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Rings of invariants for the three-dimensional modular representations of elementary abelian p-groups of rank four

Théo Pierron and R. James Shank

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We show that the rings of invariants for the three-dimensional modular representations of an elementary abelian p-group of rank four are complete intersections with embedding dimension at most five. Our results confirm the conjectures of Campbell, Shank and Wehlau (*Transform. Groups* 18 (2013), 1–22) for these representations.

Introduction

We continue the investigation of the rings of invariants of modular representations of elementary abelian p-groups initiated in [Campbell et al. 2013]. We show that the rings of invariants for three-dimensional modular representations of groups of rank four are complete intersections and we confirm the conjectures of [loc. cit., §8] for these representations.

Let V denote an n-dimensional representation of a group G over a field $\mathbb{F}$ of characteristic p for a prime number p. We will usually assume that G is finite and that p divides the order of G, in other words, that V is a *modular representation* of G. We view V as a left module over the group ring $\mathbb{F}G$ and the dual, V^*, as a right $\mathbb{F}G$-module. Let $\mathbb{F}[V]$ denote the symmetric algebra on V^*. The action of G on V^* extends to an action by degree-preserving algebra automorphisms on $\mathbb{F}[V]$. By choosing a basis $\{x_1, x_2, \dots, x_n\}$ for V^*, we identify $\mathbb{F}[V]$ with the algebra of polynomials $\mathbb{F}[x_1, x_2, \dots, x_n]$. Our convention that $\mathbb{F}[V]$ is a right $\mathbb{F}G$-module is consistent with the convention used by the invariant theory package in the computer algebra software Magma [Bosma et al. 1997]. The ring of invariants, $\mathbb{F}[V]^G$, is the subring of $\mathbb{F}[V]$ consisting of those polynomials fixed by the action of G. Note that elements of $\mathbb{F}[V]$ represent polynomial functions on V and that elements of $\mathbb{F}[V]^G$ represent polynomial functions on the set of orbits V/G. For G finite and $\mathbb{F}$ algebraically

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closed, $\mathbb{F}[V]^G$ is the ring of regular functions on the categorical quotient $V/\!\!/G$. For background on the invariant theory of finite groups, see [Benson 1993; Campbell and Wehlau 2011; Derksen and Kemper 2002; Neusel and Smith 2002].

Computing the ring of invariants for a modular representation is typically a difficult problem; the rings are often not Cohen–Macaulay. It is natural to take p-groups as a starting point and recent work of David Wehlau [2013] gives us a good understanding in the case of a cyclic group of order p. The next step is to look at elementary abelian p-groups. The rings of invariants for the two-dimensional modular representations of elementary abelian p-groups were computed in Section 2 of [Campbell et al. 2013] and the three-dimensional modular representations were classified in Section 4 of that paper. The only three-dimensional representations for which computing the ring of invariants is not straightforward are those of type (1, 1, 1), in other words, those representations for which $\dim(V^G) = 1$ and $\dim((V/V^G)^G) = 1$. Our goal here is to compute the rings of invariants for representations of type (1, 1, 1) for groups of rank four. The methods we use are essentially the same as the methods used in [loc. cit.]. As the rank increases, the complexity of the required calculations increases; we believe that it is not feasible to use the methods here for rank greater than four.

We denote by $E = \langle e_1, e_2, e_3, e_4\rangle \cong (\mathbb{Z}/p)^4$ a rank-four elementary abelian p-group. Note that E only has representations of type (1, 1, 1) if $p > 2$, so we make this assumption throughout the paper. As in Section 4 of [loc. cit.], define $\sigma : \mathbb{F}^2 \to \mathrm{GL}_3(\mathbb{F})$ by

$$\sigma(c_1, c_2) := \begin{pmatrix} 1 & 2c_1 & c_1^2+c_2 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that σ defines a representation of the group $(\mathbb{F}^2, +)$. For a matrix

$$M := \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{pmatrix}$$

with $c_{ij} \in \mathbb{F}$, the assignment $e_j \mapsto \sigma(c_{1j}, c_{2j})$ determines a three-dimensional representation of E, which we denote by V_M. The action of E on $\mathbb{F}[x, y, z]$ is given by right multiplication on $x = [0\ \ 0\ \ 1]$, $y = [0\ \ 1\ \ 0]$ and $z = [1\ \ 0\ \ 0]$. Thus $x \cdot \sigma(c_1, c_2) = x$, $y \cdot \sigma(c_1, c_2) = y + c_1 x$ and $z \cdot \sigma(c_1, c_2) = z + 2c_1 y + (c_1^2 + c_2)x$. The representation V_M is of type (1, 1, 1) if at least one c_{1j} is nonzero. Furthermore, by Proposition 4.1 of [loc. cit.], for every representation of type (1, 1, 1), there exists a choice of basis for which the action is given by some matrix M.

In this paper, we compute $\mathbb{F}[V_M]^E$ for all $M \in \mathbb{F}^{2\times 4}$. We give a stratification of $\mathbb{F}^{2\times 4}$ and show that within each stratum there is a uniform computation of $\mathbb{F}[V_M]^E$. Note that the automorphism group of E is isomorphic to $\mathrm{GL}_4(\mathbb{F}_p)$, where $\mathbb{F}_p$ denotes the field of p elements. Since $\mathbb{F}_p \subseteq \mathbb{F}$, there is a natural right action of $\mathrm{GL}_4(\mathbb{F}_p)$

on $\mathbb{F}^{2\times 4}$. If M and M' lie in the same $\mathrm{GL}_4(\mathbb{F}_p)$-orbit, then $\mathbb{F}[V_M]^E = \mathbb{F}[V_{M'}]^E$. Essentially, we study subrings of $\mathbb{F}[x, y, z]$ parametrised by points in $\mathbb{F}^{2\times 4}/\mathrm{GL}_4(\mathbb{F}_p)$ and use elements of $\mathbb{F}[\mathbb{F}^{2\times 4}]^{\mathrm{SL}_4(\mathbb{F}_p)}$ to describe the stratification.

In Section 2, we work over the field $\Bbbk := \mathbb{F}_p(x_{ij} \mid i \in \{1, 2\},\ j \in \{1, 2, 3, 4\})$ and compute $\Bbbk[V_{\mathcal{M}}]^E$ for the generic matrix

$$\mathcal{M} := \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}.$$

We show that $\Bbbk[V_{\mathcal{M}}]^E$ is a complete intersection of embedding dimension five with generators in degrees 1, p^2, p^2+2p, p^3+2 and p^4, and relations in degrees p^3+2p^2 and p^4+2p. Consider the 10×4 matrix

$$\Gamma := \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{11}^p & x_{12}^p & x_{13}^p & x_{14}^p \\ x_{21}^p & x_{22}^p & x_{23}^p & x_{24}^p \\ \vdots & \vdots & \vdots & \vdots \\ x_{11}^{p^4} & x_{12}^{p^4} & x_{13}^{p^4} & x_{14}^{p^4} \\ x_{21}^{p^4} & x_{22}^{p^4} & x_{23}^{p^4} & x_{24}^{p^4} \end{pmatrix}$$

and for a subsequence (i, j, k, ℓ) of $(1, 2, \ldots, 10)$, let $\gamma_{ijk\ell}$ denote the associated 4×4 minor of Γ. Note that $\gamma_{ijk\ell} \in \mathbb{F}[\mathbb{F}^{2\times 4}]^{\mathrm{SL}_4(\mathbb{F}_p)}$ and, for $g \in \mathrm{GL}_4(\mathbb{F}_p)$, we have $g(\gamma_{ijk\ell}) = \det(g)\gamma_{ijk\ell}$. We use zero-sets of various $\gamma_{ijk\ell}$ to define the stratification of $\mathbb{F}^{2\times 4}/\mathrm{GL}_4(\mathbb{F}_p)$. In Section 3, we show that for $M \in \mathbb{F}^{2\times 4}$ with $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) \neq 0$ and $\gamma_{1357}(M) \neq 0$, the generic calculation survives evaluation. In Sections 4 through 10, we compute the rings of invariants for the remaining strata.

Section 4: $\gamma_{1357}(M) \neq 0$, $\gamma_{1235}(M) \neq 0$, $\gamma_{1234}(M) = 0$. We show $\mathbb{F}[V_M]^E$ is a complete intersection with generators in degrees 1, $2p$, p^3, p^3+2 and p^4, and relations in degrees $2p^3$ and p^4+2p.

Section 5: $\gamma_{1357}(M) \neq 0$, $\gamma_{1235}(M) = 0$, $\gamma_{1234}(M) \neq 0$. If $\gamma_{1245}(M) \neq 0$ then $\mathbb{F}[V_M]^E$ is a complete intersection with generators in degrees 1, p^2, p^2+p, p^3+p+2 and p^4, and relations in degrees p^3+p^2 and p^4+p^2+2p. Otherwise, $\mathbb{F}[V_M]^E$ is a hypersurface with generators in degrees 1, p^2, p^2+2 and p^4, with the relation in degree p^4+2p^2.

Section 6: $\gamma_{1357}(M) = 0$, $\gamma_{1235}(M) \neq 0$, $\gamma_{1234}(M) \neq 0$. We show $\mathbb{F}[V_M]^E$ is a complete intersection with generators in degrees 1, p^2, p^2+2p, p^3+1 and p^4, and relations in degrees p^3+2p^2 and p^4+p.

Section 7: $\gamma_{1357}(M) \neq 0$, $\gamma_{1235}(M) = 0$, $\gamma_{1234}(M) = 0$. We show $\mathbb{F}[V_M]^E$ is a hypersurface. If $\gamma_{1257}(M) = 0$, then the generators are in degrees 1, 2, p^4 and p^4

and the relation is in degree $2p^4$. Otherwise, the generators are in degrees 1, p, p^3+p^2+p+2, p^4 and the relation is in degree $p^4+p^3+p^2+2p$.

Section 8: $\gamma_{1357}(M)=0$, $\gamma_{1235}(M)\neq 0$, $\gamma_{1234}(M)=0$. We show $\mathbb{F}[V_M]^E$ is a complete intersection with generators in degrees 1, $2p$, p^3, p^3+1 and p^4, with relations in degrees $2p^2$ and p^4+p.

Section 9: $\gamma_{1357}(M)=0$, $\gamma_{1235}(M)=0$, $\gamma_{1234}(M)\neq 0$. If $\gamma_{1245}(M)\neq 0$, then $\mathbb{F}[V_M]^E$ is a complete intersection with generators in degrees 1, p^2, p^2+p, p^3+1 and p^4, with relations in degrees p^3+p^2 and p^4+p. Otherwise, $\mathbb{F}[V_M]^E$ is a hypersurface with generators in degrees 1, p^2, p^2+1 and p^4, with a relation in degree p^4+p^2.

Section 10: $\gamma_{1357}(M)=0$, $\gamma_{1235}(M)=0$, $\gamma_{1234}(M)=0$. If $\gamma_{1246}(M)\neq 0$ then $\mathbb{F}[V_M]^E$ is a hypersurface with generators in degrees 1, p, p^3+1, p^4 and a relation in degree p^4+p. Otherwise, the representation is either not faithful or not of type (1, 1, 1); in either case, the invariants were computed in [Campbell et al. 2013].

