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Mutual estimates for the dyadic reverse Hölder and Muckenhoupt constants for the dyadically doubling weights

Oleksandra V. Beznosova and Temitope Ode

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Muckenhoupt and reverse Hölder classes of weights play an important role in harmonic analysis, PDEs and quasiconformal mappings. In 1974, Coifman and Fefferman showed that a weight belongs to a Muckenhoupt class A_p for some $1 < p < \infty$ if and only if it belongs to a reverse Hölder class RH_q for some $1 < q < \infty$. In 2009, Vasyunin found the exact dependence between p, q and the corresponding characteristic of the weight using the Bellman function method. The result of Coifman and Fefferman works for the dyadic classes of weights under an additional assumption that the weights are dyadically doubling. We extend Vasyunin's result to the dyadic reverse Hölder and Muckenhoupt classes and obtain the dependence between p, q, the doubling constant and the corresponding characteristic of the weight. More precisely, given a dyadically doubling weight in RH_p^d on a given dyadic interval I, we find an upper estimate on the average of the function w^q (with $q < 0$) over the interval I. From the bound on this average, we can conclude, for example, that w belongs to the corresponding $A_{q_1}^d$-class or that w^p is in $A_{q_2}^d$ for some values of q_i. We obtain our results using the method of Bellman functions. The main novelty of this paper is how we use dyadic doubling in the Bellman function proof.

1. Definitions and main results

We will be dealing with a family of dyadic intervals on the real line,

$$D := \{[n2^{-k}, (n+1)2^{-k}] : n, k \in \mathbb{Z}\}.$$

For an interval J, let $D(J)$ stand for the family of all its dyadic subintervals, $D(J) := \{I \in D : I \subset J\}$ and let $D_n(J)$ stand for the family of all dyadic subintervals

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of J of length exactly $2^{-n}|J|$. For a locally integrable function f, let $\langle f\rangle_I$ stand for the average of f over the interval I,

$$\langle f\rangle_I := \frac{1}{|I|}\int_I f(x)\,dx,$$

where $|I|$ is the Lebesgue measure of I.

Let w be a weight; i.e., w is a locally integrable, almost everywhere nonnegative function which is not identically zero. Since we will be dealing mostly with averages, we define the dyadic doubling constant of the weight w to be

$$\mathrm{Db}^d(w) := \inf_{I\in D}\{C : \langle w\rangle_{I^*} \leqslant C\langle w\rangle_I\} = \tfrac{1}{2}\inf_{I\in D}\left\{C : \int_{I^*} w(x)\,dx \leqslant C\int_I w(x)\,dx\right\},$$

where I^* is the dyadic "parent" of the interval I, i.e., the smallest dyadic interval that strictly contains the interval I. If the dyadic doubling constant of the weight w is bounded by Q, we will say that $w\in \mathrm{Db}^{d,Q}$. Note also that any weight is positive almost everywhere; therefore the dyadic doubling constant defined this way is always greater than $\frac{1}{2}$.

Our main assumption is that a weight w belongs to the dyadic reverse Hölder class on the interval J with the corresponding constant bounded by δ:

$$w\in RH_p^{\delta,d}(J) \iff [w]_{RH_p^{\delta,d}(J)} := \sup_{I\in D(J)}\{C : \langle w^p\rangle_I^{1/p} \leqslant C\langle w\rangle_I\} \leqslant \delta.$$

We define $A_q^{\delta,d}(J)$ to be the class of the dyadic Muckenhoupt weights on the interval J with the corresponding constant bounded by δ:

$$w\in A_q^{\delta,d}(J) \iff [w]_{A_q^{\delta,d}(J)} := \sup_{I\in D(J)}\langle w\rangle_I\langle w^{-1/(q-1)}\rangle_I^{q-1} \leqslant \delta.$$

Given a dyadically doubling weight $w\in RH_p^{\delta,d}(J)$, our goal in this paper is to bound the averages involved in the definitions of $w\in A_{q_1}^d$ and $w^p\in A_{q_2}^d$,

$$\langle w\rangle_J\langle w^{-1/(q_1-1)}\rangle_J^{q_1-1} \quad\text{and}\quad \langle w^p\rangle_J\langle w^{-p/(q_2-1)}\rangle_J^{q_2-1}.$$

Note that the quantities $\langle w\rangle_J$ and $\langle w^p\rangle_J$ are involved in the definition of $RH_p^{\delta,d}(J)$; therefore for our goals, it is enough to bound $\langle w^q\rangle_J$ from above for $q<0$.

