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Depths and Stanley depths of path ideals of spines

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For a special class of trees, namely trees that are themselves a path, a precise formula is given for the depth of an ideal generated by all (undirected) paths of a fixed length. The dimension of these ideals is also computed, which is used to classify which such ideals are Cohen–Macaulay. The techniques of the proofs are shown to extend to provide a lower bound on the Stanley depth of these ideals. Combining these results gives a new class of ideals for which the Stanley conjecture holds.

1. Introduction

There is a well-known correspondence between square-free monomial ideals generated in degree two and graphs. If G is a graph on n vertices, let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k in n variables and define the *edge ideal* $I = I(G)$ to be the ideal generated by all monomials of the form $x_i x_j$, where $\{x_i, x_j\}$ is an edge of G; see [Villarreal 1990]. The use of graphs to study algebraic properties of edge ideals has proven quite fruitful. A natural extension of the edge ideal of a graph is the path ideal of a graph. For each positive integer ℓ, define $P_\ell(G)$ to be the monomial ideal whose generators correspond to paths of length ℓ of G. Since the vertices of a path are distinct, $P_\ell(G)$ is a square-free monomial ideal. Various authors have used combinatorial information from the associated graphs to deduce information about depths of edge ideals [Dao et al. 2013; Dao and Schweig 2013; Fouli and Morey 2014; Herzog and Hibi 2005; Kummini 2009; Morey 2010]. The goal of this article is to examine the depth of a path ideal.

If $(R, \mathfrak{m})$ is a commutative, Noetherian, local ring and I is an ideal of R, the *depth* of R/I is an important algebraic invariant that, loosely speaking, provides one way to measure the size of R/I. More specifically, $\operatorname{depth}(R/I)$ is the maximal length of a sequence in $\mathfrak{m}$ that is regular on R/I. When $\operatorname{depth}(R/I) = \dim(R/I)$, the ring

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is said to be *Cohen–Macaulay*. There are many ways to detect depth, including the vanishing of Ext modules, local cohomology modules, or Koszul homology, to name a few. See [Herzog and Hibi 2011] for general information about depths and [Miller and Sturmfels 2005, Theorem 13.37] for a sample of the many ways the Cohen–Macaulay property can be detected. In this article, the depths of certain path ideals will be computed. Since the heights of such ideals are easily determined, the depth will be used to classify which such ideals are Cohen–Macaulay.

As a consequence of the method of proof employed, the results regarding the depth of path ideals of a special type of tree extend to a lower bound on the Stanley depth of the ideals. Let I be a monomial ideal. A *Stanley decomposition* of R/I is a direct sum decomposition $R/I = \bigoplus_{i=1}^{s} m_i R_{t_i}$ where m_i is a monomial and $R_{t_i} = k[x_{i_1}, \dots, x_{i_{t_i}}]$ is a polynomial subring of R generated over k by t_i of the variables of R. The depth of this decomposition is the minimum of the t_i, that is, the smallest number of variables used in any summand. The *Stanley depth*, denoted s-depth, of R/I is then the maximum depth of a Stanley decomposition of R/I. Introduced in [Stanley 1982], s-depth is a more geometric invariant attached to a monomial ideal, or more generally to a $\mathbb{Z}^r$-graded module. For a more detailed introduction to Stanley depths, see [Pournaki et al. 2009]. Stanley conjectured that the Stanley depth is always bounded below by the depth. By combining the bound found in Theorem 4.1 with Theorem 3.10, we prove that one class of path ideals is Stanley, that is, the Stanley conjecture holds true for this class of ideals. While other classes of Stanley ideals are known, see for instance [Pournaki et al. 2013] or [Cimpoeaş 2009], the conjecture is still largely open.

The contents of the paper are as follows. In Section 2 we provide the definitions and basic facts used throughout the paper. In Sections 3 and 4 we focus on the particular case where the tree T is a path. An exact formula for the depths of the path ideals is computed in Theorem 3.10. In Lemma 3.13, the dimension of such rings is given. Combining these results, Proposition 3.14 shows that if T is a path on n vertices, $P_\ell(T)$ is Cohen–Macaulay if and only if $n = \ell + 1$ or $n = 2\ell + 2$. In Section 4, using the techniques of Section 3, a bound is given in Theorem 4.1 for the s-depths of path ideals of T and as a result, in Corollary 4.2 these ideals are seen to be Stanley; that is, the Stanley conjecture is satisfied for this class of ideals.

2. Definitions and background

We begin by reviewing some standard notation and terminology regarding graphs and their connections to algebra. By abuse of notation, x_i will be used to denote both the vertex of a graph G and the corresponding variable of the polynomial ring R. For information regarding square-free monomial ideals, see [Villarreal 2001] and for additional background in graph theory, see [Harary 1969].

A *graph* is a vertex set $V = \{x_1, \ldots, x_n\}$ together with a set $E = E(G) \subseteq V \times V$ of edges. As previously stated, associated to any graph G is a square-free monomial ideal generated in degree two, $I = I(G)$, called the edge ideal of I. Given a graph G, there is another family of square-free monomial ideals associated to G. For each positive integer ℓ, define $P_\ell(G)$ to be the monomial ideals whose generators correspond to paths of length ℓ of G. Notice that a path of length ℓ contains $\ell + 1$ vertices, so $P_\ell(G)$ is a homogeneous ideal with generators of degree $\ell + 1$. When $\ell = 1$, $P_1(G) = I(G)$ is the edge ideal of G. Notice that since a path is defined to have distinct vertices, $P_\ell(G)$ is a square-free monomial ideal.

The concept of a graph can be easily extended to one of a *clutter*, which is also called a *simple hypergraph*. A clutter $\mathfrak{C}$ is a vertex set V together with a set E of edges, where elements of E are nonempty subsets of V, with no inclusions among elements of E. That is, if $e, f \in E$, then $e \not\subset f$. For a graph, an edge $e \in E$ consists of two vertices, while for a clutter an edge may contain any number of vertices. Since a path ideal can be viewed as a special type of clutter with edges consisting of $\ell + 1$ vertices, tools from combinatorial optimization may be applied to path ideals.