1. Preliminaries

We make extensive use of the theory of SAGBI bases to compute rings of invariants. A SAGBI basis is the subalgebra analogue of a Gröbner basis for ideals, and is a particularly nice generating set for the subalgebra. The concept was introduced independently by Robbiano and Sweedler [1990] and Kapur and Madlener [1989]; a useful reference is Chapter 11 of Sturmfels [1996]. We adopt the convention that a monomial is a product of variables and a term is a monomial with a coefficient. We use the graded reverse lexicographic order with $x<y<z$. For a polynomial $f\in\mathbb{F}[x,y,z]$, we denote the lead monomial of f by $\mathrm{LM}(f)$ and the lead term of f by $\mathrm{LT}(f)$. For $\mathcal{B}=\{h_1,\dots,h_\ell\}\subset\mathbb{F}[x,y,z]$ and $I=(i_1,\dots,i_\ell)$, a sequence of nonnegative integers, denote $\prod_{j=1}^{\ell}h_j^{i_j}$ by h^I. A *tête-à-tête* for $\mathcal{B}$ is a pair (h^I,h^J) with $\mathrm{LM}(h^I)=\mathrm{LM}(h^J)$; we say that a tête-à-tête is *nontrivial* if the support of I is disjoint from the support of J. The reduction of an S-polynomial is a fundamental calculation in the theory of Gröbner bases. The analogous calculation for SAGBI bases is the *subduction* of a tête-à-tête. For any $f\in\mathbb{F}[x,y,z]$, if there exists a sequence I such that $\mathrm{LM}(f)=\mathrm{LM}(h^I)$, we can choose $c\in\mathbb{F}$ so that $\mathrm{LT}(f)=\mathrm{LT}(ch^I)$. Then $\mathrm{LT}(f-ch^I)<\mathrm{LT}(f)$. If by iterating this process we can write f as a polynomial in the h_i, we say that f subducts to zero (using $\mathcal{B}$). For a tête-à-tête (h^I,h^J), choose c so that $\mathrm{LT}(h^I)=\mathrm{LT}(ch^J)$. We say that the tête-à-tête subducts to zero if h^I-ch^J subducts to zero. A subset $\mathcal{B}$ of a subalgebra $A\subset\mathbb{F}[x_1,\dots,x_n]$ is a SAGBI basis for A if the lead monomials of the elements of $\mathcal{B}$ generate the lead term algebra of A or, equivalently, every nontrivial tête-à-tête for $\mathcal{B}$ subducts to zero. For background material on term orders and Gröbner bases, we recommend [Adams and Loustaunau 1994].

The following specialisation of Theorem 1.1 of [Campbell et al. 2013] is our primary computational tool. Note that under the hypotheses of the theorem, $\{x, h_1, h_\ell\}$ is a homogeneous system of parameters and, therefore, $\mathbb{F}[V_M]^E$ is an integral extension of A.

Theorem 1.1. *For homogeneous $h_1, \ldots, h_\ell \in \mathbb{F}[V_M]^E$ with $\mathrm{LM}(h_1) = y^i$ for some $i > 0$, $\mathrm{LM}(h_\ell) = z^j$ for some $j > 0$ and $\mathrm{LM}(h_k) \in \mathbb{F}[y, z]$ for $k = 2, \ldots, \ell-1$, define $\mathcal{B} := \{x, h_1, \ldots, h_\ell\}$ and let A denote the algebra generated by $\mathcal{B}$. If $A[x^{-1}] = \mathbb{F}[V_M]^E[x^{-1}]$ and $\mathcal{B}$ is a SAGBI basis for A, then $A = \mathbb{F}[V_M]^E$ and $\mathcal{B}$ is a SAGBI basis for $\mathbb{F}[V_M]^E$.*

Note that, if an algebra is generated by a finite SAGBI basis, then for the corresponding presentation, the ideal of relations is generated by elements corresponding to the subductions of the nontrivial tête-à-têtes (see Corollary 11.6 of [Sturmfels 1996]). We use the term *complete intersection* to refer to an algebra with a presentation for which the ideal of relations is generated by a regular sequence. Since the Krull dimension of $\mathbb{F}[V_M]^E$ is three, the ring is a complete intersection if the number of generators minus the number of nontrivial tête-à-têtes is three.

We routinely use the *SAGBI/divide-by-x* algorithm introduced in Section 1 of [Campbell et al. 2013]. The traditional SAGBI basis algorithm proceeds by subducting tête-à-têtes and adding any nonzero subductions to the generating set. For SAGBI/divide-by-x, if a nonzero subduction is divisible by x, we divide by the highest possible power of x before adding the polynomial to the generating set. While the SAGBI algorithm extends the generating set for a given subalgebra, SAGBI/divide-by-x extends the subalgebra. If we start with a subalgebra A which contains a homogeneous system of parameters and satisfies the condition that $A[x^{-1}] = \mathbb{F}[V_M]^E[x^{-1}]$, then the SAGBI/divide-by-x algorithm will produce a generating set for $\mathbb{F}[V_M]^E$ (see Theorem 1.2 of [loc. cit.]).

For $f \in \mathbb{F}[V_M]$, we define the *norm* of f to be the orbit product

$$N_M(f) := \prod\{f \cdot g \mid g \in E\} \in \mathbb{F}[V_M]^E$$

with the action of E determined by M. When applying Theorem 1.1, we often take h_ℓ to be $N_M(z)$.

Remark 1.2. Note that the action of E restricts to an action on $\mathbb{F}[x, y]$ and that $\mathbb{F}[x, y]^E = \mathbb{F}[x, N_M(y)]$ (see Section 2 of [Campbell et al. 2013]). Therefore, if $h \in \mathbb{F}[x, y]^E$ is homogeneous with $\deg(h) = |\{y \cdot g \mid g \in E\}|$ then h is a linear combination of $N_M(y)$ and $x^{\deg(h)}$.

Define $\delta := y^2 - xz$ and observe that

$$\delta \cdot \sigma(c_1, c_2) = (y + c_1 x)^2 - x\big(z + 2c_1 y + (c_1^2 + c_2)x\big) = \delta - c_2 x^2.$$

Note that $\mathbb{F}[x, y, z][x^{-1}] = \mathbb{F}[x, y, -\delta/x][x^{-1}]$ and that the $\mathbb{F}[x, y, -\delta/x]^E$ is a polynomial algebra (see Theorem 3.9.2 of [Campbell and Wehlau 2011]). This "change of basis" can be a useful way to compute the field of fractions of $\mathbb{F}[V_M]^E$. Form the matrix $\widetilde{\Gamma}$ by augmenting Γ with the column

$$\left[\frac{y}{x}\ \left(-\frac{\delta}{x^2}\right)\ \left(\frac{y}{x}\right)^p\ \left(-\frac{\delta}{x^2}\right)^p\ \cdots\ \left(\frac{y}{x}\right)^{p^4}\ \left(-\frac{\delta}{x^2}\right)^{p^4}\right]^T.$$

For a subsequence $J = (j_1, \dots, j_5)$ of $(1, 2, \dots, 10)$, let $\tilde{f}_J \in \Bbbk[x, y, z][x^{-1}]$ denote the associated 5×5 minor of $\widetilde{\Gamma}$. Let f_J denote the element of $\Bbbk[x, y, z]$ constructed by minimally clearing the denominator of $\tilde{f}_J$. Observe that $f_J \in \Bbbk[V_{\mathcal{M}}]^E$. Furthermore, the coefficients of f_J lie in $\mathbb{F}_p[x_{ij}]^{\mathrm{SL}_4(\mathbb{F}_p)}$ and, for an arbitrary $M \in \mathbb{F}^{2\times 4}$, evaluating the coefficients of f_J at M gives an element $\bar{f}_J \in \mathbb{F}[V_M]^E$. Invariants constructed in this way are a crucial ingredient in our calculations. Define $f_1 := f_{12345}$ and observe that $\mathrm{LT}(f_1) = \gamma_{1234} y^{p^2}$. Note that $\mathrm{LT}(f_{12346}) = -\gamma_{1234} y^{2p^2}$. A straightforward calculation shows that

$$\mathrm{LT}(f_1^2 + \gamma_{1234} f_{12346}) = 2\gamma_{1234}\gamma_{1235} x^{p^2-2p} y^{p^2+2p}.$$

Therefore,

$$f_2 := \frac{f_1^2 + \gamma_{1234} f_{12346}}{2x^{p^2-2p}} \in \Bbbk[V_{\mathcal{M}}]^E$$

has lead term $\gamma_{1234}\gamma_{1235} y^{p^2+2p}$.

We make frequent use of the *Plücker relations* for the minors of Γ and $\widetilde{\Gamma}$.

Theorem 1.3. *Let N be an $n \times m$ matrix with $n > m$. Denote by $p_{i_1,\dots,i_m}$ the $m \times m$ minor of N determined by the rows $i_1, \dots, i_m$. For sequences $(i_1, \dots, i_{m-1})$ and $(j_1, \dots, j_{m+1})$, we have the following Plücker relation*

$$\sum_{a=1}^{m+1} (-1)^a p_{i_1,\dots,i_{m-1},j_a} p_{j_1,\dots,j_{a-1},j_{a+1},\dots,j_{m+1}} = 0.$$

For a proof of the above theorem, see, for example, [Lakshmibai and Raghavan 2008, §4.1.3].

Lemma 1.4. *For $2 < i < 7$,*

$$\gamma_{12i7}\gamma_{1234}^p = \gamma_{12i6}\gamma_{1235}^p - \gamma_{12i5}\gamma_{1245}^p + \gamma_{12i4}\gamma_{1345}^p - \gamma_{12i3}\gamma_{2345}^p.$$

Proof. Since taking p-th powers is $\mathbb{F}_p$-linear, $\gamma_{(i+2)(j+2)(k+2)(\ell+2)} = \gamma_{ijk\ell}^p$. For example, $\gamma_{3456} = \gamma_{1234}^p$. The desired result follows from this fact, using the $(1, 2, i)(3, 4, 5, 6, 7)$ Plücker relation for the matrix Γ. □

For $K = (k_1, k_2, \dots, k_6)$ a subsequence of $(1, 2, \dots, 10)$, let K_i denote the subsequence of K formed by omitting i and let $K_{i,j}$ denote the subsequence of K formed by omitting i and j. The following is Lemma 5.3 from [Campbell et al. 2013].

Lemma 1.5. *For any subsequence (i_1, i_2, i_3) of K,*

$$(-1)^{\epsilon_1}\gamma_{K_{i_1,i_2}}\tilde{f}_{K_{i_3}}+(-1)^{\epsilon_2}\gamma_{K_{i_2,i_3}}\tilde{f}_{K_{i_1}}+(-1)^{\epsilon_3}\gamma_{K_{i_1,i_3}}\tilde{f}_{K_{i_2}}=0$$

for some choice of $\epsilon_\ell \in \{0, 1\}$.

Remark 1.6. Note that $\gamma_{1357}(M)=0$ if and only if $\{c_{11}, c_{12}, c_{13}, c_{14}\}$ is linearly dependent over $\mathbb{F}_p$. This follows from the usual construction of the Dickson invariants; see, for example, [Wilkerson 1983]. The key observation is that $\gamma_{1357}(M)^{p-1}$ is the product of the nonzero $\mathbb{F}_p$-linear combinations of $\{c_{11}, c_{12}, c_{13}, c_{14}\}$.

2. The generic case

In this section we compute $\Bbbk[V_{\mathcal{M}}]^E$. With f_1 and f_2 defined as in Section 1, using Theorem 5.2 of [Campbell et al. 2013], we see that

$$\Bbbk[V_{\mathcal{M}}]^E[x^{-1}]=\Bbbk[x, f_1, f_2][x^{-1}].$$

Thus it is sufficient to extend $\{x, f_1, f_2, N_{\mathcal{M}}(z)\}$ to a SAGBI basis. We use the *SAGBI/divide-by-x* algorithm of [loc. cit., §1] to do this. We will show that the algorithm produces one new invariant, which we denote by f_3, and that

$$\mathrm{LT}(f_3)=\gamma_{1357}y^{p^3+2}.$$

For $p=3$ and $p=5$, this result follows from a Magma calculation. For the rest of this section, we assume $p>5$.