It is a well-known fact that $w\in A_q^d$ for some $1<q<\infty$ implies that w is a dyadically doubling weight; it is also known that in the dyadic case, the reverse Hölder classes RH_p^d contain weights that are not dyadically doubling (see [Buckley 1990]). In fact, if $w\in RH_p^{\delta,d}(J)$, nothing prevents w from being close or even equal to 0 on, say, the left half of J; the local $RH_p^d(J)$-constant can be defined for such weights. There is no way to define an $A_q^d(J)$-constant for such a weight, and even the quantity $\langle w^q\rangle_J$ is undefined for $q<0$, which is the case considered in this paper. What prevents this from happening is the doubling assumption that does not allow $\langle w\rangle_J$

to be too far from $\langle w\rangle_{J^\pm}$, and therefore if w is equal to 0 on any dyadic subinterval of J then w has to be identically 0 on the whole interval J (which is not permitted).

We are ready to define the Bellman function for our problem: for $p>1$, $q<0$ and $Q>2$, let

$$\mathcal{B}(x_1,x_2;p,q,\delta,Q):=\sup_{w\in RH_p^{\delta,d}(J),\,\mathrm{Db}^d(w)\leqslant Q}\Big\{\langle w^q\rangle_J : w \text{ is s.t. } \langle w\rangle_J=x_1,\ \langle w^p\rangle_J=x_2\Big\}.$$

The parameters p, q, δ and Q will be fixed throughout the paper, so we will skip them and write $\mathcal{B}(x_1,x_2)$. Note also that by a rescaling argument, $\mathcal{B}$ does not depend on the interval J. The constant Q corresponds to the doubling constant of the weight w. We know that for any weight, we have $\mathrm{Db}^d(w)>\frac{1}{2}$. We take $Q>2$ for technical reasons (we need it in the proof), so one may think of Q as being the maximum of the doubling constant of the weight w and 2; that is, $Q:=\max\{\mathrm{Db}^d(w),2\}$.

Then for the given p, q, δ, and Q, we have that $\mathcal{B}$ is defined on the domain

$$U_\delta:=\Big\{\vec{x}=(x_1,x_2)\ :\ \exists w\in RH_p^{\delta,d} \text{ s.t. } \mathrm{Db}^d(w)\leqslant Q \text{ and } x_1=\langle w\rangle_J,\ x_2=\langle w^p\rangle_J\Big\}.$$

In order to state the main theorem, we need to define functions $u_p^\pm(t)$. Let $u_p^\pm(t)$ be two solutions (positive and negative) of the equation

$$(1-pu)^{1/p}(1-u)^{-1}=t,\qquad 0\leqslant t\leqslant 1. \tag{1-1}$$

For $Q\geqslant 2$, we define $\varepsilon(p,\delta,Q)$ as follows. Let

$$H:=H(p,Q)=\frac{Q^p-1}{Q-1}\quad\text{and}\quad \varepsilon:=\frac{H}{p}\left(\frac{p-1}{H-1}\right)^{(p-1)/p}\delta.$$

Then we can define

$$s^\pm(\varepsilon):=u^\pm\left(\frac{1}{\varepsilon}\right)\quad\text{and}\quad r^\pm:=u^\pm\left(\frac{y^{1/p}}{\varepsilon x}\right).$$

Note that since $u^+(t)$ is a decreasing function and in our domain

$$\frac{1}{\varepsilon}\leqslant\frac{y^{1/p}}{\varepsilon x},$$

we have that $r^+\in[0,s^+]$. Similarly, since $u^-(t)$ is an increasing function, we have that $r^-\in[s^-,0]$.

