0$, $\rho'(\cdot, 0) = \rho_0^*$, and $f \prec f'$. Then we have $\rho^*(\cdot, t) \prec \rho'(\cdot, t)$ for all $t \geq 0$.

Proof. We prove this theorem by first discretely approximating ρ_t as $(\rho_{n+1} - \rho_n)/h$. Set $g = \rho_n$. We will show that $\rho^*_{n+1} \prec \rho'_{n+1}$ for every positive h. Since the Laplacian is an m-accretive operator [Barbu 2010], we can then let h go to 0 and recover Theorem 2.8. Therefore it suffices to prove the following lemma for each discrete time step. □

Lemma 2.9. *Assume $\nabla\cdot\vec{u} \leq 0$. Let ρ and ρ' solve*

$$\rho = h\Delta\rho + h\nabla\cdot(\rho\nabla(\mathcal{N} * f)) + \nabla\cdot(\rho\vec{u}) + g, \tag{2-14}$$

$$\rho' = h\Delta\rho' + h\nabla\cdot(\rho'\nabla(\mathcal{N} * f')) + g', \tag{2-15}$$

respectively, and assume that ρ', f' and g' are all radially symmetric. If $f^* \prec f'$ and $g^* \prec g'$, then we have $\rho^* \prec \rho'$.

Proof. For any $r > 0$, one can choose $s > 0$ such that $|\{\rho > s\}| = |B(0, r)|$. Thus, let $\Omega := \{\rho > s\}$ and it follows that $\Omega^* = B(0, r)$. Integrate (2-15) over $B(0, r)$, which gives

$$M_{\rho'}(r) = hM_{\Delta\rho'}(r) + h\rho'(r)M_{f'}(r) + M_{g'}(r). \tag{2-16}$$

We also integrate (2-14) over $\Omega = \{\rho > s\}$. Using Lemma 2.6 and Definition 2.1, we have

$$\begin{aligned} M_{\rho^*}(r) &= \int_\Omega \rho(x)\,dx \quad \text{(by Lemma 2.6(a))} \\ &= h\int_\Omega \Delta\rho\,dx + h\nabla\cdot(\rho\nabla(\mathcal{N} * f)) + \int_\Omega \nabla\cdot(\rho\vec{u})\,dx + \int_\Omega g\,dx \\ &\le hM_{\Delta\rho^*}(r) + h\rho(r)M_f^*(r) + M_g^*(r) \quad \text{(by Lemma 2.6(2)–(4))}. \end{aligned} \tag{2-17}$$

In the last inequality, we also used $\int_\Omega \nabla\cdot(\rho\vec{u})\,dx = \int_{\partial\Omega} \hat{n}\cdot(\rho\vec{u})\,d\sigma = s\int_\Omega \nabla\cdot\vec{u}\,dx \le 0$, where our assumption $\nabla\cdot\vec{u} \le 0$ is applied.

In order to show that $\rho^* \prec \rho'$, first notice that by sending $r \to \infty$ in (2-16) and (2-17) and using the assumption that $g^* \prec g'$, we have $\lim_{r\to\infty} M_{\rho^*}(r) - M_{\rho'}(r) \le 0$. Therefore, if $\rho^* \prec \rho'$ does not hold, there must exist some finite $r_0 > 0$, such that $M_{\rho^*}(r) - M_{\rho'}(r)$ attains a positive maximum at r_0. Note that we have $\rho'(r_0) = \rho^*(r_0)$ since $\partial_r(M_{\rho^*} - M_{\rho'})(r_0) = 0$.