Expanding the definitions of f_1, f_{12346} and f_2 gives

$$f_1=\gamma_{1234}y^{p^2}+\gamma_{1235}\delta^p x^{p^2-2p}+\gamma_{1245}x^{p^2-p}y^p+\gamma_{1345}\delta x^{p^2-2}+\gamma_{2345}x^{p^2-1}y,$$

$$f_{12346}=-\gamma_{1234}\delta^{p^2}+\gamma_{1236}\delta^p x^{2p^2-2p}+\gamma_{1246}x^{2p^2-p}y^p+\gamma_{1346}\delta x^{2p^2-2}+\gamma_{2346}x^{2p^2-1}y$$

and

$$\begin{aligned}
f_2&=\frac{f_1^2+\gamma_{1234}f_{12346}}{2x^{p^2-2p}}\\
&=\gamma_{1234}\gamma_{1235}y^{p^2}\delta^p+\gamma_{1234}\gamma_{1245}x^p y^{p^2+p}+\gamma_{1234}\gamma_{1345}\delta x^{2p-2}y^{p^2}\\
&\quad+\gamma_{1234}\gamma_{2345}x^{2p-1}y^{p^2+1}+\tfrac{1}{2}\gamma_{1234}^2x^{2p}z^{p^2}+\tfrac{1}{2}\gamma_{1235}^2\delta^{2p}x^{p^2-2p}\\
&\quad+\gamma_{1235}\gamma_{1245}\delta^p x^{p^2-p}y^p+\gamma_{1235}\gamma_{1345}\delta^{p+1}x^{p^2-2}+\gamma_{1235}\gamma_{2345}\delta^p x^{p^2-1}y\\
&\quad+\tfrac{1}{2}\gamma_{1234}\gamma_{1236}x^{p^2}\delta^p+\tfrac{1}{2}\gamma_{1245}^2x^{p^2}y^{2p}+\gamma_{1245}\gamma_{1345}\delta x^{p^2+p-2}y^p\\
&\quad+\gamma_{1245}\gamma_{2345}x^{p^2+p-1}y^{p+1}+\tfrac{1}{2}\gamma_{1234}\gamma_{1246}y^p x^{p^2+p}+\tfrac{1}{2}\gamma_{1345}^2\delta^2x^{p^2+2p-4}\\
&\quad+\gamma_{1345}\gamma_{2345}\delta x^{p^2+2p-3}y+\tfrac{1}{2}\gamma_{2345}^2x^{p^2+2p-2}y^2\\
&\quad+\tfrac{1}{2}\gamma_{1234}\gamma_{1346}\delta x^{p^2+2p-2}+\tfrac{1}{2}\gamma_{1234}\gamma_{2346}x^{p^2+2p-1}y.
\end{aligned}$$

Subducting the tête-à-tête (f_1^{p+2}, f_2^p) gives

$$\tilde{f}_3 = \underbrace{\gamma_{1235}^p f_1^{p+2}}_{T_1} - \underbrace{\gamma_{1234}^2 f_2^p}_{T_2} + \underbrace{\alpha_1 x^{p^2-2p} f_1^p f_2}_{T_3} + \underbrace{\alpha_2 x^{p^2} f_1^{p+1}}_{T_4} + \underbrace{\alpha_3 x^{2p^2-2p} f_1^{p-1} f_2}_{T_5} + \underbrace{\alpha_4 x^{2p^2-p} f_1^{(p-3)/2} f_2^{(p+1)/2}}_{T_6},$$

where

$$\alpha_1 = -2\gamma_{1235}^p, \quad \alpha_2 = \gamma_{1234}\gamma_{1245}^p, \quad \alpha_3 = \frac{\gamma_{1234}^{p+1}\gamma_{1237}}{\gamma_{1235}}, \quad \alpha_4 = \frac{\gamma_{1234}^{p+3}\gamma_{1257}}{\gamma_{1235}^{(p+3)/2}}.$$

Lemma 2.1. *For $p \geq 5$, we have $\mathrm{LT}(\tilde{f}_3) = \alpha x^{2p^2-2} y^{p^3+2}$ with*

$$\alpha = \frac{\gamma_{1234}^{p+1}}{\gamma_{1235}}\big(\gamma_{1234}\gamma_{1345}^{p+1} + \gamma_{1235}^p\gamma_{1345}\gamma_{1236} - \gamma_{1235}^{p+1}\gamma_{1346}\big) = -\frac{\gamma_{1357}\gamma_{1234}^{2p+2}}{\gamma_{1235}}.$$

Proof. We work modulo the ideal in $\Bbbk[x, y, z]$ generated by x^{2p^2-1}. By the definition of f_2, we have

$$T_1 - T_2 + T_3 = -\gamma_{1235}^p \gamma_{1234} f_1^p f_{12346} - \gamma_{1234}^2 f_2^p.$$

As $f_1^p \equiv \gamma_{1234}^p y^{p^3}$ and

$$f_2^p \equiv \gamma_{1234}^p\gamma_{1235}^p \delta^{p^2} y^{p^3} + \gamma_{1234}^p\gamma_{1245}^p x^{p^2} y^{p^3+p^2} + \gamma_{1234}^p\gamma_{1345}^p \delta^p x^{2p^2-2p} y^{p^3} + \gamma_{1234}^p\gamma_{2345}^p x^{2p^2-p} y^{p^3+p},$$

we obtain

$$\begin{aligned}T_1 - T_2 + T_3 \equiv\ & -\gamma_{1234}^{p+2}\gamma_{1245}^p x^{p^2} y^{p^3+p^2} - \gamma_{1234}^{p+1}\big(\gamma_{1234}\gamma_{1345}^p + \gamma_{1235}^p\gamma_{1236}\big)\delta^p x^{2p^2-2p} y^{p^3} \\ & - \gamma_{1234}^{p+1}\big(\gamma_{1234}\gamma_{2345}^p + \gamma_{1235}^p\gamma_{1246}\big) x^{2p^2-p} y^{p^3+p} \\ & - \gamma_{1234}^{p+1}\gamma_{1235}^p\gamma_{1346}\delta x^{2p^2-2} y^{p^3}.\end{aligned}$$

Since

$$\begin{aligned} x^{p^2} f_1^{p+1} &\equiv \gamma_{1234}^p y^{p^3} x^{p^2} f_1 \\ &\equiv \gamma_{1234}^{p+1} x^{p^2} y^{p^3+p^2} + \gamma_{1234}^p \gamma_{1235} \delta^p x^{2p^2-2p} y^{p^3} \\ &\quad + \gamma_{1234}^p\gamma_{1245} x^{2p^2-p} y^{p^3+p} + \gamma_{1234}^p\gamma_{1345}\delta x^{2p^2-2} y^{p^3},\end{aligned}$$

we see that

$$\begin{aligned}T_1 - T_2 + T_3 + T_4 \equiv\ & \gamma_{1234}^{p+1}\big(\gamma_{1235}\gamma_{1245}^p - \gamma_{1235}^p\gamma_{1236} - \gamma_{1234}\gamma_{1345}^p\big) x^{2p^2-2p} y^{p^3} \delta^p \\ & + \gamma_{1234}^{p+1}\big(\gamma_{1245}^{p+1} - \gamma_{1235}^p\gamma_{1246} - \gamma_{1234}\gamma_{2345}^p\big) x^{2p^2-p} y^{p^3+p} \\ & + \gamma_{1234}^{p+1}\big(\gamma_{1245}^p\gamma_{1345} - \gamma_{1235}^p\gamma_{1346}\big)\delta x^{2p^2-2} y^{p^3}.\end{aligned}$$

Using Lemma 1.4 for $i = 3$ and $i = 4$, along with the analogous result coming from the $(1,3,4)(3,4,5,6,7)$ Plücker relation for Γ, gives

$$T_1 - T_2 + T_3 + T_4 \equiv -\gamma_{1234}^{2p+1}\gamma_{1237}x^{2p^2-2p}y^{p^3}\delta^p \\ - \gamma_{1234}^{2p+1}\gamma_{1247}x^{2p^2-p}y^{p^3+p} - \gamma_{1234}^{2p+1}\gamma_{1347}\delta x^{2p^2-2}y^{p^3}.$$

Since $3p^2 - 4p \geq 2p^2 - 1$ for $p \geq 5$, we have $x^{2p^2-2p}f_1^{p-1} \equiv \gamma_{1234}^{p-1}y^{p^3-p^2}x^{2p^2-2p}$. Using the description of f_2 given above,

$$x^{2p^2-2p}f_2 \equiv \gamma_{1234}x^{2p^2-2p}y^{p^2}(\gamma_{1235}\delta^p + \gamma_{1245}x^p y^p + \gamma_{1345}\delta x^{2p-2}).$$

Thus

$$T_5 \equiv \alpha_3\gamma_{1234}^p y^{p^3}x^{2p^2-2p}(\gamma_{1235}\delta^p + \gamma_{1245}x^p y^p + \gamma_{1345}\delta x^{2p-2}).$$

Using the $(1,2,4)(1,2,3,5,7)$ and $(1,3,5)(1,2,3,4,7)$ Plücker relations gives

$$T_1 - T_2 + T_3 + T_4 + T_5 \equiv -\frac{\gamma_{1234}^{2p+2}\gamma_{1257}}{\gamma_{1235}}x^{2p^2-p}y^{p^3+p} - \frac{\gamma_{1234}^{2p+2}\gamma_{1357}}{\gamma_{1235}}\delta x^{2p^2-2}y^{p^3}.$$

Expanding and reducing modulo $\langle x^{2p^2-1}\rangle$, we get

$$x^{2p^2-p}f_1^{(p-3)/2} \equiv x^{2p^2-p}\gamma_{1234}^{(p-3)/2}y^{(p^3-3p^2)/2}$$

and

$$x^{2p^2-p}f_2^{(p+1)/2} \equiv \gamma_{1234}^{(p+1)/2}\gamma_{1235}^{(p+1)/2}x^{2p^2-p}y^{(p^3+3p^2)/2+p}.$$

Thus

$$\frac{T_6}{\alpha_4} \equiv \gamma_{1234}^{p-1}\gamma_{1235}^{(p+1)/2}x^{2p^2-p}y^{p^3+p}$$

and

$$\tilde{f}_3 = T_1 - T_2 + T_3 + T_4 + T_5 + T_6 \equiv \alpha x^{2p^2-2}y^{p^3+2}.$$

Using the $(1,2,3)(1,3,4,5,6)$ and $(1,3,5)(3,4,5,6,7)$ Plücker relations, we obtain

$$\alpha = \frac{\gamma_{1234}^{p+2}}{\gamma_{1235}}(\gamma_{1345}^{p+1} - \gamma_{1356}\gamma_{1235}^p) = -\frac{\gamma_{1234}^{2p+2}\gamma_{1357}}{\gamma_{1235}}$$

and, since we are using the grevlex term order with $x < y < z$, the result follows. □

Define

$$f_3 := -\tilde{f}_3\frac{\gamma_{1235}}{\gamma_{1234}^{2p+2}x^{2p^2-2}}$$

so that $\mathrm{LT}(f_3) = \gamma_{1357}y^{p^3+2}$. Looking at the exponents of y modulo p, it is clear that there is only one new nontrivial tête-à-tête: $(f_3^p, f_2 f_1^{p^2-1})$. In order to prove that $\mathcal{B} := \{x, f_1, f_2, f_3, N_{\mathcal{M}}(z)\}$ is a SAGBI basis for $\Bbbk[V_{\mathcal{M}}]^E$, it is sufficient to show that this tête-à-tête subducts to zero. However, $N_{\mathcal{M}}(z)$ is rather complicated and it is more convenient to take an indirect approach. Subducting the tête-à-tête using only $\{x, f_1, f_2, f_3\}$ gives

$$\begin{aligned}\tilde{f}_4 := \underbrace{\beta_1 f_3^p}_{T_1'} - \underbrace{\beta_2 f_1^{p^2-1} f_2}_{T_2'} + \underbrace{\beta_3 x^p f_1^{p^2-(p+3)/2} f_2^{(p+1)/2}}_{T_3'}& \\ + \underbrace{\beta_4 x^{2p-2} f_1^{p^2-p} f_3}_{T_4'} + \underbrace{\beta_5 x^{2p-1} f_1^{(p^2-1)/2-p} f_2^{(p-1)/2} f_3^{(p+1)/2}}_{T_5'}&,\end{aligned}$$

where

$$\beta_1 := \gamma_{1235}\gamma_{1234}^{p^2}, \quad \beta_2 := \gamma_{1357}^p, \quad \beta_3 := \frac{\gamma_{1234}(\gamma_{1245}\gamma_{1357}^p - \gamma_{1235}\gamma_{2357}^p)}{\gamma_{1235}^{(p+1)/2}},$$
$$\beta_4 := \gamma_{1234}^p\gamma_{1345}\gamma_{1357}^{p-1}, \quad \beta_5 := -\gamma_{1234}^{(p^2+p+2)/2}\gamma_{1235}^{(p+3)/2}\gamma_{1357}^{(p-3)/2}.$$

The lemma below proves that $\{x, f_1, f_2, f_3, \tilde{f}_4/x^{2p}\}$ is a SAGBI basis. We then use this in the proof of Theorem 2.3.

Lemma 2.2. *For $p \geq 5$, we have* $\mathrm{LT}(\tilde{f}_4) = \frac{1}{2}\gamma_{1234}^{p^2}\gamma_{1235}^{p+1}x^{2p}z^{p^4}$.