Some basic notions from graph theory will be used throughout the paper and so are presented here for completeness. If $V' \subset V$ is a subset of the vertices of a graph G, the *induced subgraph* on V' is the graph G' given by $V(G') = V'$ and $E(G') = \{e \in E \mid e \subset V'\}$. That is, the edges of G' are precisely the edges of G with both endpoints in V'. If $x \in V(G)$, the *neighbor set* $N(x)$ is the set of all vertices that are adjacent to x, that is, $N(x) = \{y \in V(G) \mid \{x, y\} \in E(G)\}$. The *degree* of a vertex x is the cardinality of $N(x)$. A *leaf* is a vertex of degree one, and a *tree* is a connected graph where every induced subgraph has a leaf. A walk of length s is a collection of vertices and edges $x_0, e_1, x_1, e_2, \ldots, e_s, x_s$ where $e_i = x_{i-1}x_i$ for $1 \le i \le s$. A walk without repeated vertices is a *path*. If T is a tree, then for any vertices $x, y \in V(G)$, there is a unique path between x and y. The length of this path is the *distance* between x and y, which is denoted by $d(x, y)$. In a general graph, $d(x, y)$ is the minimum of the lengths of all paths connecting x and y. A *forest* is a collection of trees. An *isolated vertex* is a vertex x with $N(x) = \varnothing$. Since $k[x_1, \ldots, x_n, y]/(I, y) \cong k[x_1, \ldots, x_n]/I$ for any monomial ideal I whose generators lie in $k[x_1, \ldots, x_n]$, the graphs throughout this paper are generally assumed to be free of isolated vertices.

There are two common constructions used in combinatorial optimization that take a clutter or graph and produce smaller, related clutters or graphs. One is the *deletion* $\mathfrak{C} \setminus x$, which is formed by removing x from the vertex set of $\mathfrak{C}$ and deleting any edge in $\mathfrak{C}$ that contains x. This has the effect of setting $x = 0$, or of passing to the quotient ring $R/(x)$. The other operation is the *contraction*, $\mathfrak{C}/x$. This is performed by removing x from the vertex set and removing x from any edge that contains x. When $\mathfrak{C} = G$ is a graph, this will result in each vertex in $N(x)$ becoming an isolated vertex since if $y \in N(x)$ then removing x from the edge $\{x, y\}$ isolates y. Any additional edges containing y are removed since the definition of a clutter does not allow containments among edges. This operation has the effect of setting $x = 1$, or of passing to the localization R_x. A *minor* of a graph is formed by performing any combination of deletions and contractions.

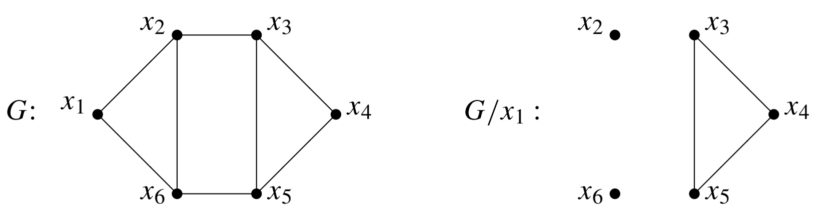
Example 2.1. The graph G shown below corresponds to the ideal

$$I = I(G) = (x_1x_2, x_1x_6, x_2x_3, x_2x_6, x_3x_4, x_3x_5, x_4x_5, x_5x_6).$$

Inverting x_1 yields

$$I_{x_1} = (x_2, x_6, x_2x_3, x_2x_6, x_3x_4, x_3x_5, x_4x_5, x_5x_6) = (x_2, x_6, x_3x_4, x_3x_5, x_4x_5),$$

which corresponds to the graph G/x_1:



If G is a graph, a *minimal vertex cover* of G is a set $C \subset V$ such that for every $e \in E$, $e \cap C \neq \varnothing$ and C is minimal with respect to this property, meaning if C' is any proper subset of C, then there exists an edge $e \in E$ with $e \cap C' = \varnothing$. The minimum cardinality of a minimal vertex cover of G (or $\mathfrak{C}$) is denoted by $\alpha_0 = \alpha_0(G)$. A prime ideal P is a *minimal prime of an ideal I* if $I \subset P$ and if Q is a prime ideal with $I \subset Q \subset P$, then $Q = P$. It is straightforward to check that C is a minimal vertex cover of G if and only if the prime ideal P generated by the variables corresponding to vertices of C is a minimal prime of $I(G)$. Thus $\alpha_0 = \text{height}(I)$.

The definition of depth is usually given for a local ring or with respect to a particular prime ideal. Although $R = k[x_1, \dots, x_n]$ is not a local ring, it has a unique homogeneous maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Since all ideals in this article are homogeneous ideals contained in $\mathfrak{m}$, R may be treated as a local ring. All depths will be taken with respect to $\mathfrak{m}$.

3. Depths of path ideals of spines

In general, it can be quite difficult to determine the precise depth of an ideal. In this section, we give an exact formula for the depth of a path ideal of a tree that does not branch. When combined with the height of the ideal, this formula allows us to determine which path ideals are Cohen–Macaulay. By noting that in this special case the directed path ideal is the same as the path ideal, and by using the Auslander–Buchsbaum formula, the depth formula found can be used to recover the projective dimension result of [He and Van Tuyl 2010, Theorem 4.1] which was also recovered in [Bouchat et al. 2011, Corollary 5.1]. However, the method of proof will allow us in Section 4 to extend the depth result to a bound on the Stanley depths of the ideals, as was done in [Pournaki et al. 2013] for powers of edge ideals. This bound shows that these ideals are Stanley.