Theorem 1.1 (main theorem). *Let $p>1$, $q<0$, $Q\geqslant 2$ and $\delta>1$; let $s^-:=s^-(\varepsilon)$ for $\varepsilon(p,\delta,Q)$ defined above. If $q\in(1/s^-,0)$ then*

$$\mathcal{B}(x_1,x_2;p,q,\delta)\leqslant x_1^q\,\frac{1-qr^-}{1-qs^-}\left(\frac{1-s^-}{1-r^-}\right)^q=x_2^{q/p}\,\frac{1-qr^-}{1-qs^-}\left(\frac{1-ps^-}{1-pr^-}\right)^{q/p}.$$

The proof of Theorem 1.1 can be found in Section 2.

Note that the result from [Vasyunin 2008] assumes that the reverse Hölder inequality for the weight w holds for any interval $I \subset J$, while our Theorem 1.1 only uses dyadic subintervals $I \in D(J)$ and the doubling constant. Therefore our result is more general (in the sense that if a weight is in the continuous reverse Hölder class, it has to be in the dyadic class and it has to be doubling, so our theorem applies). Unfortunately, we lose the sharpness. Note also that Theorem 1.1 is not a straight-forward extension of Vasyunin's result because it fails in the case when the weight w is not dyadically doubling. The latter is easy to see: in his thesis, Buckley gave examples of weights in RH_p-classes that are not dyadically doubling and therefore do not belong to any of the A_p^d.

Let us consider the following simple example. Let $w(x) = \chi_{J^+}(x)$. Then $w \in RH_p^{d,2^{1-1/p}}(J)$ for all $1 < p < \infty$. At the same time, it is clearly impossible to bound $\langle w^q \rangle_J$ for $q < 0$. Note that this weight is not dyadically doubling, so the doubling assumption in Theorem 1.1 is necessary, and we have to find a way to use doubling in the Bellman function argument. Most of Vasyunin's proof works in the dyadic setting; it is Lemma 4 in his paper that fails and does not have a full size dyadic analogue. We replace Lemma 4 using a technique from [Pereyra 2009] to incorporate the doubling property of the weight in the Bellman function proof.

As a consequence of Theorem 1.1, we obtain the following corollary.

Corollary 1.2 (RH_p vs. A_q). *Let w be a reverse Hölder dyadically doubling weight with $[w]_{RH_p^d} = \delta$ and $Q := \max\{\mathrm{Db}^d(w), 2\}$. Let $\varepsilon(p, \delta, Q)$ be defined as above. Let $s^- = s^-(\varepsilon)$. Then*:

(i) *For every $q_1 > 1 - s^-$, we have $w \in A_{q_1}^d$, and moreover,*

$$[w]_{A_{q_1}^d} \leqslant \left(\frac{q_1 - 1}{q_1 - 1 + s^-}\right)^{q_1 - 1}.$$

(ii) *For every $q_2 > 1 - ps^-$, we have $w^p \in A_{q_2}^d$, and moreover,*

$$[w^p]_{A_{q_2}^d} \leqslant \left(\frac{q_2 - 1}{q_2 - 1 + ps^-}\right)^{q_2 - 1}.$$

Above, $s^-(\varepsilon)$ is the negative solution of the equation $(1 - ps^-)^{1/p}(1 - s^-)^{-1} = 1/\varepsilon$.

A result similar to the second part of the above corollary was used in [Beznosova et al. 2014] (without a proof) for the sharp norms of t-Haar multiplier operators. The difference is that in [loc. cit.], the ε was taken to be $\varepsilon_1 = Q\delta$, which is an upper bound for our $\varepsilon(p, \delta, Q)$.