Proof. We work modulo the ideal in $\Bbbk[x, y, z]$ generated by x^{2p+1} and $x^{2p}y$, which we denote by $\mathfrak{n}$. Since $p \geq 5$, we have $p^2 - 2p \geq 2p + 1$. Therefore, using the expressions for f_1 and f_2 given above, we have $f_1 \equiv_{\mathfrak{n}} \gamma_{1234}y^{p^2}$ and

$$\begin{aligned}f_2 \equiv_{\mathfrak{n}} \gamma_{1234}\gamma_{1235}y^{p^2}\delta^p + \gamma_{1234}\gamma_{1245}y^{p^2+p}x^p + \gamma_{1234}\gamma_{1345}\delta x^{2p-2}y^{p^2}& \\ + \gamma_{1234}\gamma_{2345}x^{2p-1}y^{p^2+1} + \tfrac{1}{2}\gamma_{1234}^2x^{2p}z^{p^2}&.\end{aligned}$$

We will need expressions modulo $\mathfrak{n}$ for f_3^p, $x^{2p-2}f_3$ and $x^{2p-1}f_3^{(p+1)/2}$. Let $\mathfrak{m}$ denote the ideal generated by x^2y and x^3. Reworking the calculations of the proof of Lemma 2.1 to keep additional terms of f_3 gives

$$f_3 \equiv_{\mathfrak{m}} \gamma_{1357}\delta y^{p^3} + \gamma_{2357}xy^{p^3+1} + \tfrac{1}{2}\gamma_{1235}x^2z^{p^3}.$$

Thus

$$\begin{aligned}f_3^p &\equiv_{\mathfrak{n}} \gamma_{1357}^p\delta^p y^{p^4} + \gamma_{2357}^p x^p y^{p^4+p} + \tfrac{1}{2}\gamma_{1235}^p x^{2p}z^{p^4}, \\ x^{2p-2}f_3 &\equiv_{\mathfrak{n}} \gamma_{1357}\delta x^{2p-2}y^{p^3} + \gamma_{2357}x^{2p-1}y^{p^3+1} + \tfrac{1}{2}\gamma_{1235}x^{2p}z^{p^3}, \\ x^{2p-1}f_3^{(p+1)/2} &\equiv_{\mathfrak{n}} \gamma_{1357}^{(p+1)/2}x^{2p-1}y^{(p^3+2)(p+1)/2}.\end{aligned}$$

Therefore

$$\begin{aligned}T_1' - T_2' \equiv_{\mathfrak{n}} \gamma_{1234}^{p^2}(\gamma_{1235}\gamma_{2357}^p - \gamma_{1245}\gamma_{1357}^p)x^p y^{p^4+p} - \gamma_{1234}^{p^2}\gamma_{1345}\gamma_{1357}^p\delta x^{2p-2}y^{p^4}& \\ -\gamma_{1234}^{p^2}\gamma_{2345}\gamma_{1357}^p x^{2p-1}y^{p^4+1} + \tfrac{1}{2}\gamma_{1234}^{p^2}\gamma_{1235}^{p+1}x^{2p}z^{p^4}&.\end{aligned}$$

Since $x^p f_2^{(p+1)/2} \equiv_{\mathfrak{n}} \gamma_{1234}^{(p+1)/2}\gamma_{1235}^{(p+1)/2}x^p y^{(p^3+3p^2)/2+p}$, we have

$$T_1' - T_2' + T_3' \equiv_{\mathfrak{n}} -\gamma_{1234}^{p^2}\gamma_{1345}\gamma_{1357}^{p}\delta x^{2p-2}y^{p^4} - \gamma_{1234}^{p^2}\gamma_{2345}\gamma_{1357}^{p}x^{2p-1}y^{p^4+1} + \tfrac{1}{2}\gamma_{1234}^{p^2}\gamma_{1235}^{p+1}x^{2p}z^{p^4}.$$

Using the description of $x^{2p-2}f_3$ given above, we see that

$$T_1' - T_2' + T_3' + T_4' \equiv_{\mathfrak{n}} \gamma_{1234}^{p^2}\gamma_{1357}^{p-1}(\gamma_{1345}\gamma_{2357} - \gamma_{1357}\gamma_{2345})x^{2p-1}y^{p^4+1} + \tfrac{1}{2}\gamma_{1234}^{p^2}\gamma_{1235}^{p+1}x^{2p}z^{p^4}.$$

The $(2,3,5)(1,3,4,5,7)$ Plücker relation gives

$$\gamma_{2345}\gamma_{1357} - \gamma_{2357}\gamma_{1345} = -\gamma_{1235}\gamma_{3457}.$$

Thus

$$T_1' - T_2' + T_3' + T_4' \equiv_{\mathfrak{n}} \gamma_{1234}^{p^2}\gamma_{1357}^{p-1}\gamma_{1235}\gamma_{3457}x^{2p-1}y^{p^4+1} + \tfrac{1}{2}\gamma_{1234}^{p^2}\gamma_{1235}^{p+1}x^{2p}z^{p^4}.$$

Observe that

$$x^{2p-1}f_1^{(p^2-1)/2-p} \equiv_{\mathfrak{n}} \gamma_{1234}^{(p^2-1)/2-p}x^{2p-1}y^{(p^4-p^2)/2-p^3}$$

and

$$x^{2p-1}f_2^{(p-1)/2} \equiv_{\mathfrak{n}} \gamma_{1234}^{(p-1)/2}\gamma_{1235}^{(p-1)/2}x^{2p-1}y^{(p^3+p^2)/2-p}.$$

Therefore, using the description of $x^{2p-1}f_3^{(p+1)/2}$ given above, we obtain

$$\tilde{f}_4 := T_1' - T_2' + T_3' + T_4' + T_5' \equiv_{\mathfrak{n}} \tfrac{1}{2}\gamma_{1234}^{p^2}\gamma_{1235}^{p+1}x^{2p}z^{p^4},$$

and, since we are using the grevlex term order with $x < y < z$, the result follows. □

Theorem 2.3. *The set $\mathcal{B} := \{x, f_1, f_2, f_3, N_{\mathcal{M}}(z)\}$ is a SAGBI basis, and hence a generating set, for $\Bbbk[V_{\mathcal{M}}]^E$. Furthermore, $\Bbbk[V_{\mathcal{M}}]^E$ is a complete intersection with generating relations coming from the subduction of the tête-à-têtes (f_2^p, f_1^{p+2}) and $(f_3^p, f_2 f_1^{p^2-1})$.*

Proof. Define $f_4 := \tilde{f}_4/x^{2p}$, $\mathcal{B}' := \{x, f_1, f_2, f_3, f_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-têtes for $\mathcal{B}'$ are (f_2^p, f_1^{p+2}) and $(f_3^p, f_2 f_1^{p^2-1})$. From Lemmas 2.1 and 2.2, these tête-à-têtes subduct to zero. Therefore $\mathcal{B}'$ is a SAGBI basis for A. From Theorem 5.2 of [Campbell et al. 2013], $\Bbbk[V_{\mathcal{M}}]^E[x^{-1}] = \Bbbk[x, f_1, f_2][x^{-1}]$. Thus $A[x^{-1}] = \Bbbk[V_{\mathcal{M}}]^E[x^{-1}]$. Note that $\mathrm{LM}(f_4) = z^{p^4}$. Therefore, by Theorem 1.1, $A = \Bbbk[V_{\mathcal{M}}]^E$ and $\mathcal{B}'$ is a SAGBI basis for $\Bbbk[V_{\mathcal{M}}]^E$. Hence the lead term algebra of $\Bbbk[V_{\mathcal{M}}]^E$ is generated by $\{x, y^{p^2}, y^{p^2+2p}, y^{p^3+2}, z^{p^4}\}$. Since the orbit of z has size p^4, we see that $\mathrm{LM}(N_{\mathcal{M}}(z)) = z^{p^4}$. Thus $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$ and $\mathcal{B}$ is also a SAGBI basis for $\Bbbk[V_{\mathcal{M}}]^E$. For any subalgebra with a SAGBI basis, the relations are generated by the nontrivial tête-à-tête. Hence (f_2^p, f_1^{p+2}) and $(f_3^p, f_2 f_1^{p^2-1})$ generate the ideal of relations and $\Bbbk[V_{\mathcal{M}}]^E$ is a complete intersection with embedding dimension five. □

3. The essentially generic case

In this section we consider representations V_M for $M \in \mathbb{F}^{2\times 4}$ for which $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) \neq 0$ and $\gamma_{1357}(M) \neq 0$. With this restriction on M, we can evaluate the coefficients of the polynomials $\{f_i \mid i = 1, 2, 3, 4\}$, as defined in Section 2, at M to get $\{\bar{f}_i \mid i = 1, 2, 3, 4\} \subset \mathbb{F}[V_M]^E$. Note that $\mathrm{LT}(\bar{f}_1) = \gamma_{1234}(M)y^{p^2}$ so that $\mathrm{LM}(\bar{f}_1) = y^{p^2}$. Similarly $\mathrm{LM}(\bar{f}_2) = y^{p^2+2p}$, $\mathrm{LM}(\bar{f}_3) = y^{p^3+2}$ and $\mathrm{LM}(\bar{f}_4) = z^{p^4}$. Also, note that $\gamma_{1357}(M) = 0$ if and only if $\{c_{11}, c_{12}, c_{13}, c_{14}\}$ is linearly dependent over $\mathbb{F}_p$. Thus, if $\gamma_{1357}(M) \neq 0$, the orbit of z has size p^4 and $\mathrm{LM}(N_M(z)) = z^{p^4}$.

Theorem 3.1. *If $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) \neq 0$ and $\gamma_{1357}(M) \neq 0$, then the set $\mathcal{B} := \{x, \bar{f}_1, \bar{f}_2, \bar{f}_3, N_M(z)\}$ is a SAGBI basis, and hence a generating set, for $\mathbb{F}[V_M]^E$. Furthermore, $\mathbb{F}[V_M]^E$ is a complete intersection with generating relations coming from the subduction of the tête-à-têtes $(\bar{f}_2^p, \bar{f}_1^{p+2})$ and $(\bar{f}_3^p, \bar{f}_2\bar{f}_1^{p^2-1})$.*

Proof. Define $\mathcal{B}' := \{x, \bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-têtes for $\mathcal{B}'$ are $(\bar{f}_2^p, \bar{f}_1^{p+2})$ and $(\bar{f}_3^p, \bar{f}_2\bar{f}_1^{p^2-1})$. The calculations in the proofs of Lemmas 2.1 and 2.2 survive evaluation at M, proving that these tête-à-têtes subduct to zero and $\mathcal{B}'$ is a SAGBI basis for A. Thus, to use Theorem 1.1 to prove $A = \mathbb{F}[V_M]^E$, we need only show that $A[x^{-1}] = \mathbb{F}[V_M]^E[x^{-1}]$.

Consider

$$f_{12357} = \gamma_{1235}y^{p^3} - \gamma_{1237}y^{p^2}x^{p^3-p^2} + \gamma_{1257}y^p x^{p^3-p} + \gamma_{1357}\delta x^{p^3-2} + \gamma_{2357}yx^{p^3-1}$$

and evaluate the coefficients at M to get $\bar{f}_{12357} \in \mathbb{F}[V_M]^E$ with lead monomial y^{p^3}. Since $\gamma_{1357}(M) \neq 0$, we know that $\bar{f}_{12357}$ has degree one as a polynomial in z. Furthermore, the coefficient of z is $-\gamma_{1357}(M)x^{p^3-1}$. Therefore, using Theorem 2.4 of [Campbell and Chuai 2007], $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), \bar{f}_{12357}][x^{-1}]$. Thus, to prove $A = \mathbb{F}[V_M]^E$, it is sufficient to show that $\{N_M(y), \bar{f}_{12357}\} \subset A[x^{-1}]$.