Subtracting (2-17) and (2-16) at r_0 gives

$$\begin{aligned} M_{\rho^*-\rho'}(r_0) &\le hM_{\Delta(\rho^*-\rho')}(r_0) + h\rho'(r_0)M_{f^*-f'}(r_0) + M_{g^*-g'}(r_0) \\ &\le hM_{\Delta(\rho^*-\rho')}(r_0), \end{aligned} \tag{2-18}$$

where in the last step we use the assumptions that $f^* \prec f'$ and $g^* \prec g'$. Since ρ^* and ρ' are radially symmetric, we can simplify $M_{\Delta(\rho^*-\rho')}$ as

$$M_{\Delta(\rho^*-\rho')} = \partial_{rr}M_{\rho^*-\rho'} - \frac{\partial_r M_{\rho^*-\rho'}}{r}. \tag{2-19}$$

Since $M_{\rho^*-\rho'}$ achieves a maximum at r_0, it follows from the above expression that $M_{\Delta(\rho^*-\rho')}(r_0) \le 0$, and combining it with (2-18), we have $M_{\rho^*-\rho'}(r_0) \le 0$, leading to a contradiction. This yields $\rho^* \prec \rho'$. □

Once we have Theorem 2.8, our global regularity result follows from the same iteration argument as in [Kim and Yao 2012], which we sketch below.

Theorem 2.10. *Let ρ solve* (1-4) *with smooth, subcritical initial data ρ_0. Then ρ is globally regular and bounded for all time.*

Proof. Let $\rho_1(\cdot, t) := \rho^*(\cdot, t)$. For any $n \ge 1$, we iteratively define $\rho_{n+1}(\cdot, t)$ by

$$\partial_t\rho_{n+1} = \Delta\rho_{n+1} + \nabla\cdot(\rho_{n+1}\nabla(\mathcal{N} * \rho_n)), \tag{2-20}$$

with initial data ρ_0^*.

By Lemma 2.9, $\rho_1 \prec \rho_2$. Once we have this, one can then iteratively apply Lemma 2.9 (with zero velocity field) to obtain $\rho_n \prec \rho_{n+1}$ for all $n \ge 1$. Thus, we have an increasing sequence of mass functions. Finally, along a subsequence,

$\rho_n \to \bar{\rho}$ (hence $\rho^* = \rho_1 \prec \bar{\rho}$ too), where $\bar{\rho}$ solves

$$\partial_t \bar{\rho} = \Delta \bar{\rho} + \nabla \cdot (\bar{\rho} \nabla (\mathcal{N} * \bar{\rho})). \tag{2-21}$$

Details for this can be found in [Kim and Yao 2012]. Since $\bar{\rho}$ solves the PKS equation without advection, Corollary 2.4 gives that $\bar{\rho}(\cdot, t)$ is bounded and regular globally in time. Since $\rho(\cdot, t) \prec \bar{\rho}(\cdot, t)$ for all $t \geq 0$, the same holds for $\rho(\cdot, t)$ as desired. □

3. Numerical study of solutions with supercritical mass

Numerical methods. We follow the Euler–Maruyama stochastic particle approximation to the 2D Keller–Segel equation that is explained in detail in [Haškovec and Schmeiser 2009], henceforth abbreviated [HS]. The key benefit of this approach, as opposed to using finite element or finite volume, is that the system can still be analyzed after a singularity blow-up forms. It is also relatively simple to code, which was our main concern for this investigation. The only significant difference between our method and the one described in detail in [HS] is the addition of a deterministic advection step at every time interval. Below we lay out the basic aspects of the numerical method from [HS] and the added advection term.

For all of the simulations, we use a system of $N = 1000$ particles located at $x_1, \ldots, x_N$ with mass sizes $M_1, \ldots, M_N$ respectively. The "light" particles, with mass less than or equal to 8π, approximate the smooth part of the solution, and the "heavy" particles, with mass greater than 8π, approximate delta functions when the solution has blown up.

Step 1: Advection. To model the extra advection term at every time step, the particles move according to advection, which gives $\mathrm{d}x_n = \vec{u}(x_n)\, dt$.

Step 2: Aggregation. Using the stochastic particle approximation from [HS], the nonlocal, chemotactic interaction term of the PDE is

$$\mathrm{d}x_n = -\frac{1}{2\pi} \sum_{m \neq n} M_m \frac{x_n - x_m}{|x_n - x_m|^2}\, dt.$$

Then our set of particles undergoes the processes of collision and splitting. This part is the same as in [HS], which is explained further in the next subsection for the sake of completeness.