The proof of Corollary 1.2 is very simple. Note that since $r^- \in [s^-, 0]$, we have $1 - r^- \leqslant 1 - s^-$; therefore

$$\frac{1 - s^-}{1 - r^-} \geqslant 1.$$

So, since $q < 0$, we have that

$$\left(\frac{1-s^-}{1-r^-}\right)^q \leqslant 1.$$

We also have that

$$\left(\frac{1-ps^-}{1-pr^-}\right)^{q/p} \leqslant 1$$

since p is positive. At the same time, since both q and r^- are negative, qr^- is positive and $1-qr^- \leqslant 1$. Therefore, for our choice of parameters, we have that

$$\langle w^q\rangle_J \leqslant \frac{\min\{\langle w\rangle_J^q, \langle w^p\rangle_J^{q/p}\}}{1-qs^-}.$$

Using this rough estimate in the definition of the corresponding Muckenhoupt constant, we get the desired bounds.

2. Proof of Theorem 1.1

In this section, we essentially follow the proof from [Vasyunin 2008]. Unfortunately, we cannot use the full proof from Vasyunin's paper since it relies on Lemma 4 from his paper, which fails in the dyadic case. We will sketch the proof, referring to Vasyunin's results whenever possible, and replace his Lemma 4 with our dyadically doubling analogue, Lemma 2.4.

We fix $p > 1$, , $q < 0$, $Q \geqslant 2$, $\delta > 1$ and let

$$\mathcal{B}(x_1, x_2; p, q, \delta, Q) := \sup_{w \in RH_p^{\delta,d}(J),\, \mathrm{Db}^d(w) \leqslant Q} \left\{\langle w^q\rangle_J : w \text{ is s.t. } \langle w\rangle_J = x_1, \langle w^p\rangle_J = x_2\right\}$$

and

$$B_{\max} = B_{\max}(x_1, x_2; p, q, \delta, Q) := x_1^q \frac{1-qr^-}{1-qs^-}\left(\frac{1-s^-}{1-r^-}\right)^q$$

be defined on the domains

$$U_\delta = \left\{\vec{x} = (x_1, x_2) : \exists w \in RH_p^{\delta,d} \text{ s.t. } \mathrm{Db}^d(w) \leqslant Q \text{ and } x_1 = \langle w\rangle_J, x_2 = \langle w^p\rangle_J\right\}$$

and

$$\Omega_\varepsilon := \left\{\vec{x} = (x_1, x_2) : x_i > 0,\ x_1^p \leqslant x_2 \leqslant \varepsilon^p x_1^p\right\}$$

respectively. Please note that U_δ and Ω_ε here are two domains defined in two different ways. In Lemma 2.4, we show that one is contained in the other and any line segment that connects points in U_δ that correspond to the same weight and dyadic interval has to lie inside the enlarged domain Ω_ε. This part is the main difference between the continuous and the dyadic case.

Note that

$$x_1^q \frac{1-qr^-}{1-qs^-}\left(\frac{1-s^-}{1-r^-}\right)^q = x_2^{q/p}\frac{1-qr^-}{1-qs^-}\left(\frac{1-ps^-}{1-pr^-}\right)^{q/p}$$

by the definitions of s^- and r^-.

Our goal is to show that $\mathcal{B} \leqslant B_{\max}$. We will prove it using the Bellman function method. The proof consists of the following parts, which we will now state in the form of lemmata.

Lemma 2.1. *If the function $B_{\max}$, defined above, is concave on the domain Ω_δ, i.e.,*

$$B_{\max}\left(\frac{x^- + x^+}{2}\right) \geqslant \frac{B_{\max}(x^-) + B_{\max}(x^+)}{2} \tag{2-1}$$

for any x^+ and x^- such that there exists a weight $w \in RH_p^{\delta,d}$ with $\mathrm{Db}^d(w) \leqslant Q$, where $x^+ = (\langle w\rangle_{J^+}, \langle w^p\rangle_{J^+})$ and $x^- = (\langle w\rangle_{J^-}, \langle w^p\rangle_{J^-})$, then Theorem 1.1 holds.

Lemma 2.2. *The function $B_{\max}$ is locally concave on the domain Ω_ε; i.e., its Hessian matrix*

$$d^2 B_{\max} = \left\{\frac{\partial^2 B_{\max}}{\partial x_1 \partial x_2}\right\}$$

is not positive definite.