Using Lemma 1.5 for the subsequence $(1, 2, 4)$ of $(1, 2, 3, 4, 5, 7)$ shows that

$$\gamma_{1235}^p \tilde{f}_{12357} = \gamma_{3457}\tilde{f}_{12357} \in \mathrm{Span}_{\mathbb{F}_p}\{\gamma_{2357}\tilde{f}_{13457}, \gamma_{1357}\tilde{f}_{23457}\}.$$

Thus $\bar{f}_{12357} \in \mathrm{Span}_{\mathbb{F}[x,x^{-1}]}\{\bar{f}_{13457}, \bar{f}_{23457}\}$. Similarly, using the $(1, 6, 7)$ subsequence of $(1, 3, 4, 5, 6, 7)$, we have that $\bar{f}_{13457} \in \mathrm{Span}_{\mathbb{F}[x,x^{-1}]}\{\bar{f}_{13456}, \bar{f}_{12345}^p\}$. Iterating this process gives

$$\bar{f}_{12357} \in \mathrm{Span}_{\mathbb{F}[x,x^{-1}]}\{\bar{f}_{12345}, \bar{f}_{12345}^p, \bar{f}_{12346}\}.$$

Since $\bar{f}_{12345} = \bar{f}_1$ and $\bar{f}_{12346} = 2\bar{f}_2x^{p^2-2p} - \bar{f}_1^2$, we see that $\bar{f}_{13457} \in A[x^{-1}]$. A similar argument shows that

$$\bar{f}_{13579} \in \mathrm{Span}_{\mathbb{F}[x,x^{-1}]}\big\{\bar{f}_{12345}^{p^i}, \bar{f}_{12346}^{p^j} \;\big|\; i, j \in \{0, 1, 2\}\big\},$$

giving $\bar{f}_{13579} \in A[x^{-1}]$. Since $\bar{f}_{13579} = \gamma_{1357}(M)N_M(y)$ (see Remark 1.2), we have $N_M(y) \in A[x^{-1}]$. Therefore $A = \mathbb{F}[V_M]^E$. As in the proof of Theorem 2.3, observe that $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

Remark 3.2. Lemmas 2.1 and 2.2 are only valid for $p > 5$. However, for the Magma calculations used to verify Theorem 2.3 for $p = 3$ and $p = 5$, only γ_{1234} and γ_{1235} are inverted. Thus Theorem 3.1 remains valid for $p = 3$ and $p = 5$.

4. The $\gamma_{1234} = 0$, $\gamma_{1235} \neq 0$, $\gamma_{1357} \neq 0$ stratum

In this section we consider representations V_M for $M \in \mathbb{F}^{2\times 4}$ for which $\gamma_{1234}(M) = 0$, $\gamma_{1235}(M) \neq 0$ and $\gamma_{1357}(M) \neq 0$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$. Evaluating coefficients gives

$$\bar{f}_1 = \bar{\gamma}_{1235}\delta^p x^{p^2-2p} + \bar{\gamma}_{1245} y^p x^{p^2-p} + \bar{\gamma}_{1345}\delta x^{p^2-2} + \bar{\gamma}_{2345} y x^{p^2-1}.$$

Define

$$h_1 := \frac{\bar{f}_1}{\bar{\gamma}_{1235} x^{p^2-2p}} \quad \text{and} \quad h_2 := \frac{\bar{f}_{12357}}{\bar{\gamma}_{1235}}$$

so that $\mathrm{LT}(h_1) = y^{2p}$ and $\mathrm{LT}(h_2) = y^{p^3}$. Note that $h_1, h_2 \in \mathbb{F}[V_M]^E$. Furthermore, arguing as in the proof of Theorem 3.1, $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), h_2][x^{-1}]$.

Lemma 4.1. $$N_M(y) = h_2^p + \left(\frac{\bar{\gamma}_{1237}^p}{\bar{\gamma}_{1235}^p} - \frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}}\right) h_2 x^{p^4-p^3} - \frac{\bar{\gamma}_{1357}^p}{\bar{\gamma}_{1235}^p} h_1 x^{p^4-2p}.$$

Proof. Since $\bar{f}_{13579} = \bar{\gamma}_{1357} N_M(y)$ (see Remark 1.2), we have

$$N_M(y) = y^{p^4} - \frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}} y^{p^3} x^{p^4-p^3} + \frac{\bar{\gamma}_{1379}}{\bar{\gamma}_{1357}} y^{p^2} x^{p^4-p^2} - \frac{\bar{\gamma}_{1579}}{\bar{\gamma}_{1357}} y^p x^{p^4-p} + \frac{\bar{\gamma}_{3579}}{\bar{\gamma}_{1357}} y x^{p^4-1}.$$

Using the definition gives

$$h_2 := y^{p^3} - \frac{\bar{\gamma}_{1237}}{\bar{\gamma}_{1235}} y^{p^2} x^{p^3-p^2} + \frac{\bar{\gamma}_{1257}}{\bar{\gamma}_{1235}} y^p x^{p^3-p} + \frac{\bar{\gamma}_{1357}}{\bar{\gamma}_{1235}} \delta x^{p^3-2} + \frac{\bar{\gamma}_{2357}}{\bar{\gamma}_{1235}} y x^{p^3-1}.$$

Thus

$$\begin{aligned} N_M(y) - h_2^p = {} & \left(\frac{\bar{\gamma}_{1237}^p}{\bar{\gamma}_{1235}^p} - \frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}}\right) y^{p^3} x^{p^4-p^3} - \left(\frac{\bar{\gamma}_{1257}^p}{\bar{\gamma}_{1235}^p} - \frac{\bar{\gamma}_{1379}}{\bar{\gamma}_{1357}}\right) y^{p^2} x^{p^4-p^2} \\ & - \left(\frac{\bar{\gamma}_{2357}^p}{\bar{\gamma}_{1235}^p} + \frac{\bar{\gamma}_{1579}}{\bar{\gamma}_{1357}}\right) y^p x^{p^4-p} - \frac{\bar{\gamma}_{1357}^p}{\bar{\gamma}_{1235}^p} \delta^p x^{p^4-2p} + \frac{\bar{\gamma}_{3579}}{\bar{\gamma}_{1357}} y x^{p^4-1}. \end{aligned}$$

Using the $(1,3,5)(3,4,5,7,9)$, $(1,3,7)(3,4,5,7,9)$ and $(1,5,7)(3,4,5,7,9)$ Plücker relations gives

$$\begin{aligned} N_M(y) - h_2^p = {} & \frac{\bar{\gamma}_{1357}^{p-1}}{\bar{\gamma}_{1235}^p} \big(\bar{\gamma}_{1345} y^{p^3} x^{p^4-p^3} - \bar{\gamma}_{1347} y^{p^2} x^{p^4-p^2} \\ & - \bar{\gamma}_{1457} y^p x^{p^4-p} - \bar{\gamma}_{1357} \delta^p x^{p^4-2p}\big) + \frac{\bar{\gamma}_{3579}}{\bar{\gamma}_{1357}} y x^{p^4-1}. \end{aligned}$$

Using the $(1,2,3)(1,3,4,5,7)$ and $(1,2,5)(1,3,4,5,7)$ Plücker relations,

$$\bar{\gamma}_{1347} = \frac{\bar{\gamma}_{1237}\bar{\gamma}_{1345}}{\bar{\gamma}_{1235}} \quad \text{and} \quad \bar{\gamma}_{1457} = \frac{\bar{\gamma}_{1245}\bar{\gamma}_{1357} - \bar{\gamma}_{1257}\bar{\gamma}_{1345}}{\bar{\gamma}_{1235}}.$$

Thus

$$N_M(y) = h_2^p + \frac{\bar{\gamma}_{1357}^{p-1}}{\bar{\gamma}_{1235}^p}\left(\bar{\gamma}_{1345}h_2 x^{p^4-p^3} - \frac{\bar{\gamma}_{1357}\bar{\gamma}_{1245}}{\bar{\gamma}_{1235}} y^p x^{p^4-p} - \frac{\bar{\gamma}_{1357}\bar{\gamma}_{1345}}{\bar{\gamma}_{1235}}\delta x^{p^4-2}\right) - \frac{\bar{\gamma}_{1357}^p}{\bar{\gamma}_{1235}^p}\delta^p x^{p^4-2p} + \left(\frac{\bar{\gamma}_{3579}}{\bar{\gamma}_{1357}} - \frac{\bar{\gamma}_{1357}^{p-1}\bar{\gamma}_{1345}\bar{\gamma}_{2357}}{\bar{\gamma}_{1235}^{p+1}}\right) y x^{p^4-1}.$$

Using the $(1,3,5)(2,3,4,5,7)$ Plücker relation, $\bar{\gamma}_{1345}\bar{\gamma}_{2357} = \bar{\gamma}_{1357}\bar{\gamma}_{2345} + \bar{\gamma}_{1235}^{p+1}$, giving

$$\frac{\bar{\gamma}_{1357}^{p-1}\bar{\gamma}_{1345}\bar{\gamma}_{2357}}{\bar{\gamma}_{1235}^{p+1}} = \frac{\bar{\gamma}_{2345}\bar{\gamma}_{1357}^p}{\bar{\gamma}_{1235}^p} + \bar{\gamma}_{1357}^{p-1}.$$

From the definition of h_1,

$$N_M(y) = h_2^p + \frac{\bar{\gamma}_{1357}^{p-1}\bar{\gamma}_{1345}}{\bar{\gamma}_{1235}^p} h_2 x^{p^4-p^3} - \frac{\bar{\gamma}_{1357}^p}{\bar{\gamma}_{1235}^p} h_1 x^{p^4-2p} + \left(\frac{\bar{\gamma}_{3579}}{\bar{\gamma}_{1357}} - \bar{\gamma}_{1357}^{p-1}\right) y x^{p^4-1}.$$

The result follows from the fact that $\bar{\gamma}_{3579} = \bar{\gamma}_{1357}^p$. □

As a consequence of the lemma, $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, h_1, h_2][x^{-1}]$. Thus applying the SAGBI/divide-by-x algorithm to $\{x, h_1, h_2, N_M(z)\}$ produces a generating set for $\mathbb{F}[V_M]^E$. Subducting the tête-à-tête $(h_2^2, h_1^{p^2})$ gives

$$\tilde{h}_3 := h_2^2 - h_1^{p^2} + 2\frac{\bar{\gamma}_{1237}}{\bar{\gamma}_{1235}} h_1^{p(p+1)/2} x^{p^3-p^2} - 2\frac{\bar{\gamma}_{1257}}{\bar{\gamma}_{1235}} h_1^{(p^2+1)/2} x^{p^3-p}.$$

Lemma 4.2.
$$\mathrm{LT}(\tilde{h}_3) = \frac{2\bar{\gamma}_{1357}}{\bar{\gamma}_{1235}} y^{p^3+2} x^{p^3-2}.$$

Proof. We work modulo the ideal in $\mathbb{F}[x, y, z]$ generated by x^{p^3-1}. Therefore $h_1^{p^2} \equiv y^{2p^3}$, $h_1 x^{p^3-p} \equiv y^{2p} x^{p^3-p}$ and

$$h_2^2 \equiv y^{2p^3} - 2\frac{\bar{\gamma}_{1237}}{\bar{\gamma}_{1235}} y^{p^3+p^2} x^{p^3-p^2} + 2\frac{\bar{\gamma}_{1257}}{\bar{\gamma}_{1235}} y^{p^3+p} x^{p^3-p} + 2\frac{\bar{\gamma}_{1357}}{\bar{\gamma}_{1235}} \delta y^{p^3} x^{p^3-2}.$$

Since $x^{p^3-p^2} h_1^p \equiv x^{p^3-p^2} y^{2p^2}$, we have $(h_1^p)^{(p+1)/2} x^{p^3-p^2} \equiv x^{p^3-p^2} y^{p^3+p^2}$. Thus

$$h_2^2 \equiv h_1^{p^2} - 2\frac{\bar{\gamma}_{1237}}{\bar{\gamma}_{1235}} h_1^{p(p+1)/2} x^{p^3-p^2} + 2\frac{\bar{\gamma}_{1257}}{\bar{\gamma}_{1235}} h_1^{(p^2+1)/2} x^{p^3-p} + 2\frac{\bar{\gamma}_{1357}}{\bar{\gamma}_{1235}} \delta y^{p^3} x^{p^3-2}.$$

Hence $\tilde{h}_3 \equiv 2(\bar{\gamma}_{1357}/\bar{\gamma}_{1235})\delta y^{p^3} x^{p^3-2}$, and the result follows. □

Define $h_3 := \bar\gamma_{1235}\tilde h_3/(2\bar\gamma_{1357}x^{p^3-2})$ so that $\mathrm{LT}(h_3) = y^{p^3+2}$. Subducting the tête-à-tête $(h_3^p, h_2^p h_1)$ gives

$$\tilde h_4 := h_3^p - h_1h_2^p - \alpha_1 x^p h_1^{(p^3+1)/2} + \alpha_2 x^{2p-2}h_3h_1^{(p^3-p^2)/2} - \alpha_3 x^{2p-1}h_1^{(p^2-1)/2}h_2^{(p-3)/2}h_3^{(p+1)/2},$$

with

$$\alpha_1 := \left(\frac{\bar\gamma_{2357}}{\bar\gamma_{1357}}\right)^p - \frac{\bar\gamma_{1245}}{\bar\gamma_{1235}},\quad \alpha_2 := \frac{\bar\gamma_{1345}}{\bar\gamma_{1235}},\quad \alpha_3 := \alpha_2\frac{\bar\gamma_{2357}}{\bar\gamma_{1357}} - \frac{\bar\gamma_{2345}}{\bar\gamma_{1235}}.$$