The primary tool we will employ for computing depths is to form a family of short exact sequences and then apply the depth lemma (see, for example, [Bruns and Herzog 1993, Proposition 1.2.9], or [Villarreal 2001, Lemma 1.3.9]). In particular, if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of finitely generated R modules with homogeneous maps and $\operatorname{depth}(C) > \operatorname{depth}(A)$, then $\operatorname{depth}(B) = \operatorname{depth}(A)$. Note that the method used in this section is a variation of the method used in [Hà and Morey 2010; Morey 2010], where instead of using the left term of one sequence to form the subsequent sequence, the right-hand term is used. Starting with the standard short exact sequence

$$0 \to R/(I:z) \xrightarrow{f} R/I \xrightarrow{g} R/(I,z) \to 0$$

and making judicious choices for $z \in R$, we form a family of sequences

$$\begin{array}{ccccccccc} 0 & \to & R/K_1 & \to & R/I & \to & R/C_1 & \to & 0, \\ & & \vdots & & \vdots & & \vdots & & \\ 0 & \to & R/K_i & \to & R/C_{i-1} & \to & R/C_i & \to & 0, \\ & & \vdots & & \vdots & & \vdots & & \\ 0 & \to & R/K_s & \to & R/C_{s-1} & \to & R/C_s & \to & 0, \end{array} \tag{3-1}$$

where $C_0 = I$, $K_i = (C_{i-1} : z_i)$, and $C_i = (C_{i-1}, z_i)$ for $1 \le i \le s$. The goal is to find bounds on the depths of K_i for $1 \le i \le s$ and for C_s. Then applying the depth lemma starting with the last sequence and working back to the first will yield a bound on the depth of R/I. In this section, it will be easier to describe the sequence $\{z_i\}$ using a double index, so the ideals playing the roles of K_i and C_i will be doubly indexed as well.

A tree that does not branch is traditionally referred to as a *path*, however, to avoid the confusion of dealing with path ideals of paths, we will refer to such a graph as a spine. To be precise, we define a *spine* of length $n-1$ to be a set of n distinct vertices $x_1, \dots, x_n$ together with $n-1$ edges $x_i x_{i+1}$ for $1 \le i \le n-1$. We denote such a spine by S_n and we will use $R = k[x_1, \dots, x_n]$ to denote the polynomial ring associated to S_n, or more generally, any graph on n vertices. As subrings of R will be used, define $R_t = k[x_1, \dots, x_t]$ for $t \le n$. While working with these ideals, it will often be convenient to work with subideals generated by selected paths. To facilitate this, define $P_{(\ell,s)}$ to be the ideal generated by the monomials corresponding to all paths of length ℓ of the spine connecting x_1 to x_s. For example, $P_{(2,5)} = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5)$. Using this notation, $P_\ell(S_n) = P_{(\ell,n)}$.

We first handle a special case.

Lemma 3.1. *Let S_n be a spine on n vertices. If $n \leq \ell$, then* $\operatorname{depth}(R/P_{(\ell,n)}) = n$.

Proof. As $\ell \geq n$ we see that S_n does not contain a path of length ℓ. Thus $P_{(\ell,n)} = P_\ell(S_n) = (0)$ and we have $\operatorname{depth}(R/P_{(\ell,n)}) = \operatorname{depth}(R/(0)) = \operatorname{depth}(R) = n$. □

We now fix ℓ and n. In order to define the monomials that will serve the role of z_i above, it is useful to apply the division algorithm to produce unique integers b and c with $0 \leq c < \ell + 2$ and $n - \ell - 1 = b(\ell + 2) + c$. It will often be convenient to write $n = (\ell + 1) + b(\ell + 2) + c$ throughout the paper. For $1 \leq c \leq \ell + 1$, define a sequence $\{a_{(j,k)}\}$ by

$$a_{(j,k)} = \prod_{t=n-\ell-k+1}^{n-j-k+1} x_t$$

for $1 \leq j \leq \min\{c, \ell\}$ and $1 \leq k \leq c - j + 1$. Note that for $c = 0$, the sequence is defined to be empty. The order in which the terms appear in this sequence is crucial to the definition of the family of sequences above. Specifically, the terms are ordered $a_{(1,1)}, a_{(1,2)}, a_{(1,3)}, \ldots, a_{(1,c)}, a_{(2,1)}, a_{(2,2)}, \ldots, a_{(2,c-1)}, a_{(3,1)}, \ldots$.

Example 3.2. Suppose $n = 18$ and $\ell = 6$. We then have $b = 1$ and $c = 3$ so our sequence of monomials $\{a_{(j,k)}\}$ is

$$\begin{aligned} a_{(1,1)} &= x_{12}x_{13}x_{14}x_{15}x_{16}x_{17}, & a_{(2,1)} &= x_{12}x_{13}x_{14}x_{15}x_{16}, \\ a_{(1,2)} &= x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}, & a_{(2,2)} &= x_{11}x_{12}x_{13}x_{14}x_{15}, \\ a_{(1,3)} &= x_{10}x_{11}x_{12}x_{13}x_{14}x_{15}, & a_{(3,1)} &= x_{12}x_{13}x_{14}x_{15}. \end{aligned}$$

Using this sequence, we now define the ideals that will play the roles of C_i and K_i in the sequences above when $I = P_{(\ell,n)}$. Notice that since the sequence used is doubly indexed, the ideals C_i and K_i will require double indices as well, with the same ranges on the indices as above. We first define the ideals $C_{(j,k)} = (I, a_{(1,1)}, a_{(1,2)}, \ldots, a_{(j,k)})$. Note that for $c = 0$, the sequence was defined to be empty, and the only ideal defined is $C_{(0,k)} = P_{(\ell,n)}$ for all k. In general, the sequence of $a_{(j,k)}$ was selected so that many of the terms of $C_{(j,k)} = (I, a_{(1,1)}, a_{(1,2)}, \ldots, a_{(j,k)})$ will be redundant.

Next we define the ideals $K_{(j,k)}$, with the same bounds on j, k as before, by

$$K_{(j,k)} = \begin{cases} (C_{(j-1,c-(j-1)+1)} : a_{(j,1)}) & \text{if } k = 1, \\ (C_{(j,k-1)} : a_{(j,k)}) & \text{if } k > 1. \end{cases} \tag{3-2}$$

Notice that each $K_{(j,k)}$ is formed by taking the quotient ideal of the next term in the sequence with the preceding C ideal. It is straightforward to obtain an explicit formula for $K_{(j,k)}$ (see Proposition 3.4). The selection of the sequence $a_{(j,k)}$ was designed so that these quotient ideals will each have two elements of degree one, and these elements will make all paths of length less than ℓ redundant as generators.