Lemma 2.3. *Let $x^o, x^+, x^- \in U_\delta$, where $x^o = \frac{1}{2}(x^+ + x^-)$ and the line segment connecting x^+ and x^- lies completely inside the larger domain Ω_ε. Suppose that the function $B_{\max}$ is locally convex on Ω_ε; i.e., on Ω_ε we have that the Hessian $d^2 B_{\max}$ is not positive definite. Then the inequality (2-1) holds.*

Lemma 2.4. *Let x^o, x^+, and x^- be three points in Ω_δ with the property that $x^o = \frac{1}{2}(x^+ + x^-)$ such that there is a weight $w \in RH_p^{\delta,d}$ with $\mathrm{Db}^d(w) \leqslant Q$ and a dyadic interval I such that*

$$x_1^o = \langle w\rangle_I, \qquad x_2^o = \langle w^p\rangle_I,$$
$$x_1^\pm = \langle w\rangle_{I^\pm}, \quad x_2^\pm = \langle w^p\rangle_{I^\pm}.$$

Then the line segment connecting x^+ and x^- lies completely inside the larger domain Ω_ε.

Proof of Lemma 2.1. First, observe that if a weight w is constant on the interval J, say $w = c$, then $\langle w^q\rangle_J = \langle w\rangle_J^q = \langle w^p\rangle_J^{q/p}$; therefore in this case, $\mathcal{B} \leqslant B_{\max}$.

Now let w be a step function. Note that for any dyadic interval I, we have that $\langle w\rangle_I = \frac{1}{2}\big(\langle w\rangle_{I^+} + \langle w\rangle_{I^-}\big)$ and $\langle w^p\rangle_I = \frac{1}{2}\big(\langle w^p\rangle_{I^+} + \langle w^p\rangle_{I^-}\big)$. This, together with the concavity of $B_{\max}$, gives

$$\begin{aligned}
&|J|\,B_{\max}\big(\langle w\rangle_J, \langle w^p\rangle_J\big)\\
&\qquad \geqslant |J^-|\,B_{\max}\big(\langle w\rangle_{J^-}, \langle w^p\rangle_{J^-}\big) + |J^+|\,B_{\max}\big(\langle w\rangle_{J^+}, \langle w^p\rangle_{J^+}\big)\\
&\qquad \geqslant |J^{--}|\,B_{\max}\big(\langle w\rangle_{J^{--}}, \langle w^p\rangle_{J^{--}}\big) + |J^{-+}|\,B_{\max}\big(\langle w\rangle_{J^{-+}}, \langle w^p\rangle_{J^{-+}}\big)\\
&\qquad\quad + |J^{+-}|\,B_{\max}\big(\langle w\rangle_{J^{+-}}, \langle w^p\rangle_{J^{+-}}\big) + |J^{++}|\,B_{\max}\big(\langle w\rangle_{J^{++}}, \langle w^p\rangle_{J^{++}}\big)\\
&\qquad \geqslant \cdots \geqslant \sum_{I\in D_n(J)} |I|\,B_{\max}\big(\langle w\rangle_I, \langle w^p\rangle_I\big).
\end{aligned}$$

Now note that since w is a step function, it has at most finitely many jumps. Let the number of jumps be m. For n large enough, in the last formula we have that w is constant on $2^n - m$ subintervals $I \in D_n(J)$ (we will call these subintervals "good") and has jump discontinuities on the other m subintervals (we will call these subintervals "bad"). On good subintervals, w is constant, so for such intervals, we have that $|I|\,B_{\max}(\langle w\rangle_I, \langle w^p\rangle_I) \geqslant |I|\langle w^q\rangle_I$. For the bad intervals, we know that $B_{\max}$ is a continuous function and the set of points $\{x = (\langle w\rangle_I, \langle w^p\rangle_I) : I \in D(J)\}$ is a compact subset of Ω_ε, so $B_{\max}(\langle w\rangle_I, \langle w^p\rangle_I)$ for bad intervals $\{I_k\}_{k=1,\dots,m}$ are bounded by a uniform constant M. So the whole sum differs from $|J|\langle w^q\rangle_J$ by at most $M\sum_{I \text{ bad}} |I|$, which tends to 0 as $n \to \infty$.