Lemma 4.3.
$$\mathrm{LT}(\tilde h_4) = \left(\frac{\bar\gamma_{1235}}{\bar\gamma_{1357}}\right)^p x^{2p}z^{p^4}.$$

Proof. We work modulo the ideal $\mathfrak{n} := \langle x^{2p+1}, x^{2p}y\rangle$. Using the definition of h_3 and methods analogous to the proof of Lemma 4.2, it is not hard to show that

$$h_3 \equiv_{\langle x^3,x^2y\rangle} \delta y^{p^3} + \frac{\bar\gamma_{2357}}{\bar\gamma_{1357}}xy^{p^3+1} + \frac{\bar\gamma_{1235}}{\bar\gamma_{1357}}x^2z^{p^3}.$$

Thus

$$h_3^p \equiv_{\mathfrak{n}} \delta^p y^{p^4} + \left(\frac{\bar\gamma_{2357}}{\bar\gamma_{1357}}\right)^p x^p y^{p^4+p} + \left(\frac{\bar\gamma_{1235}}{\bar\gamma_{1357}}\right)^p x^{2p}z^{p^4}.$$

Since $h_2 \equiv_{\mathfrak{n}} y^{p^3}$, we have

$$h_1h_2^p \equiv_{\mathfrak{n}} y^{p^4}\left(\delta^p + \frac{\bar\gamma_{1245}}{\bar\gamma_{1235}}y^px^p + \frac{\bar\gamma_{1345}}{\bar\gamma_{1235}}\delta x^{2p-2} + \frac{\bar\gamma_{2345}}{\bar\gamma_{1235}}yx^{2p-1}\right).$$

Furthermore, since $x^ph_1 \equiv_{\mathfrak{n}} x^p\delta^p$, expanding gives $x^ph_1^{(p^3+1)/2} \equiv_{\mathfrak{n}} x^py^{p^4+p}$. Thus

$$h_3^p - h_1h_2^p - \alpha_1x^ph_1^{(p^3+1)/2} \equiv_{\mathfrak{n}} -\frac{\bar\gamma_{1345}}{\bar\gamma_{1235}}\delta x^{2p-2}y^{p^4} - \frac{\bar\gamma_{2345}}{\bar\gamma_{1235}}x^{2p-1}y^{p^4+1} + \left(\frac{\bar\gamma_{1235}}{\bar\gamma_{1357}}\right)^p x^{2p}z^{p^4}.$$

Note that $x^{2p-2}h_1^{(p^3-p^2)/2} \equiv_{\mathfrak{n}} x^{2p-2}y^{p^4-p^3}$. Thus

$$x^{2p-2}h_3h_1^{(p^3-p^2)/2} \equiv_{\mathfrak{n}} x^{2p-2}\left(y^{p^4}\delta + \frac{\bar\gamma_{2357}}{\bar\gamma_{1357}}xy^{p^4+1}\right).$$

Hence

$$h_3^p - h_1h_2^p - \alpha_1x^ph_1^{(p^3+1)/2} + \alpha_2x^{2p-2}h_3h_1^{(p^3-p^2)/2} \equiv_{\mathfrak{n}} \alpha_3x^{2p-1}y^{p^4+1} + \left(\frac{\bar\gamma_{1235}}{\bar\gamma_{1357}}\right)^p x^{2p}z^{p^4}.$$

Since $x^{2p-1}h_1^{(p^2-1)/2}h_2^{(p-3)/2}h_3^{(p+1)/2} \equiv_{\mathfrak{n}} x^{2p-1}y^{p^4+1}$, the result follows. □

Define $h_4 := \bar\gamma_{1357}\tilde h_4/(\bar\gamma_{1357}x^{2p})$ so that $\mathrm{LT}(h_4) = z^{p^4}$.

Theorem 4.4. *If* $\gamma_{1234}(M) = 0$, $\gamma_{1235}(M) \neq 0$ *and* $\gamma_{1357}(M) \neq 0$, *then the set* $\mathcal{B} := \{x, h_1, h_2, h_3, N_M(z)\}$ *is a SAGBI basis, and hence a generating set, for* $\mathbb{F}[V_M]^E$. *Furthermore,* $\mathbb{F}[V_M]^E$ *is a complete intersection with generating relations coming from the subduction of the tête-à-têtes* $(h_2^2, h_1^{p^2})$ *and* $(h_3^p, h_1 h_2^p)$.

Proof. Define $\mathcal{B}' := \{x, h_1, h_2, h_3, h_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-têtes for $\mathcal{B}'$ are $(h_2^2, h_1^{p^2})$ and $(h_3^p, h_1 h_2^p)$. Using Lemmas 4.2 and 4.3, these tête-à-têtes subduct to zero, proving that $\mathcal{B}'$ is a SAGBI basis for A. Since $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, h_1, h_2][x^{-1}]$, using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Finally, observe that $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

5. The $\gamma_{1234} \neq 0$, $\gamma_{1235} = 0$, $\gamma_{1357} \neq 0$ strata

In this section we consider representations V_M for $M \in \mathbb{F}^{2\times 4}$ for which $\gamma_{1235}(M) = 0$, $\gamma_{1234}(M) \neq 0$ and $\gamma_{1357}(M) \neq 0$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$.

Lemma 5.1. *If* $\bar{\gamma}_{1234} \neq 0$, $\bar{\gamma}_{1235} = 0$ *and* $\bar{\gamma}_{1357} \neq 0$, *then* $\bar{\gamma}_{1345} \neq 0$.

Proof. Let r_i denote row i of the matrix $\Gamma(M)$. Since $\bar{\gamma}_{1234} \neq 0$, the set $\{r_1, r_2, r_3, r_4\}$ is linearly independent. Using this and the hypothesis that $\bar{\gamma}_{1235} = 0$, we conclude that r_5 is a linear combination of $\{r_1, r_2, r_3\}$, say $r_5 = a_1 r_1 + a_2 r_2 + a_3 r_3$. Since r_3 is nonzero and the entries of r_5 are the p-th powers of the entries of r_3, we see that r_5 is nonzero. Suppose, by way of contradiction, that $\bar{\gamma}_{1345} = 0$. Then r_5 is a nonzero linear combination of $\{r_1, r_3, r_4\}$, say $r_5 = b_1 r_1 + b_3 r_3 + b_4 r_4$. Thus $b_1 r_1 + b_3 r_3 + b_4 r_4 = a_1 r_1 + a_2 r_2 + a_3 r_3$. Since $\{r_1, r_2, r_3, r_4\}$ is linearly independent, $b_4 = a_2 = 0$, $a_1 = b_1$, $a_3 = b_3$ and $r_5 = a_1 r_1 + a_3 r_3$, contradicting the assumption that $\bar{\gamma}_{1357} \neq 0$. □

Take f_1 as defined in Section 2, evaluate coefficients and divide by $\bar{\gamma}_{1234}$ to get

$$\hat{f}_1 := y^{p^2} + \frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1234}} y^p x^{p^2-p} + \frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1234}} \delta x^{p^2-2} + \frac{\bar{\gamma}_{2345}}{\bar{\gamma}_{1234}} y x^{p^2-1}.$$

Note that $\hat{f}_1$ is of degree one in z with coefficient $x^{p^2-2}\bar{\gamma}_{1345}/\bar{\gamma}_{1234}$ and so, using Theorem 2.4 of [Campbell and Chuai 2007], $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), \hat{f}_1][x^{-1}]$. Define

$$\tilde{h}_2 := N_M(y) - \hat{f}_1^{p^2} + \alpha_1 \hat{f}_1^p x^{p^4-p^3} + \alpha_2 \hat{f}_1^2 x^{p^4-2p^2},$$

with

$$\alpha_1 := \frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}} + \frac{\bar{\gamma}_{1245}^{p^2}}{\bar{\gamma}_{1234}^{p^2}} \quad\text{and}\quad \alpha_2 := \frac{\bar{\gamma}_{1345}^{p^2}}{\bar{\gamma}_{1234}^{p^2}}.$$

We work modulo the ideal $\mathfrak{n} := \langle x^{p^4-p^2-1}\rangle$. Since $\bar{\gamma}_{1357} N_M(y) = \bar{f}_{13579}$ (see Remark 1.2), we have $N_M(y) \equiv_{\mathfrak{n}} y^{p^4} - (\bar{\gamma}_{1359}/\bar{\gamma}_{1357}) y^{p^3} x^{p^4-p^3}$. Therefore

$$N_M(y) - \hat{f}_1^{p^2} \equiv_{\mathfrak{n}} -\left(\frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}} + \frac{\bar{\gamma}_{1245}^{p^2}}{\bar{\gamma}_{1234}^{p^2}}\right) y^{p^3} x^{p^4-p^3} - \frac{\bar{\gamma}_{1345}^{p^2}}{\bar{\gamma}_{1234}^{p^2}} \delta^{p^2} x^{p^4-2p^2}.$$

Thus

$$N_M(y) - \hat{f}_1^{p^2} + \alpha_1 \hat{f}_1^p x^{p^4-p^3} \equiv_{\mathfrak{n}} -\frac{\bar{\gamma}_{1345}^{p^2}}{\bar{\gamma}_{1234}^{p^2}} \delta^{p^2} x^{p^4-2p^2} \equiv_{\mathfrak{n}} -\frac{\bar{\gamma}_{1345}^{p^2}}{\bar{\gamma}_{1234}^{p^2}} y^{2p^2} x^{p^4-2p^2}.$$

Hence

$$\begin{aligned}\tilde{h}_2 &= N_M(y) - \hat{f}_1^{p^2} + \alpha_1 \hat{f}_1^p x^{p^4-p^3} + \alpha_2 \hat{f}_1^2 x^{p^4-2p^2}\\ &\equiv_{\mathfrak{n}} \frac{2\alpha_2}{\bar{\gamma}_{1234}} (\bar{\gamma}_{1245} y^{p^2+p} x^{p^4-p^2-p} + \bar{\gamma}_{1345} y^{p^2+2} x^{p^4-p^2-2}).\end{aligned}$$

We first consider the case $\bar{\gamma}_{1245} \neq 0$. Define

$$h_2 := \bar{\gamma}_{1234}^{p^2+1} \tilde{h}_2 / (2x^{p^4-p^2-p} \bar{\gamma}_{1345}^{p^2} \bar{\gamma}_{1245})$$

so that $\mathrm{LT}(h_2) = y^{p^2+p}$. Since $N_M(y) \in \mathbb{F}[x, \hat{f}_1, h_2]$, we have

$$\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, \hat{f}_1, h_2][x^{-1}].$$

Subducting the tête-à-tête $(h_2^p, \hat{f}_1^{p+1})$ gives

$$\tilde{h}_3 := \hat{f}_1^{p+1} - h_2^p + \left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^p \hat{f}_1^{p-2} h_2^2 x^{p^2-2p}.$$

Lemma 5.2. $$\mathrm{LT}(\tilde{h}_3) = 2\left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^{p+1} y^{p^3+p+2} x^{p^2-p-2}.$$

Proof. We work modulo the ideal $\langle x^{p^2-p-1} \rangle$. Thus $\hat{f}_1 \equiv y^{p^2}$. Reviewing the definition of h_2, we see that

$$h_2^p \equiv y^{p^3+p^2} + \left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^p y^{p^3+2p} x^{p^2-2p}$$

and

$$h_2^2 x^{p^2-2p} \equiv y^{2p^2+2p} x^{p^2-2p} + 2\left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right) y^{2p^2+p+2} x^{p^2-p-2}.$$

Thus

$$\hat{f}_1^{p+1} - h_2^p + \left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^p \hat{f}_1^{p-2} h_2^2 x^{p^2-2p} \equiv 2\left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^{p+1} y^{p^3+p+2} x^{p^2-p-2}$$

and the result follows. □

Define $h_3 := \bar{\gamma}_{1245}^{p+1} \tilde{h}_3 / (2\bar{\gamma}_{1345}^{p+1} x^{p^2-p-2})$ so that $\mathrm{LT}(h_3) = y^{p^3+p+2}$.