Example 3.3. Assume $n = 18$ and $\ell = 6$. Then $b = 1$, $c = 3$ and the sequence $\{a_{(j,k)}\}$ is given in Example 3.2. Set $I = P_{(6,18)}$. Then

$$I = (x_1x_2x_3x_4x_5x_6x_7, x_2x_3x_4x_5x_6x_7x_8, \dots, x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}x_{17}, x_{12}x_{13}x_{14}x_{15}x_{16}x_{17}x_{18}).$$

By definition,

$$C_{(2,1)} = (I, a_{(1,1)}, a_{(1,2)}, a_{(1,3)}, a_{(2,1)}),$$
$$C_{(2,2)} = (I, a_{(1,1)}, a_{(1,2)}, a_{(1,3)}, a_{(2,1)}, a_{(2,2)}),$$
$$C_{(3,1)} = (I, a_{(1,1)}, a_{(1,2)}, a_{(1,3)}, a_{(2,1)}, a_{(2,2)}, a_{(3,1)}).$$

Removing redundant generators yields

$$C_{(2,1)} = (x_1x_2x_3x_4x_5x_6x_7, \dots, x_8x_9x_{10}x_{11}x_{12}x_{13}x_{14}, x_{10}x_{11}x_{12}x_{13}x_{14}x_{15}, x_{12}x_{13}x_{14}x_{15}x_{16}),$$
$$C_{(2,2)} = (x_1x_2x_3x_4x_5x_6x_7, \dots, x_8x_9x_{10}x_{11}x_{12}x_{13}x_{14}, x_{12}x_{13}x_{14}x_{15}x_{16}, x_{11}x_{12}x_{13}x_{14}x_{15}),$$
$$C_{(3,1)} = (x_1x_2x_3x_4x_5x_6x_7, \dots, x_8x_9x_{10}x_{11}x_{12}x_{13}x_{14}, x_{12}x_{13}x_{14}x_{15}).$$

Now by definition $K_{(2,2)} = (C_{(2,1)} : a_{(2,2)})$ and $K_{(3,1)} = (C_{(2,2)} : a_{(3,1)})$. Notice that $x_{10}a_{(2,2)} = a_{(1,3)} \in C_{(2,1)}$ and $x_{16}a_{(2,2)} = a_{(1,2)} \in C_{(2,1)}$. By the definitions of $a_{(j,k)}$ and of a path, all generators of $C_{(j,k)}$ will be products of consecutive vertices for all allowable j, k. Thus if $ma_{(2,2)} \in C_{(2,1)}$ for some monomial m, either $m \in C_{(2,1)}$, or m is divisible by x_{10} or x_{16} since those are the two vertices adjacent to the consecutive vertices appearing in $a_{(2,2)}$. Thus

$$K_{(2,2)} = (x_1x_2x_3x_4x_5x_6x_7, x_2x_3x_4x_5x_6x_7x_8, x_3x_4x_5x_6x_7x_8x_9, x_{10}, x_{16}).$$

Similarly $x_{11}a_{(3,1)} = a_{(2,2)} \in C_{(2,2)}$ and $x_{16}a_{(3,1)} = a_{(2,1)} \in C_{(2,2)}$, so $x_{11}, x_{16} \in K_{(3,1)}$, and

$$K_{(3,1)} = (x_1x_2x_3x_4x_5x_6x_7, \dots, x_4x_5x_6x_7x_8x_9x_{10}, x_{11}, x_{16}).$$

Proposition 3.4. *The family of ideals $K_{(j,k)}$ has the explicit formulation*

$$K_{(j,k)} = (P_{(\ell,n-\ell-k-1)}, x_{n-\ell-k}, x_{n-j-k+2}). \tag{3-3}$$

Proof. First notice that for $1 \le k \le c$, both $x_{n-\ell-k}a_{(1,k)}$ and $x_{n-k+1}a_{(1,k)}$ are generators of $I = P_{(\ell,n)}$, so $(C_{(1,k-1)}, x_{n-\ell-k}, x_{n-k+1}) \subseteq (C_{(1,k-1)} : a_{(j,k)})$, where $C_{(1,0)} = P_{(\ell,n)}$. The other inclusion is straightforward, so removing redundant elements from the list of generators yields the desired result for $j = 1$. Assume $j \geq 2$. By the definition of the sequence $\{a_{(j,k)}\}$, we have

$$x_{n-\ell-1}\, a_{(j,1)} = x_{n-\ell-1} \prod_{t=n-\ell}^{n-j} x_t = \prod_{t=n-\ell-1}^{n-(j-1)-2+1} x_t = a_{(j-1,2)},$$

and similarly $x_{n-j+1}\, a_{(j,1)} = a_{(j-1,1)}$. These equalities show that $x_{n-\ell-1}, x_{n-j+1} \in K_{(j,1)} = (C_{(j-1,c-(j-1)+1)} : a_{(j,1)})$. Since all generators of $C_{(j-1,c-(j-1)+1)}$ are products of consecutive vertices and $x_{n-\ell-1}, x_{n-j+1}$ are the only vertices that extend the consecutive path of vertices of $a_{(j,1)}$, we have

$$K_{(j,1)} = (C_{(j-1,c-(j-1)+1)}, x_{n-\ell-1}, x_{n-j+1}) = (P_{(\ell,n-\ell-1-1)}, x_{n-\ell-1}, x_{n-j+1}),$$

as desired, where the last equality follows from removing redundant generators.

Finally, if $k \geq 2$, we have $(a_{(j-1,k+1)} : a_{(j,k)}) = (x_{n-\ell-k})$ and $(a_{(j-1,k)} : a_{(j,k)}) = (x_{n-j-k+2})$. Thus for $k \geq 2$, $x_{n-\ell-k}, x_{n-j-k+2} \in K_{(j,k)} = (C_{(j,k-1)} : a_{(j,k)})$. Thus as before

$$K_{(j,k)} = (C_{(j,k-1)}, x_{n-\ell-k}, x_{n-j-k+2}) = (P_{(\ell,n-\ell-k-1)}, x_{n-\ell-k}, x_{n-j-k+2}). \qquad \square$$

Given this explicit form for $K_{(j,k)}$, it is easy to see that the depth of $K_{(j,k)}$ can be found inductively from the depth of the path ideal of a shorter spine. Thus the lemma below will allow us to simultaneously control the depth of each of the left-hand terms of the series of sequences. The proof is a direct application of [Morey 2010, Lemma 2.2] and thus is omitted.