Step 3: Diffusion. After the aggregation step, each light particle undergoes a random-walk step, giving $\mathrm{d}x_n = \sqrt{2\, dt}\, \mathcal{N}_{(0,1)}$, where $\mathcal{N}_{(0,1)}$ denotes the Gaussian distribution with mean 0 and variance 1.

Particle collisions and splitting. In our method, particle collisions and splitting are handled in the same way as in [HS], which we describe below for the sake of

completeness. In the aggregation step, it is easy to see that as the distance between two particles goes to zero, the interaction kernel grows to infinity. Hence we allow two sufficiently close particles $\{x_1, x_2\}$ to collide and form one new particle, with the new mass $M' = M_1 + M_2$. The criterion, from [HS], for two particles to collide during a time interval $(0, \Delta T)$ is given by

$$\|x_1 - x_2\|^2 \leq \frac{M_1 + M_2}{\pi} \Delta T.$$

At each discrete time step, this inequality is evaluated for each particle pair and the particles satisfying this condition are fused. This inevitably leads to single particles accumulating more and more mass as they pull in other particles towards them.

The main issue with fusing particles is that as particles collide, the effective grid spacing coarsens, as there are fewer and fewer particles in the domain. To compensate for the particles lost to collisions, particles are randomly split at each time step so that the total number of particles remains constant. This helps maintain a proper discretization of the space and helps more accurately model the desired effects. The exact method of particle collisions and splitting is explained in detail in [HS]. The random-walk step is evaluated after the splitting occurs so that when two particles are split, they remain at the same position but are then redistributed by Brownian motion.

Numerical results. We run numerical simulations for a variety of different flows $\vec{u}$ to find whether blow-up could be delayed or prevented with the presence of a flow. The flows we investigate are

$$\text{shear flow:} \quad \vec{u}(x, y) = (e^{-y^2}, 0), \tag{3-1}$$

$$\text{strain flow:} \quad \vec{u}(x, y) = (-x, y), \tag{3-2}$$

$$\text{diverging flow:} \quad \vec{u}(r, \theta) = (1/r, \theta), \tag{3-3}$$

which are compared against the no-flow case. Note that the shear flow and strain flow are both incompressible. Although the diverging flow is not incompressible, we also test it and it turns out that it is effective at preventing blow-up with a relatively weak flow strength. Velocity fields for these flows are illustrated in Figure 1.

We use 1000 particles in our simulations, which are initially randomly distributed in a disk $B(0, \sqrt{0.5})$. An initial mass $M = 16\pi$ (twice the critical mass) is distributed evenly across all the particles. We choose the time step as $\Delta t = 5 \cdot 10^{-5}$ and run simulations for 2000 time steps until $t = 0.1$. With no flow, our simulation shows that blow-up occurs at $t = 0.0325$, as shown in Figure 2.

In order to see how different flow strengths affect the result, we multiply the flow $\vec{u}$ by a constant C, so that the stochastic particle approximation for the advection