Lemma 5.3. *Subducting the tête-à-tête $(h_3^p, \hat{f}_1^{p^2-1} h_2^2)$ gives an invariant with lead term $-\bar{\gamma}_{1245}^p \bar{\gamma}_{1234}^{p^2} z^{p^4} x^{p^2+2p} / (4\bar{\gamma}_{1345}^{p^2+p})$.*

Proof. Modulo the ideal $\langle x^{p^2+2p+1}, x^{p^2+2p}y\rangle$, the expression

$$\begin{aligned} h_3^p - \hat{f}_1^{p^2-1}h_2^2 + \beta_1 h_3 \hat{f}_1^{p^2-p+1}x^{p-2} + \beta_2 h_2 \hat{f}_1^{p^2} x^p + \beta_3 \hat{f}_1^{p^2+1}x^{2p} \\ + \beta_4 h_2^4 \hat{f}_1^{p^2-4}x^{p^2-2p} + \beta_5 h_3 h_2^2 \hat{f}_1^{p^2-p-2}x^{p^2-p-2} \\ + \beta_6 h_3^2 \hat{f}_1^{p^2-2p}x^{p^2-4} + \beta_7 h_2^2 \hat{f}_1^{p^2-2}x^{p^2} + \beta_8 h_3 \hat{f}_1^{p^2-p}x^{p^2+p-2} \\ + \beta_9 h_2 \hat{f}_1^{p^2-1}x^{p^2+p} + \beta_{10} h_3 h_2^{p-1}\hat{f}_1^{p^2-2p}x^{p^2+2p-2} \\ + \beta_{11} h_3^{(p+1)/2} h_2^{(p-3)/2} \hat{f}_1^{(p^2+1)/2-p}x^{p^2+2p-1}, \end{aligned}$$

with

$$\beta_1 := 2\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}, \quad \beta_2 := -\left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1234}}\right)^{p+1}\left(\frac{\bar{\gamma}_{1234}}{\bar{\gamma}_{1345}}\right)^p,$$

$$\beta_3 := \frac{1}{2}\left(\frac{\bar{\gamma}_{1245,}}{\bar{\gamma}_{1345}}\right)^p\left(\left(\frac{\bar{\gamma}_{2345}}{\bar{\gamma}_{1345}}\right)^{p^2} - \left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1345}}\right)^{p^2}\left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1234}}\right)^p\right),$$

$$\beta_4 := -\frac{1}{2}\left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^p, \quad \beta_5 := 2\left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^{p+1}, \quad \beta_6 := -2\left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^{p+2},$$

$$\beta_7 := \frac{1}{2}\left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1345}}\right)^p\left(\frac{\bar{\gamma}_{1234}}{\bar{\gamma}_{1345}}\right)^{p^2-p}\frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}}, \quad \beta_8 := -\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1234}}\beta_7,$$

$$\beta_9 := -\frac{1}{2}\left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1345}}\right)^p\left(\frac{\bar{\gamma}_{1234}}{\bar{\gamma}_{1345}}\right)^{p^2}\left(\frac{\bar{\gamma}_{1245}\bar{\gamma}_{1379}}{\bar{\gamma}_{1234}\bar{\gamma}_{1357}} + \frac{\bar{\gamma}_{1579}}{\bar{\gamma}_{1357}}\right),$$

$$\beta_{10} := \frac{1}{2}\left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1345}}\right)^{p-1}\left(\frac{\bar{\gamma}_{1234}}{\bar{\gamma}_{1345}}\right)^{p^2}\frac{\bar{\gamma}_{1379}}{\bar{\gamma}_{1357}}, \quad \beta_{11} := \frac{1}{2}\left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1345}}\right)^p\left(\frac{\bar{\gamma}_{1234}}{\bar{\gamma}_{1345}}\right)^{p^2}\frac{\bar{\gamma}_{3579}}{\bar{\gamma}_{1357}},$$

is congruent to $-\bar{\gamma}_{1245}^p\bar{\gamma}_{1234}^{p^2}z^{p^4}x^{p^2+2p}/(4\bar{\gamma}_{1345}^{p^2+p})$. □

Theorem 5.4. *If $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) = 0$, $\gamma_{1357}(M) \neq 0$ and $\gamma_{1245}(M) \neq 0$, then the set $\mathcal{B} := \{x, \hat{f}_1, h_2, h_3, N_M(z)\}$ is a SAGBI basis for $\mathbb{F}[V_M]^E$. Furthermore, $\mathbb{F}[V_M]^E$ is a complete intersection with generating relations coming from the subduction of the tête-à-têtes $(h_2^p, \hat{f}_1^{p+1})$ and $(h_3^p, \hat{f}_1^{p^2-1}h_2^2)$.*

Proof. Use the subduction of $(h_3^p, \hat{f}_1^{p^2-1}h_2^2)$ given in Lemma 5.3 to construct an invariant h_4 with lead term z^{p^4}. Define $\mathcal{B}' := \{x, \hat{f}_1, h_2, h_3, h_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-têtes for $\mathcal{B}'$ are $(h_2^p, \hat{f}_1^{p+1})$ and $(h_3^p, \hat{f}_1^{p^2-1}h_2^2)$. Using Lemmas 5.2 and 5.3, these tête-à-têtes subduct to zero, proving that $\mathcal{B}'$ is a SAGBI basis for A. Since $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, \hat{f}_1, h_2][x^{-1}]$, using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Finally, observe that $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

We now consider the case $\bar{\gamma}_{1245} = 0$. Define $\hat{h}_2 := \bar{\gamma}_{1234}^{p^2+1}\tilde{h}_2/(2x^{p^4-p^2-2}\bar{\gamma}_{1345}^{p^2+1})$ so that $\mathrm{LT}(\hat{h}_2) = y^{p^2+2}$. Since $N_M(y) \in \mathbb{F}[x, \hat{f}_1, \hat{h}_2]$, we have $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, \hat{f}_1, \hat{h}_2][x^{-1}]$.

Lemma 5.5. *Subducting the tête-à-tête* $(\hat{h}_2^{p^2}, \hat{f}_1^{p^2+2})$ *gives an invariant with lead term* $z^{p^4}(\bar{\gamma}_{1234}x^2/(2\bar{\gamma}_{1345}))^{p^2}$.

Proof. Modulo the ideal $\langle x^{p^2+1}, x^{p^2}y\rangle$, the expression

$$\begin{aligned}\hat{f}_1^{p^2+2} - \hat{h}_2^{p^2} - \big(&\alpha_1\hat{h}_2\hat{f}_1^{p^2}x^{p^2-2} + \alpha_2\hat{f}_1^{p^2+1}x^{p^2}\\ &+\alpha_3\hat{h}_2^{p}\hat{f}_1^{p^2-p}x^{2p^2-2p} + \alpha_4\hat{h}_2^{p(p+1)/2}\hat{f}_1^{(p^2-p-2)/2}x^{2p^2-p}\\ &+\alpha_5\hat{h}_2\hat{f}_1^{p^2-1}x^{2p^2-2} + \alpha_6\hat{h}_2^{(p^2+1)/2}\hat{f}_1^{(p^2-3)/2}x^{2p^2-1}\big),\end{aligned}$$

with

$$\alpha_1 := \frac{2\bar{\gamma}_{1345}}{\bar{\gamma}_{1234}}, \qquad \alpha_2 := -\frac{\bar{\gamma}_{1379}\bar{\gamma}_{1234}^{p^2}}{\bar{\gamma}_{1357}\bar{\gamma}_{1345}^{p^2}}, \quad \alpha_3 := -\frac{\bar{\gamma}_{1359}\bar{\gamma}_{1234}^{p^2-p}}{\bar{\gamma}_{1357}\bar{\gamma}_{1345}^{p^2-p}},$$

$$\alpha_4 := \frac{\bar{\gamma}_{1579}\bar{\gamma}_{1234}^{p^2}}{\bar{\gamma}_{1357}\bar{\gamma}_{1345}^{p^2}}, \quad \alpha_5 := \frac{\bar{\gamma}_{1379}\bar{\gamma}_{1234}^{p^2-1}}{\bar{\gamma}_{1357}\bar{\gamma}_{1345}^{p^2-1}}, \quad \alpha_6 := -\frac{\bar{\gamma}_{3579}\bar{\gamma}_{1234}^{p^2}}{\bar{\gamma}_{1357}\bar{\gamma}_{1345}^{p^2}},$$

is congruent to $z^{p^4}(\bar{\gamma}_{1234}x^2/(2\bar{\gamma}_{1345}))^{p^2}$. □

Theorem 5.6. *If* $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) = 0$, $\gamma_{1357}(M) \neq 0$ *and* $\gamma_{1245}(M) = 0$, *then the set* $\mathcal{B} := \{x, \hat{f}_1, \hat{h}_2, N_M(z)\}$ *is a SAGBI basis for* $\mathbb{F}[V_M]^E$. *Furthermore,* $\mathbb{F}[V_M]^E$ *is a hypersurface with the relation coming from the subduction of the tête-à-tête* $(\hat{h}_2^{p^2}, \hat{f}_1^{p^2+2})$.

Proof. Use the subduction of $(\hat{h}_2^{p^2}, \hat{f}_1^{p^2+2})$ given in Lemma 5.5 to construct an invariant $\hat{h}_3$ with lead term z^{p^4}. Define $\mathcal{B}' := \{x, \hat{f}_1, \hat{h}_2, \hat{h}_3\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-tête for $\mathcal{B}'$ is $(\hat{h}_2^{p^2}, \hat{f}_1^{p^2+2})$, which subducts to zero using Lemma 5.5. Thus $\mathcal{B}'$ is a SAGBI basis for A. Since $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, \hat{f}_1, \hat{h}_2][x^{-1}]$, using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Finally, observe that $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

6. The $\gamma_{1234} \neq 0$, $\gamma_{1235} \neq 0$, $\gamma_{1357} = 0$ stratum

In this section we consider representations V_M for $M \in \mathbb{F}^{2\times 4}$ for which $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) \neq 0$ and $\gamma_{1357}(M) = 0$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$. Evaluating the coefficients of f_1 and dividing by $\bar{\gamma}_{1234}$ gives $\hat{f}_1$ with lead term y^{p^2}. Since $\bar{\gamma}_{1357} = 0$ and $\bar{\gamma}_{1235} \neq 0$, the orbit of y has size p^3 and $N_M(y) = \bar{f}_{12357}/\bar{\gamma}_{1235}$ (see Remark 1.2). For convenience, write

$$N_M(y) = y^{p^3} + \alpha_2 y^{p^2}x^{p^3-p^2} + \alpha_1 y^p x^{p^3-p} + \alpha_0 y x^{p^3-1}$$

and

$$\hat{f}_1 = y^{p^2} + \beta_3\delta^p x^{p^2-2p} + \beta_2 y^p x^{p^2-p} + \beta_1\delta x^{p^2-2} + \beta_0 y x^{p^2-1},$$

with

$$\alpha_2 = -\frac{\bar{\gamma}_{1237}}{\bar{\gamma}_{1235}}, \quad \alpha_1 = \frac{\bar{\gamma}_{1257}}{\bar{\gamma}_{1235}}, \quad \alpha_0 = \frac{\bar{\gamma}_{2357}}{\bar{\gamma}_{1235}},$$

$$\beta_3 = \frac{\bar{\gamma}_{1235}}{\bar{\gamma}_{1234}}, \quad \beta_2 = \frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1234}}, \quad \beta_1 = \frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1234}}, \quad \beta_0 = \frac{\bar{\gamma}_{2345}}{\bar{\gamma}_{1234}}.$$

Subducting $N_M(y)$ gives

$$\tilde{h}_2 := N_M(y) - \hat{f}_1^p + \beta_3^p x^{p^3-2p^2} \hat{f}_1^2.$$

Lemma 6.1. $$\mathrm{LT}(\tilde{h}_2) = 2\left(\frac{\bar{\gamma}_{1235}}{\bar{\gamma}_{1234}}\right)^{p+1} y^{p^2+2p} x^{p^3-p^2-2p}.$$

Proof. We work modulo the ideal $\langle x^{p^3-p^2-p}\rangle$. Using the definitions of f_{12357} and f_{12345}, we have $N_M(y) \equiv y^{p^3}$ and $\hat{f}_1^p \equiv y^{p^3} + (\bar{\gamma}_{1235}/\bar{\gamma}_{1234})^p y^{2p^2} x^{p^3-2p^2}$. The result follows from the observation that

$$\hat{f}_1 x^{p^3-2p^2} \equiv y^{p^2} x^{p^3-2p^2} + \left(\frac{\bar{\gamma}_{1235}}{\bar{\gamma}_{1234}}\right) y^{2p} x^{p^3-p^2-2p}. \qquad \square$$