Lemma 3.5. *For all j and k,*

$$\operatorname{depth}(R/K_{(j,k)}) = \operatorname{depth}(R_{n-\ell-k-1}/P_{(\ell,n-\ell-k-1)}) + \ell + k - 1.$$

We now need to control the depth of the final term of the final sequence. The nature of this proof will allow us to simultaneously handle the case $c = 0$, which was omitted above. For convenience, we will denote the final $C_{(j,k)}$ by $I_{(1)}$ and the final $a_{(j,k)}$ by $a_{(1)}$ since the final values of j and k depend on the relationship between c and ℓ. Explicitly, define

$$I_{(1)} = \begin{cases} I & \text{if } c = 0, \\ C_{(c,1)} & \text{if } 1 \leq c \leq \ell, \\ C_{(\ell,2)} & \text{if } c = \ell + 1, \end{cases} \qquad a_{(1)} = \begin{cases} a_{(c,1)} & \text{if } 1 \leq c \leq \ell, \\ a_{(\ell,2)} & \text{if } c = \ell + 1, \end{cases}$$

Note that since $I_{(1)}$ is used to denote the final $C_{(j,k)}$, we have

$$I_{(1)} = (I, a_{(1,1)}, a_{(1,2)}, \ldots, a_{(j,k)}),$$

where $I = P_{(\ell,n)}$ and all elements of the sequence $\{a_{(j,k)}\}$ are included in $I_{(1)}$.

The first two cases to consider follow directly from the definition of $C_{(j,k)}$ and an application of [Morey 2010, Lemma 2.2].

Lemma 3.6. *If* $c=\ell$, *then* $\operatorname{depth}(R/I_{(1)})=\operatorname{depth}(R_{n-\ell-1}/P_{(\ell,n-\ell-1)})+\ell$.

Proof. Notice that when $c=\ell$, $a_{(c,1)}=x_{n-\ell}$. Also note that $n-\ell-k+1\le n-\ell$ and since $k\le c-j+1$, $n-j-k+1\ge n-\ell$ when $c=\ell$. Thus $a_{(j,k)}=\prod_{t=n-\ell-k+1}^{n-j-k+1} x_t$ is a multiple of $x_{n-\ell}$ for all j,k when $c=\ell$. Thus $I_{(1)}=C_{(c,1)}=(P_{(\ell,n-\ell-1)},x_{n-\ell})$ and the result follows from [Morey 2010, Lemma 2.2]. □

Lemma 3.7. *If* $c=\ell+1$, *then* $\operatorname{depth}(R/I_{(1)})=\operatorname{depth}(R_{n-\ell-2}/P_{(\ell,n-\ell-2)})+\ell+1$.

Proof. Notice that when $c=\ell+1$, $a_{(\ell,1)}=x_{n-\ell}$ and $a_{(\ell,2)}=x_{n-\ell-1}$. As before, $a_{(j,k)}$ is a multiple of $x_{n-\ell}$ or of $x_{n-\ell-1}$ for all j,k, and thus the result follows from [Morey 2010, Lemma 2.2]. □

Finding the depth of $I_{(1)}$ for $0\le c\le \ell-1$ will require another family of short exact sequences. Define a sequence of monomials by $b_{(h)}=\prod_{t=n-\ell+h}^{n-c} x_t$ for $1\le h\le \ell-c$.

Example 3.8. As in Example 3.2 assume $n=18$, $\ell=6$, $b=1$, and $c=3$. Then $\{b_{(h)}\}=\{x_{13}x_{14}x_{15},\ x_{14}x_{15},\ x_{15}\}$.

We again form a family of short exact sequences using the sequence $\{b_{(h)}\}$. For convenience, define $J_{(0)}=I_{(1)}$. Now define $J_{(h)}$ and $L_{(h)}$ by $J_{(h)}=(J_{(h-1)},b_{(h)})$ and $L_{(h)}=(J_{(h-1)}:b_{(h)})$. Then as in (3-1), we have the following family of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \to & R/L_{(1)} & \to & R/I_{(1)} & \to & R/J_{(1)} & \to & 0,\\
0 & \to & R/L_{(2)} & \to & R/J_{(1)} & \to & R/J_{(2)} & \to & 0,\\
0 & \to & R/L_{(3)} & \to & R/J_{(2)} & \to & R/J_{(3)} & \to & 0,\\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \to & R/L_{(l-c-1)} & \to & R/J_{(l-c-2)} & \to & R/J_{(l-c-1)} & \to & 0,\\
0 & \to & R/L_{(l-c)} & \to & R/J_{(l-c-1)} & \to & R/J_{(l-c)} & \to & 0.
\end{array} \tag{3-4}$$

Note that for each h, $b_{(h)}=x_{n-\ell+h}b_{(h+1)}$ and $a_{(1)}=x_{n-\ell}b_{(1)}$ where $a_{(1)}$ is the final term for the original sequence when $0<c\le\ell-1$ and $a_{(1)}=\prod_{t=n-\ell}^{n} x_t$ is the last generator of I when $c=0$. Now $J_{(\ell-c)}=(I_{(1)},b_{(1)},\dots,b_{(\ell-c)})=(I_{(1)},x_{n-c})$ since $b_{(\ell-c)}=x_{n-c}$ and $b_{(h)}$ is a multiple of $b_{(\ell-c)}$ for all other h. Now each $a_{(j,k)}$ is a multiple of x_{n-c}, and $I_{(1)}=(I,a_{(1,1)},\dots,a_{(j,k)})$, so removing redundant elements from the generating set yields $J_{(\ell-c)}=(P_{(\ell,n-c-1)},x_{n-c})$. Similarly $L_{(h)}=(P_{(\ell,n-\ell+h-2)},x_{n-\ell+h-1})$. Using these explicit forms of $J_{(\ell-c)}$ and $L_{(h)}$, combined with [Morey 2010, Lemma 2.2], we are able to express the depths of all of the left-hand terms and the final right-hand term of the sequences in (3-4) in terms of the depths of path ideals of shorter spines. Note that by the definition of $b_{(h)}$, we will assume $c\le\ell-1$ whenever we are dealing with $J_{(h)}$ or $L_{(h)}$.

Lemma 3.9. *For all* h, $\operatorname{depth}(R/L_{(h)}) = \operatorname{depth}(R_{n-\ell+h-2}/P_{(\ell,n-\ell+h-2)}) + \ell - h + 1$ *and* $\operatorname{depth}(R/J_{(\ell-c)}) = \operatorname{depth}(R_{n-c-1}/P_{(\ell,n-c-1)}) + c$.