This implies that $\langle w\rangle_J \leqslant B_{\max}(\langle w\rangle_J, \langle w^p\rangle_J)$ for all step functions w.

Next we extend this result to all weights w_m that are bounded from above and from below, say $m \leqslant w \leqslant M$. We take a sequence of step functions w_n that pointwise converge to w_m. By the Lebesgue dominated convergence theorem, Lemma 2.1 should hold for w_m.

The result of [Reznikov et al. 2010] extends our argument to an arbitrary weight w, which completes the proof of the Lemma 2.1. □

Proof of Lemma 2.2. We want to show that the matrix of second derivatives of $B_{\max}$ is not positive definite. We will just refer to [Vasyunin 2008], where it is shown in a more general case. □

Proof of Lemma 2.3. For the fixed points x^o, x^+ and x^- in the domain $U_\delta \subset \Omega_\varepsilon$ with $x^o = \frac{1}{2}(x^- + x^+)$ and such that the line segment connecting x^+ and x^- lies inside the domain Ω_ε, we introduce the function $b(t) := B(x_t)$, where $x_t := \frac{1}{2}(1+t)x^+ + \frac{1}{2}(1-t)x^-$. Note that defined this way, $B(x^o) = b(0)$, while $B(x^+) = b(1)$ and $B(x^-) = b(-1)$. Note also that

$$b''(t) = \begin{bmatrix} \frac{dx}{dt} & \frac{dy}{dt} \end{bmatrix} d^2 B_{\max} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}.$$

So, since $-d^2 B_{\max}$ is not negative definite, $-b''(t) \geqslant 0$ for all $-1 \leqslant t \leqslant 1$.

On the other hand,

$$B_{\max}(x^o)-\frac{B_{\max}(x^+)+B_{\max}(x^-)}{2}=b(0)-\frac{b(1)+b(-1)}{2}=-\frac{1}{2}\int_{-1}^{1}(1-|t|)b''(t)\,dt.$$

The second part of the above formula is a simple calculus exercise of integrating by parts twice.

Clearly, since $-b''(t)$ is nonnegative,

$$B_{\max}(x^o)-\frac{B_{\max}(x^+)+B_{\max}(x^-)}{2}\geqslant 0,$$

which completes the proof of Lemma 2.3. $\square$

Proof of Lemma 2.4. Let x^o, x^+ and x^- be three points in

$$U_\delta:=\left\{\vec{x}=(x_1,x_2)\,:\,\exists w\in RH_p^{\delta,d}\cap \mathrm{Db}^{Q,d}\text{ s.t. }x_1=\langle w\rangle_J,\,x_2=\langle w^p\rangle_J\right\}$$

that correspond to the same weight w and interval I; i.e., there is a weight $w\in RH_p^{\delta,d}$ with $\mathrm{Db}^d(w)\leqslant Q$ and a dyadic interval I such that

$$\begin{aligned} x_1^o&=\langle w\rangle_I, & x_2^o&=\langle w^p\rangle_I,\\ x_1^\pm&=\langle w\rangle_{I^\pm}, & x_2^\pm&=\langle w^p\rangle_{I^\pm}. \end{aligned}$$

Note that the reverse Hölder property for the weight w implies that $x_1^p\leqslant x_2\leqslant \delta^p x_1^p$ for all three points x^o, x^+ and x^-, and the fact that w is almost everywhere positive implies that $x_1, x_2>0$. At the same time, the fact that w is dyadically doubling with a doubling constant at most Q implies that

$$x_1^o\leqslant Qx_1^\pm,\quad x_1^\pm\leqslant 2x_1^o,\quad\text{and}\quad x_1^\mp\leqslant (Q-1)x_1^\pm.$$

Without loss of generality, we will assume that $x_1^-<x_1^+$. Then we know that $x_1^o\leqslant Qx_1^-$, $x_1^+\leqslant 2x_1^o$ and $x_1^+\leqslant (Q-1)x_1^-$.