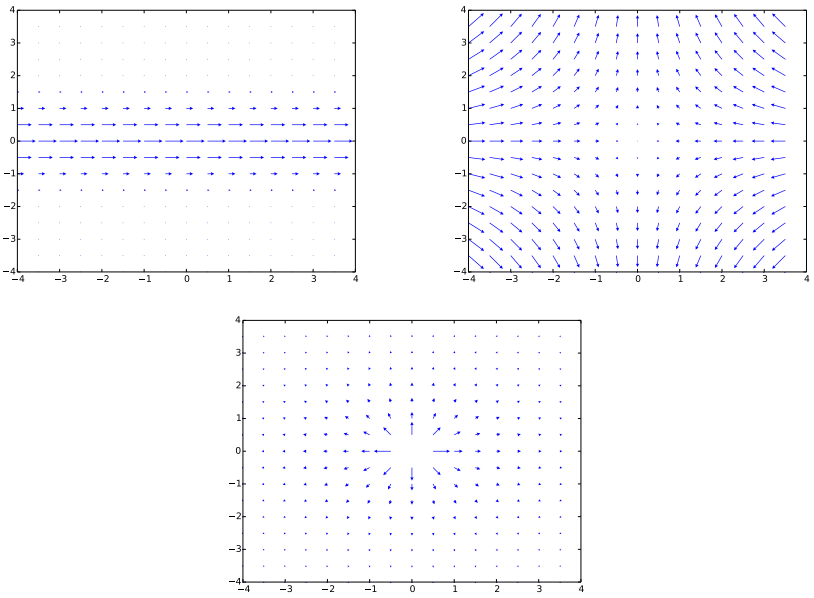

Figure 1. Illustration of the three types of flows tested in our numerical study. Top left: shear flow (3-1). Top right: strain flow (3-2). Bottom: diverging flow (3-3).

term is

$$\mathrm{d}x_n = C\vec{u}(x_n)\,dt.$$

Shear flow is tested with a flow strength, C, of 100 and 1000, strain flow with strengths of 10 and 100, and diverging flow with strengths of 1 and 10. From now on, we will refer to the smaller of the two flow strengths as "weak flow" and the larger of the two as "strong flow" for each flow configuration. Our numerical results

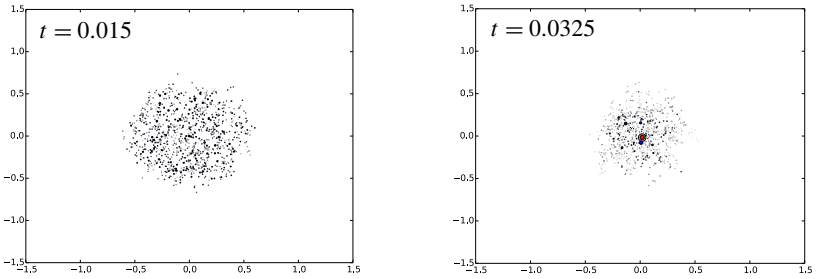

Figure 2. Numerical simulation for the PKS equation with no advection term at different times. Blow-up occurs at $t = 0.0325$. The red dot in the second picture indicates the location of blow-up, where we have a mass concentration of more than 8π.

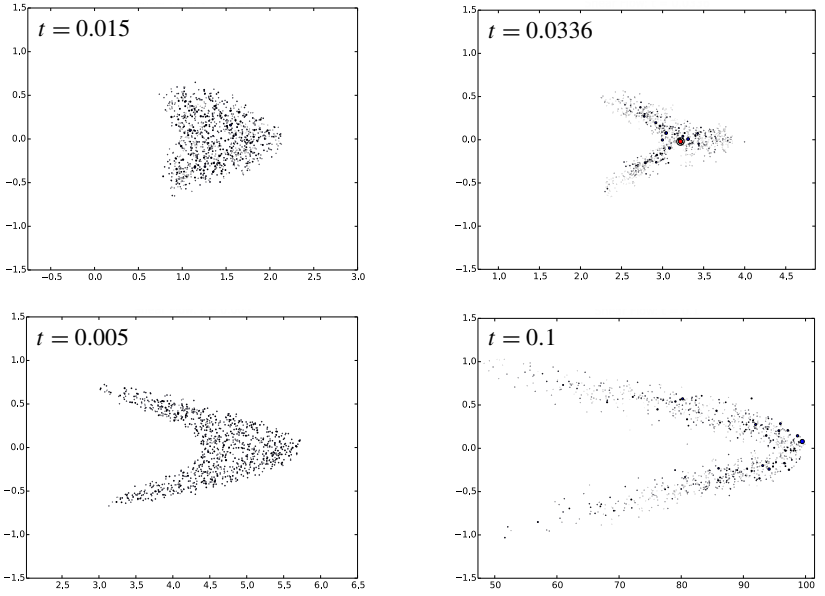

Figure 3. Shear flow (3-1) with strengths $C = 100$ (top row) and 1000 (bottom row). For $C = 100$, blow-up occurs at $t = 0.0336$ (the red dot indicates the location of blow-up). For $C = 1000$, blow-up does not occur before $t = 0.1$.

are shown in Figures 3–5. The results suggest that all of these three flows seem to be able to prevent blow-up when the flow strength is chosen to be sufficiently large.