We are now able to prove the main result regarding the depth of a path ideal of a spine.

Theorem 3.10. *Let* S_n *be a spine of* n *vertices. Then*

$$\operatorname{depth}(R/P_\ell(S_n)) = \operatorname{depth}(R/P_{(\ell,n)}) = \begin{cases} \ell(b+1) & \text{if } c = 0, \\ \ell(b+1) + c - 1 & \text{if } c > 0. \end{cases}$$

Proof. We assume ℓ is fixed and induct on n. If $n \leq \ell$, we have $b = -1$ and $c = n+1$. By Lemma 3.1, we have $\operatorname{depth}(R/P_{(\ell,n)}) = n$ and $\ell(b+1) + c - 1 = \ell(0) + n + 1 - 1 = n$, so the result holds.

Assume $n \geq \ell + 1$. When writing $n = (\ell+1) + b(\ell+2) + c$, notice that for $n \geq 0$, $b = -1$ if and only if $n \leq \ell$. Thus for $n \geq \ell+1$, $b \geq 0$. In the proof that follows, we will be working with $n - t$ for various values of t. When $b = 0$, this will often result in $n - t \leq \ell$. While this situation can easily be handled using separate cases, allowing $b - 1 = -1$ creates a more streamlined proof.

Suppose $0 \leq c \leq \ell - 1$. Then by Lemma 3.9,

$$\operatorname{depth}(R/L_{(h)}) = \operatorname{depth}(R_{n-\ell+h-2}/P_{(\ell,n-\ell+h-2)}) + \ell - h + 1,$$
$$\operatorname{depth}(R/J_{(\ell-c)}) = \operatorname{depth}(R_{n-c-1}/P_{(\ell,n-c-1)}) + c.$$

Recall that $P_{(\ell,n-\ell+h-2)} = P_\ell(S_{n-\ell+h-2})$. Since $h \leq \ell - c$, we have $n - \ell + h - 2 < n$. As ℓ has remained fixed, our inductive hypothesis on the number of vertices for a fixed path length applies. Thus by induction, if the division algorithm is used to write $n - \ell + h - 2 = \ell + 1 + b'(\ell+2) + c'$ for some integers b' and c' with $0 \leq c' < \ell + 2$, then $\operatorname{depth}(R_{n-\ell+h-2}/P_{(\ell,n-\ell+h-2)}) = \ell(b'+1) + c' - 1$ if $c' > 0$. Since $1 \leq h \leq \ell - c$ then $0 < c + h \leq \ell$. Now $n - \ell + h - 2 = \ell + 1 + (b-1)(\ell+2) + c + h$. Thus by induction,

$$\operatorname{depth}(R_{n-\ell+h-2}/P_{(\ell,n-\ell+h-2)}) = \ell((b-1)+1) + (c+h) - 1,$$

so Lemma 3.9 yields

$$\operatorname{depth}(R/L_{(h)}) = \ell(b) + c + h - 1 + \ell - h + 1 = \ell(b+1) + c.$$

Also by induction, using a similar argument on the number of vertices,

$$\operatorname{depth}(R_{n-c-1}/P_{(\ell,n-c-1)}) = \ell(b-1+1) + (\ell+1) - 1 = \ell(b+1)$$

since $n - c - 1 = (\ell+1) + (b-1)(\ell+2) + \ell + 1$, so $\operatorname{depth}(R/J_{(\ell-c)}) = \ell(b+1) + c$. Now repeated use of the depth lemma applied to (3-4) yields $\operatorname{depth} R/I_{(1)} = \ell(b+1) + c$.

Suppose $c = \ell$. Then by Lemma 3.6 we have

$$\operatorname{depth}(R/I_{(1)}) = \operatorname{depth}(R_{n-\ell-1}/P_{(\ell,n-\ell-1)}) + \ell.$$

Then $n-\ell-1 = \ell+1+b(\ell+2)+\ell-\ell-1 = \ell+1+(b-1)(\ell+2)+\ell+1$. Thus applying the inductive hypothesis with $b' = b-1$ and $c' = \ell+1$ yields

$$\operatorname{depth}(R_{n-\ell-1}/P_{(\ell,n-\ell-1)}) = \ell(b-1+1)+(\ell+1)-1,$$

so $\operatorname{depth}(R/I_{(1)}) = \ell b+\ell+\ell = \ell(b+1)+c$.

If $c = \ell+1$, then by Lemma 3.7 we have

$$\operatorname{depth}(R/I_{(1)}) = \operatorname{depth}(R_{n-\ell-2}/P_{(\ell,n-\ell-2)}) + \ell+1.$$

Then $n-\ell-2 = \ell+1+(b-1)(\ell+2)+c$, so by induction,

$$\operatorname{depth}(R_{n-\ell-2}/P_{(\ell,n-\ell-2)}) = \ell(b-1+1)+c-1,$$

and $\operatorname{depth}(R/I_{(1)}) = \ell(b)+c-1+\ell+1 = \ell(b+1)+c$.

We now have $\operatorname{depth}(R/I_{(1)}) = \ell(b+1)+c$ for all possible values of c. Notice that if $c = 0$, we have $P_{(\ell,n)} = I_{(1)}$ and $\operatorname{depth}(R/P_{(\ell,n)}) = \ell(b+1)$ for any b, and the result holds. Thus we may now assume $c > 0$ for the remainder of the proof.

By Lemma 3.5, for all j, k,

$$\operatorname{depth}(R/K_{(j,k)}) = \operatorname{depth}(R_{n-\ell-k-1}/P_{(\ell,n-\ell-k-1)}) + \ell+k-1.$$

Now if $n = (\ell+1)+b(\ell+2)+c$, then $n-\ell-k-1 = (\ell+1)+(b-1)(\ell+2)+c-k+1$. Notice that $c-k+1 > 0$ since $k \le c-j+1$. Thus we have

$$\operatorname{depth}(R_{n-\ell-k-1}/P_{(\ell,n-\ell-k-1)}) = \ell(b-1+1)+c-k+1-1 = \ell(b)+c-k$$

by induction. Then $\operatorname{depth}(R/K_{(j,k)}) = \ell(b)+c-k+\ell+k-1 = \ell(b+1)+c-1$. Now repeated application of the depth lemma to the sequences in (3-1) yields $\operatorname{depth}(R/P_{(\ell,n)}) = \ell(b+1)+c-1$ when $c > 0$. □

There are some interesting reformulations of the depth found in Theorem 3.10. They are stated here without proof as the proofs are basic computations and summation arguments.