Therefore

$$U_\delta\subset\Omega_\delta:=\left\{\vec{x}=(x_1,x_2)\,:\,x_1^p\leqslant x_2\leqslant\delta^p x_1^p\right\}\subset\Omega_\varepsilon,$$

and the points x^o, x^+ and $x^-\in U_\delta$ are such that

$$\begin{gathered} x^o=\tfrac{1}{2}(x^++x^-),\quad x_1^-<x_1^o<x_1^+,\\ x_1^o\leqslant Qx_1^-,\quad x_1^+\leqslant 2x_1^o,\quad x_1^+\leqslant(Q-1)x_1^-. \end{gathered}$$

We need to show that the line interval connecting x^+ and x^- lies inside the domain Ω_ε.

First observe that the worst case scenario is when the central point x^o and one of the endpoints lie on the upper boundary of U_δ, $x_2=\delta^p x_1^p$, while the other endpoint lies on the lower boundary of U_δ, $x_2=x_1^p$. There are two possibilities, so let us consider the two cases separately.

Case 1: x^o and x^- are on the upper boundary and x^+ is on the lower boundary. This means that

$$x^o = (x_1^o, \delta^p (x_1^o)^p), \quad x^- = (x_1^-, \delta^p (x_1^-)^p), \quad x^+ = (x_1^+, (x_1^+)^p).$$

We need to minimize the function $f(x) = x_2^{1/p} x_1^{-1}$ over the line that passes through the points x^o, x^+ and x^-. We are not going to use all of the conditions on our points. To simplify the problem, we will drop the condition that the point x^+ is on the lower boundary. We will only be using the points x^o and x^- and we will use the fact that $x_1^o \leqslant Q x_1^-$.

Again, in the worst case, which may be unattainable, $x_1^o = Q x_1^-$. The line through the points $x^- = (x_1^-, \delta^p (x_1^-)^p)$ and $x^o = (Qx^-, Q^p \delta^p (x_1^-)^p)$ has slope

$$\frac{\delta^p (x_1^-)^p (Q^p - 1)}{Q - 1}.$$

Therefore the equation is

$$x_2 - \delta^p (x_1^-)^p - \delta^p (x_1^-)^{p-1} \frac{Q^p - 1}{Q - 1} (x_1 - x_1^-) = 0.$$

So we need to solve the optimization problem

$$\begin{cases} f(x) = x_2^{1/p} x_1^{-1} \to \max, \\ x_2 - \delta^p (x_1^-)^p - \delta^p (x_1^-)^{p-1} \dfrac{Q^p - 1}{Q - 1} (x_1 - x_1^-) = 0. \end{cases}$$

The problem can be solved, for example, using method of Lagrange multipliers. If we let $H := (Q^p - 1)/(Q - 1)$ then

$$f_{\max} = \left(\frac{p-1}{H-1} \right)^{(p-1)/p} \frac{H}{p} \delta,$$

which is exactly our choice of ε.

Case 2: x^o and x^+ are on the upper boundary and x^- is on the lower boundary. In this case, we will drop the condition that x^- is on the lower boundary. Since the coordinates of our points are positive,

$$x_1^o = \frac{x_1^+ + x_1^-}{2} \geqslant \frac{x_1^+}{2},$$

so $x_1^+ \leqslant 2x_1^o$. Therefore this case is similar to Case 1 with $Q = 2$. Since $Q \geqslant 2$, this case is covered as well. This is the only place where we use that $Q \geqslant 2$.

This completes the proof of Lemma 2.4 and Theorem 1.1. □

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ovbeznosova@ua.edu *Department of Mathematics, Box 870350, University of Alabama, Tuscaloosa, AL 35487-0350, United States*

temitope_ode@baylor.edu *Baylor University, Waco, TX 76798, United States*

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