Remarks on possible future improvements. We now point out some possible improvements that can be made with our current numerical scheme. First, computation time is an issue when many different flows need to be tested. Note that the most time-consuming step is the aggregation step, where for N particles, one has to perform on the order of N^2 calculations to compute their pairwise interactions. Haškovec and Schmeiser [2009] comment on one potential heuristic for speeding up the chemotaxis calculation, where the program, while calculating the step for a given particle, averages the masses of many distant, but close together particles, and uses the essential "center of mass", instead of the contribution from each individual particle.

Currently, our numerical scheme is not sensitive enough to show a difference between an initial mass that is only slightly above or below $M = 8\pi$. It also is not able to show that for flows $-d/r^\alpha$, $\alpha < 1$, with subcritical initial data, the flows will not blow up. These insensitivities may be solved with trying smaller time steps Δt, but it is likely that another numerical method is needed. To help support these claims, it might be better to implement a finite element or finite volume approach, but this is outside the scope of this paper.

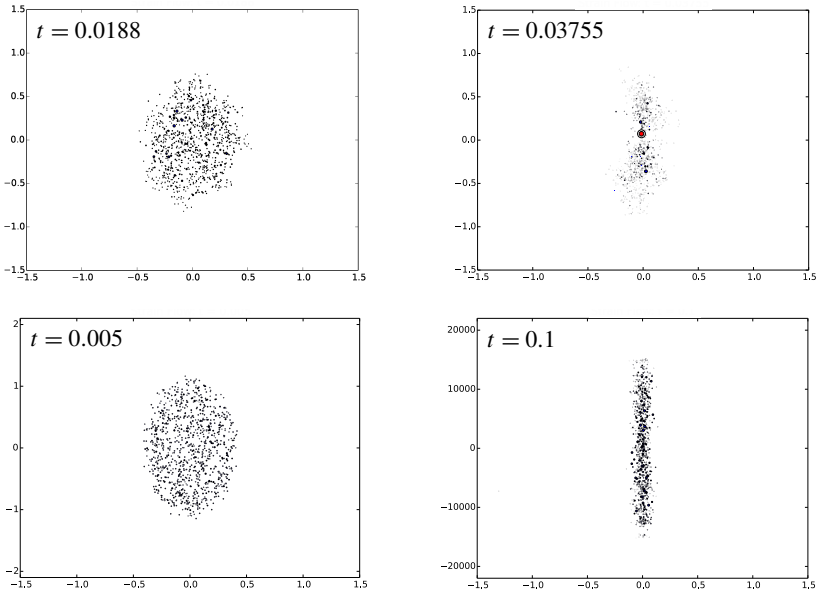

Figure 4. Strain flow (3-2) with strengths $C = 10$ (top row) and 100 (bottom row). For $C = 10$, blow-up occurs at $t = 0.03755$. For $C = 100$, blow-up does not occur before $t = 0.1$.