Corollary 3.11. *Theorem 3.10 can be reformulated as*

$$\operatorname{depth}(R/P_{(\ell,n)}) = \begin{cases} m\ell & \text{if } \ell \le (n-2m+2)/m, \\ n-2m+2 & \text{if } \ell > (n-2m+2)/m, \end{cases}$$

where $m = \lceil n/(\ell+2) \rceil$, *or as*

$$\operatorname{depth}(R/P_{(\ell,n)}) = \sum_{i=0}^{\ell-1} \left\lceil \frac{n-i}{\ell+2} \right\rceil.$$

Notice that when ℓ is large relative to n, the depth of $R/P_{(\ell,n)}$ is large. If $\ell > n$, then the depth is n, as was noted in Lemma 3.1. However it is interesting to note that as long as ℓ is roughly half of n or larger, the depth remains quite large.

Corollary 3.12. *If* $\ell \geq (n-2)/2$, *then* $\operatorname{depth}(R/P_{(\ell,n)}) = n-2$ *for* $\ell \neq n-1$ *and for* $\ell = n-1$, $\operatorname{depth}(R/P_{(\ell,n)}) = n-1$.

Proof. Since $\ell \geq (n-2)/2$, we have $b = 0$, where $n = (\ell+1) + b(\ell+2) + c$ and $c \leq \ell+1$. By Theorem 3.10, if $c = 0$, $\operatorname{depth}(R/P_{(\ell,n)}) = \ell(b+1) = \ell = n-1$ and if $c > 0$, then $\operatorname{depth}(R/P_{(\ell,n)}) = \ell(b+1) + c - 1 = \ell + c - 1 = n-2$. □

To determine when the ideal is Cohen–Macaulay, the dimension is first needed. Since that is of independent interest, it is stated separately.

Lemma 3.13. *If* $I = P_{(\ell,n)}$, *then* $\dim(R/I) = n - \lfloor n/(\ell+1) \rfloor$.

Proof. Let $m = \lfloor n/(\ell+1) \rfloor$. The set of vertices $M = \{x_{\ell+1}, x_{2\ell+2}, \ldots, x_{m\ell+m}\}$ forms a minimal vertex cover of minimal cardinality of $I = P_{(\ell,n)}$, so $\operatorname{height}(I) = \lfloor n/(\ell+1) \rfloor$. Since R is a polynomial ring of dimension n, $\dim(R/I) = n - \lfloor n/(\ell+1) \rfloor$. □

Proposition 3.14. *Let* $I = P_{(\ell,n)}$. *Then* R/I *is Cohen–Macaulay if and only if* $n = \ell+1$ *or* $n = 2\ell+2$.

Proof. Let $I = P_{(\ell,n)}$. If $n = 2\ell+2$, then by Lemma 3.13,

$$\dim(R/I) = n - \left\lfloor \frac{2\ell+2}{\ell+1} \right\rfloor = n-2,$$

and by Corollary 3.12, $\operatorname{depth}(R/P_{(\ell,n)}) = n-2$. Thus $R/P_{(\ell,n)}$ is Cohen–Macaulay. For $n = \ell+1$, Lemma 3.13 yields

$$\dim(R/P_{(\ell,n)}) = n - \left\lfloor \frac{\ell+1}{\ell+1} \right\rfloor = n-1,$$

and Corollary 3.12 gives $\operatorname{depth}(R/P_{(\ell,n)}) = n-1$, which again shows that $R/P_{(\ell,n)}$ is Cohen–Macaulay.

For the converse, consider $n = \ell+1+b(\ell+2)+c$ with $0 \leq c < \ell+2$. By Lemma 3.13,

$$\dim(R/I) = n - \left\lfloor \frac{n}{\ell+1} \right\rfloor = \left\lceil \frac{n(\ell+1)-n}{\ell+1} \right\rceil = (b+1)\ell + \left\lceil \frac{(b+c)\ell}{\ell+1} \right\rceil.$$

If $c = 0$ then $\operatorname{depth}(R/I) = \ell(b+1)$ by Theorem 3.10. If R/I is Cohen–Macaulay, then $(b+1)\ell + \lceil b\ell/(\ell+1) \rceil = \ell(b+1)$. Thus $\lceil b\ell/(\ell+1) \rceil = 0$, or $b = 0$. Since $b = c = 0$, we have $n = \ell+1$.

If $c > 0$, then $\operatorname{depth}(R/I) = \ell(b+1) + c - 1$ by Theorem 3.10. If R/I is Cohen–Macaulay, then

$$(b+1)\ell + \left\lceil \frac{(b+c)\ell}{\ell+1} \right\rceil = \ell(b+1) + c - 1 \quad \text{or} \quad \left\lceil \frac{(b+c)\ell}{\ell+1} \right\rceil = c-1.$$

Now

$$\left\lceil \frac{(b+c)\ell}{\ell+1} \right\rceil = b+c-\left\lfloor \frac{b+c}{\ell+1} \right\rfloor,$$

so we have $b - \lfloor (b+c)/(\ell+1) \rfloor = -1$. If $b \geq 1$, the left side of this equation is nonnegative, a contradiction. If $b = 0$, then $-\lfloor (b+c)/(\ell+1) \rfloor = -1$ if and only if $c = \ell+1$ since $c < \ell+2$. Thus if $c > 0$, R/I is Cohen–Macaulay if and only if $b = 0$ and $c = \ell+1$. In this case $n = 2\ell+2$.

Thus if R/I is Cohen–Macaulay, $n = \ell+1$ or $n = 2\ell+2$. □

Proposition 3.14 is particularly interesting when compared to [Campos et al. 2014, Theorem 3.8]. In fact, in each of the two instances where the path ideal of a spine is Cohen–Macaulay, the graph can be viewed as a suspension. When $n = 2\ell+2$, $P_{(\ell,n)}$ is the suspension of length ℓ of a graph that consists of a single edge connecting two vertices $(x_{n/2}, x_{n/2+1})$ and when $n = \ell+1$, $P_{(\ell,n)}$ is the suspension of length ℓ of a graph that consists of a single isolated vertex (x_n).

Note that the arguments in Proposition 3.14 can be used to determine the *Cohen–Macaulay defect*, that is $\dim(R/I) - \operatorname{depth}(R/I)$, for a path ideal. For example, if $\ell+1 < n < 2\ell+2$, $\operatorname{depth}(R/P_{(\ell,n)}) = n-2$ and $\dim(R/P_{(\ell,n)}) = n-1$ so the Cohen–Macaulay defect is 1.

4. Stanley depths of path ideals of spines

As remarked before, Theorem 3.10 together with the Auslander–Buchsbaum formula, recovers the projective dimension found in [He and Van Tuyl 2010, Theorem 4.1] and in [Bouchat et al. 2011, Corollary 5.1]. However, the method of proof has the advantage of also yielding information about the Stanley depth. There are three key factors that allow us to extend the depth result to a lower bound on the Stanley depth, or s-depth for brevity. The first two are well-known basic facts. If I is a monomial ideal of a polynomial ring R and y is an indeterminate, then

$$\text{s-depth}(R[y]/IR[y]) = \text{s-depth}(R/I) + 1, \tag{4-1}$$

and $\text{s-depth}(R) = n$ when R is a polynomial ring in n variables. The third result we will need is that s-depth satisfies a partial version of the depth lemma. In particular, it was shown in [Rauf 2010, Lemma 2.2] that if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of finitely generated R modules then

$$\text{s-depth}(B) \geq \min\{\text{s-depth}(A), \text{s-depth}(C)\}.$$

Now by carefully examining the proof of Theorem 3.10, we are able to extend the result to a lower bound on the s-depth of the path ideal of a spine. Note that the explicit calculations closely follow those of Theorem 3.10 and so details have been condensed in the proof.

Theorem 4.1. *Let S_n be a spine on n vertices. Then*

$$\text{s-depth}(R/P_\ell(S_n)) = \text{s-depth}(R/P_{(\ell,n)}) \geq \begin{cases} \ell(b+1) & \text{if } c=0, \\ \ell(b+1)+c-1 & \text{if } c>0. \end{cases}$$

Proof. We assume ℓ is fixed and induct on n. Write $n=(\ell+1)+b(\ell+2)+c$. If $n \leq \ell$, $\text{s-depth}(R/P_{(\ell,n)}) = \text{s-depth}(R) = n$ and $\ell(b+1)+c-1 = \ell(0)+n+1-1 = n$ and the result holds. Define the sequences $a_{(j,k)}$ and $b_{(h)}$ and the related ideals $K_{(j,k)}$, $C_{(j,k)}$, $L_{(h)}$, $J_{(h)}$ and $I_{(1)}$ as before. By Proposition 3.4,

$$\text{s-depth}(R/K_{(j,k)}) = \text{s-depth}(R_{n-\ell-k-1}/P_{(\ell,n-\ell-k-1)}) + \ell + k - 1,$$

and by induction

$$\text{s-depth}(R_{n-\ell-k-1}/P_{(\ell,n-\ell-k-1)}) \geq \ell(b-1+1)+c-k+1-1 = \ell(b)+c-k,$$

so $\text{s-depth}(R/K_{(j,k)}) \geq \ell(b)+c-k+\ell+k-1 = \ell(b+1)+c-1$.

If $c=\ell$ or $c=\ell+1$, then as in Lemma 3.6 or Lemma 3.7 with [Morey 2010, Lemma 2.2] replaced by (4-1),

$$\text{s-depth}(R/I_{(1)}) = \text{s-depth}(R_{n-\ell-1}/P_{(\ell,n-\ell-1)}) + \ell$$

when $c=\ell$, and

$$\text{s-depth}(R/I_{(1)}) = \text{s-depth}(R_{n-\ell-2}/P_{(\ell,n-\ell-2)}) + \ell + 1$$

when $c=\ell+1$. In either case, applying the inductive hypothesis as in Theorem 3.10 yields

$$\text{s-depth}(R/I_{(1)}) \geq \ell(b+1)+c.$$

Suppose $0 \leq c \leq \ell-1$. Then as in Lemma 3.9 with [loc. cit., Lemma 2.2] replaced by (4-1),

$$\text{s-depth}(R/L_{(h)}) = \text{s-depth}(R_{n-\ell+h-2}/P_{(\ell,n-\ell+h-2)}) + \ell - h + 1,$$
$$\text{s-depth}(R/J_{(\ell-c)}) = \text{s-depth}(R_{n-c-1}/P_{(\ell,n-c-1)}) + c.$$

As in Theorem 3.10, applying the inductive hypothesis yields

$$\text{s-depth}(R_{n-\ell+h-2}/P_{(\ell,n-\ell+h-2)}) \geq \ell((b-1)+1)+(c+h)-1,$$

so $\text{s-depth}(R/L_{(h)}) \geq \ell(b+1)+c$. Also by induction

$$\text{s-depth}(R_{n-c-1}/P_{(\ell,n-c-1)}) \geq \ell(b-1+1)+(\ell+1)-1 = \ell(b+1),$$

so $\text{s-depth}(R/J_{(\ell-c)}) \geq \ell(b+1)+c$. Now repeated use of [Rauf 2010, Lemma 2.2] applied to (3-4) yields

$$\text{s-depth } R/I_{(1)} \geq \ell(b+1)+c.$$

Notice that if $c = 0$, we have $P_{(\ell,n)} = I_{(1)}$ and s-depth$(R/P_{(\ell,n)}) \geq \ell(b+1)$ for any b, and the result holds. For $c > 0$, repeated application of [loc. cit., Lemma 2.2] to the sequences in (3-1) yields s-depth$(R/P_{(\ell,n)}) \geq \ell(b+1)+c-1$. □

A monomial ideal I is a Stanley ideal if the Stanley conjecture holds for I. That is, if s-depth$(R/I) \geq$ depth(R/I). Due to the general difficulty of computing the Stanley depth, very few classes of Stanley ideals are known. It is interesting to note that Theorem 4.1 provides a new class of Stanley ideals.

Corollary 4.2. *Let S_n be a spine of n vertices. Then $P_\ell(S_n)$ is a Stanley ideal.*

Proof. This follows directly from Theorems 3.10 and 4.1. □

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