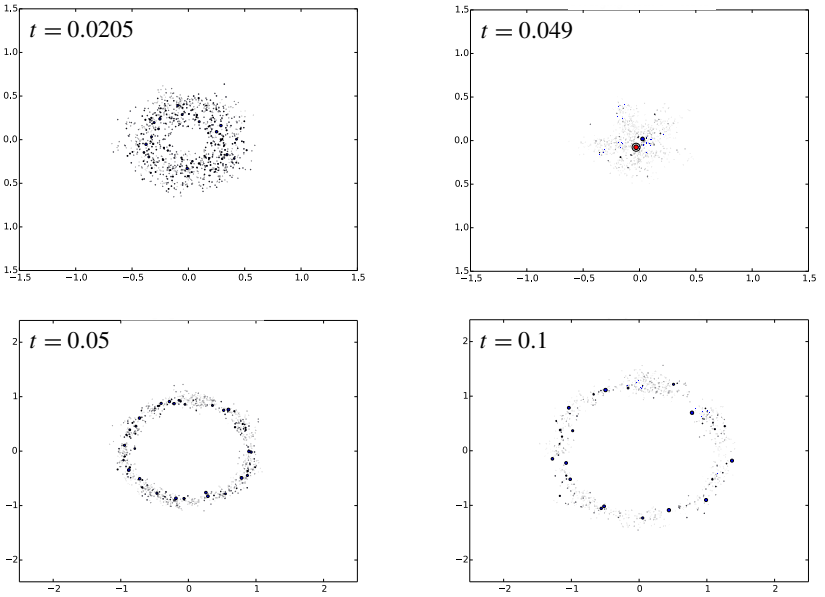

Figure 5. Diverging flow (3-3) with strengths $C = 1$ (top row) and 10 (bottom row). For $C = 1$, blow-up occurs at $t = 0.0490$. For $C = 100$, blow-up does not occur before $t = 0.1$.

Acknowledgements

This research was conducted at the University of Wisconsin–Madison during the summer of 2013 as a part of its Analysis and Differential Equations REU. We would like to thank Professor Alexander Kiselev for helping direct our research and teaching us about the chemotaxis equations. We would also like to thank Tam Do for helping us with the proofs of the theorems and providing guidance.

References

[Barbu 2010] V. Barbu, *Nonlinear differential equations of monotone types in Banach spaces*, Springer, New York, 2010. MR 2011d:34001 Zbl 1197.35002

[Burchard 2009] A. Burchard, "A short course on rearrangement inequalities", Lecture notes, 2009, available at http://www.math.utoronto.ca/almut/rearrange.pdf.

[Haškovec and Schmeiser 2009] J. Haškovec and C. Schmeiser, "Stochastic particle approximation for measure valued solutions of the 2D Keller–Segel system", *J. Stat. Phys.* **135**:1 (2009), 133–151. MR 2010f:92014 Zbl 1173.82021

[Horstmann 2003] D. Horstmann, "From 1970 until present: the Keller–Segel model in chemotaxis and its consequences, I", *Jahresber. Deutsch. Math.-Verein.* **105**:3 (2003), 103–165. MR 2005f:35163 Zbl 1071.35001

[Keller and Segel 1970] E. F. Keller and L. A. Segel, "Initiation of slime mold aggregation viewed as an instability", *J. Theor. Biol.* **26**:3 (1970), 399–415. Zbl 1170.92306

[Keller and Segel 1971] E. F. Keller and L. A. Segel, "Model for chemotaxis", *J. Theor. Biol.* **30**:2 (1971), 225–234. Zbl 1170.92307

[Kim and Yao 2012] I. Kim and Y. Yao, "The Patlak–Keller–Segel model and its variations: properties of solutions via maximum principle", *SIAM J. Math. Anal.* **44**:2 (2012), 568–602. MR 2914242 Zbl 1261.35080

[Kiselev and Ryzhik 2012] A. Kiselev and L. Ryzhik, "Biomixing by chemotaxis and enhancement of biological reactions", *Comm. Partial Differential Equations* **37**:2 (2012), 298–318. MR 2876833 Zbl 1236.35190

[Patlak 1953] C. S. Patlak, "Random walk with persistence and external bias", *Bull. Math. Biophys.* **15** (1953), 311–338. MR 18,424f Zbl 1296.82044

[Perthame 2007] B. Perthame, *Transport equations in biology*, Birkhäuser, Basel, 2007. MR 2007j:35004 Zbl 1185.92006

Received: 2014-08-25 Revised: 2015-01-08 Accepted: 2015-01-09

smk508@nyu.edu *Department of Mathematics, New York University, New York, NY 10012-1185, United States*

johnson.jayrichard@utexas.edu *Department of Mechanical Engineering, University of Texas at Austin, Austin, TX 78712, United States*

evc34@cornell.edu *Department of Mathematics, Cornell University, Ithaca, NY 14850, United States*

yaoyao@math.wisc.edu *Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53703, United States*

involve

msp.org/involve

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2016 is US $160/year for the electronic version, and $215/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2016 vol. 9 no. 1

1944-4176(2016)9:1;1-4