Define $h_2 := \tilde{h}_2 \bar{\gamma}_{1234}^{p+1}/(2\bar{\gamma}_{1235}^{p+1} x^{p^3-p^2-2p})$ so that $\mathrm{LT}(h_2) = y^{p^2+2p}$ and

$$h_2 \equiv_{\langle x^{2p}\rangle} y^{p^2}\left(\delta^p + \frac{\beta_2}{\beta_3} y^p x^p + \frac{\beta_1}{\beta_3}\delta x^{2p-2} + \frac{\beta_0}{\beta_3} y x^{2p-1}\right). \tag{1}$$

Lemma 6.2. $$\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, \hat{f}_1, h_2][x^{-1}].$$

Proof. Since $\bar{\gamma}_{1357} = 0$ and the first row of M is nonzero, we can use a change of coordinates, see [Campbell et al. 2013, §4], and the $\mathrm{GL}_4(\mathbb{F}_p)$-action to write

$$M = \begin{pmatrix} 1 & c_{12} & c_{13} & 0 \\ 0 & c_{22} & c_{23} & c_{24} \end{pmatrix}.$$

Since $\bar{\gamma}_{1235} \neq 0$, we have $c_{24} \neq 0$. With this choice of generators for E, let H denote the subgroup generated by e_1 and e_4. Using the calculation of $\mathbb{F}[x, y, z]^H$ from Theorem 6.4 of [loc. cit.], we see that $\mathbb{F}[V_M]^H[x^{-1}] = \mathbb{F}[x, N_H(y), N_H(\delta)][x^{-1}]$ with $N_H(y) := y^p - yx^{p-1}$ and $N_H(\delta) = \delta^p - \delta(c_{24}x^2)^{p-1}$. Thus, to compute $\mathbb{F}[V_M]^G[x^{-1}] = (\mathbb{F}[V_M]^H[x^{-1}])^{G/H}$, it is sufficient to compute

$$(\mathbb{F}[x, N_H(y), N_H(\delta)][x^{-1}])^{G/H} = \mathbb{F}[x, N_H(y)/x^{p-1}, N_H(\delta)/x^{2p-1}]^{G/H}[x^{-1}].$$

Note that $\deg(N_H(y)/x^{p-1}) = \deg(N_H(\delta)/x^{2p-1}) = 1$. Furthermore

$$\mathbb{F}[x, N_H(y)/x^{p-1}]^{G/H} = \mathbb{F}[x, N_{G/H}(N_H(y)/x^{p-1})]$$

and $N_{G/H}(N_H(y)/x^{p-1}) = N_M(y)/x^{p^3-p^2}$. Using the form of M given above, we see that $\bar{\gamma}_{1345} = -c_{24}^{p-1}\bar{\gamma}_{1235}$. If we evaluate $\widetilde{\Gamma}$ at M and set $x = 1$, $y = 1$

and $z=1$, then first and last columns of the resulting matrix are equal. Thus $\bar{f}_{12345}(1,1,1)=\bar{\gamma}_{1234}+\bar{\gamma}_{1245}+\bar{\gamma}_{2345}=0$. Using these two relations, we can write

$$\hat{f}_1 = N_H(y)^p - \frac{\bar{\gamma}_{2345}}{\bar{\gamma}_{1234}} N_H(y) x^{p^2-p} + \frac{\bar{\gamma}_{1235}}{\bar{\gamma}_{1234}} N_H(\delta) x^{p^2-2p}.$$

Thus we have $\hat{f}_1/x^{p^2-p} \in \mathbb{F}[x, N_H(y)/x^{p-1}, N_H(\delta)/x^{2p-1}]^{G/H}$ is of degree one in $N_H(\delta)/x^{2p-1}$ with coefficient $x^{p-1}\bar{\gamma}_{1235}/\bar{\gamma}_{1234}$. Thus by Theorem 2.4 of [Campbell and Chuai 2007], we have

$$\mathbb{F}[x, N_H(y)/x^{p-1}, N_H(\delta)/x^{2p-1}]^{G/H}[x^{-1}] = \mathbb{F}[x, N_M(y)/x^{p^3-p^2}, \hat{f}_1/x^{p^2-p}][x^{-1}].$$

Therefore $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), \hat{f}_1][x^{-1}]$. The result then follows from the fact that $N_M(y) \in \mathbb{F}[x, \hat{f}_1, h_2]$. □

Subducting the tête-à-tête $(h_2^p, \hat{f}_1^{p+2})$ gives

$$\begin{aligned}\tilde{h}_3 := h_2^p - \hat{f}_1^{p+2} + 2\beta_3 \hat{f}_1^p h_2 x^{p^2-2p} \\ - \beta_3^{-p}(\alpha_2 \hat{f}_1^{p+1} x^{p^2} - \alpha_2\beta_3 \hat{f}_1^{p-1} h_2 x^{2p^2-2p} + \alpha_1 \hat{f}_1^{(p-3)/2} h_2^{(p+1)/2} x^{2p^2-p})\end{aligned}$$

for $p \geq 5$ and

$$\begin{aligned}\tilde{h}_3 := h_2^3 - \hat{f}_1^5 + 2\beta_3 \hat{f}_1^3 h_2 x^3 \\ - (\alpha_2\beta_3^{-3} + \beta_3^3)(\hat{f}_1^4 x^9 - \beta_3 \hat{f}_1^2 h_2 x^{12}) - (\alpha_1\beta_3^{-3} + \alpha_2\beta_3^{-1} + \beta_3^5) h_2^2 x^{15}\end{aligned}$$

for $p=3$.

Lemma 6.3. $\mathrm{LT}(\tilde{h}_3) = \alpha_0 \beta_3^{-p} y^{p^3+1} x^{2p^2-1}$.

Proof. For $p=3$, this is a Magma calculation. Suppose $p \geq 5$. We work modulo the ideal $\langle x^{2p^2}\rangle$. Since $p^3-2p^2 > 2p^2$, we have $\hat{f}_1^p \equiv y^{p^3}$. Furthermore, $3p^2-4p > 2p^2$, giving $\hat{f}_1 x^{2p^2-2p} \equiv y^{p^2} x^{2p^2-2}$. Using congruence (1) given above, we have

$$h_2 x^{2p^2-2p} \equiv x^{2p^2-2p} y^{p^2}\left(\delta^p + \frac{\beta_2}{\beta_3} y^p x^p + \frac{\beta_1}{\beta_3}\delta x^{2p-2} + \frac{\beta_0}{\beta_3} y x^{2p-1}\right)$$

and

$$h_2^p \equiv y^{p^3}\left(\delta^p + \frac{\beta_2}{\beta_3} y^p x^p + \frac{\beta_1}{\beta_3}\delta x^{2p-2} + \frac{\beta_0}{\beta_3} y x^{2p-1}\right)^p.$$

Using the definition of h_2, we get

$$\begin{aligned}\hat{f}_1^2 - 2\beta_3 h_2 x^{p^2-2p} &= \beta_3^{-p} x^{2p^2-p^3}(\hat{f}_1^p - N_M(y)) \\ &= \delta^{p^2} + \beta_3^{-p}\big((\beta_2^p - \alpha_2) y^{p^2} x^{p^2} + \beta_1^p \delta^p x^{2p^2-2p} \\ &\qquad + (\beta_0^p - \alpha_1) y^p x^{2p^2-p} - \alpha_0 y x^{2p^2-1}\big).\end{aligned}$$

Thus

$$h_2^p - \hat{f}_1^p(\hat{f}_1^2 - 2\beta_3 h_2 x^{p^2-2p}) \equiv \frac{y^{p^3}}{\beta_3^p}(\alpha_2 y^{p^2} x^{p^2} + \alpha_1 y^p x^{2p^2-p} + \alpha_0 y x^{2p^2-1}).$$

Furthermore, using the above expressions,

$$\hat{f}_1^{p+1}x^{p^2}-\beta_3\hat{f}_1^{p-1}h_2x^{2p^2-2p}\equiv y^{p^3-p^2}x^{p^2}(y^{p^2}\hat{f}_1-\beta_3h_2x^{p^2-2p})\equiv x^{p^2}y^{p^3+p^2}.$$

Therefore

$$h_2^p-\hat{f}_1^p(\hat{f}_1^2-2\beta_3h_2x^{p^2-2p})-\frac{\alpha_2}{\beta_3^p}(\hat{f}_1^{p+1}x^{p^2}-\beta_3\hat{f}_1^{p-1}h_2x^{2p^2-2p})\\ \equiv\frac{y^{p^3}}{\beta_3^p}(\alpha_1y^px^{2p^2-p}+\alpha_0yx^{2p^2-1}).$$

Note that $h_2x^{2p^2-p}\equiv y^{p^2+2p}x^{2p^2-p}$ and $\hat{f}_1x^{2p^2-p}\equiv y^{p^2}x^{2p^2-p}$. Hence

$$\hat{f}_1^{(p-3)/2}h_2^{(p+1)/2}x^{2p^2-p}\equiv y^{p^3+p}x^{2p^2-p},$$

giving $\tilde{h}_3\equiv\alpha_0y^{p^3+1}x^{2p^2-1}/\beta_3^p$, as required. □

Note that $\alpha_0/\beta_3^p=\bar{\gamma}_{2357}\bar{\gamma}_{1234}^p/\bar{\gamma}_{1235}^{p+1}$. Since $\bar{\gamma}_{1357}=0$, $\bar{\gamma}_{1235}\neq 0$ and $\bar{\gamma}_{3457}=\bar{\gamma}_{1235}^p\neq 0$, arguing as in the proof of Lemma 5.1, we see that $\bar{\gamma}_{2357}\neq 0$. Define $h_3:=\bar{\gamma}_{1235}^{p+1}\tilde{h}_3/(x^{2p^2-1}\bar{\gamma}_{2357}\bar{\gamma}_{1234}^p)$ so that $\mathrm{LT}(h_3)=y^{p^3+1}$.

Lemma 6.4. $\mathrm{LM}(h_3^p-h_2^{(p^2+1)/2}\hat{f}_1^{(p^2-2p-1)/2})=x^pz^{p^4}$.

Proof. Working modulo the ideal $\mathfrak{n}:=\langle x^{p+1},x^py\rangle$, we see that $\hat{f}_1\equiv_{\mathfrak{n}}y^{p^2}$ and $h_2\equiv_{\mathfrak{n}}y^{p^2+2p}$, giving $h_3^p-h_2^{(p^2+1)/2}\hat{f}_1^{(p^2-2p-1)/2}\equiv_{\mathfrak{n}}h_3^p-y^{p^4+p}$. Thus it is sufficient to identify the lead monomial of $h_3-y^{p^3+1}$. Note that y^{p^3+1} and xz^{p^3} are consecutive monomials in the grevlex term order. Therefore, if xz^{p^3} appears with nonzero coefficient in h_3, then $\mathrm{LM}(h_3-y^{p^3+1})=xz^{p^3}$, and the result follows. Work modulo the ideal $\mathfrak{m}:=\langle y\rangle$. Then $\hat{f}_1\equiv_{\mathfrak{m}}-\beta_3z^px^{p^2-p}-\beta_1zx^{p^2-1}$ and $N_M(y)\equiv_{\mathfrak{m}}0$. Therefore

$$h_2\equiv_{\mathfrak{m}}\frac{1}{2\beta_3}\left(z^{p^2}x^{2p}+\frac{\beta_1^p}{\beta_3^p}z^px^{p^2+p}+x^{p^2}(\beta_3z^p+\beta_1zx^{p-1})^2\right).$$

Hence h_3 has degree p^3 as a polynomial in z, with leading coefficient $x/2\alpha_0$ and the result follows. □

Theorem 6.5. *If* $\gamma_{1234}(M)\neq 0$, $\gamma_{1235}(M)\neq 0$ *and* $\gamma_{1357}(M)=0$, *then the set* $\mathcal{B}:=\{x,\hat{f}_1,h_2,h_3,N_M(z)\}$ *is a SAGBI basis for* $\mathbb{F}[V_M]^E$. *Furthermore,* $\mathbb{F}[V_M]^E$ *is a complete intersection with generating relations coming from the subduction of the tête-à-têtes* $(h_2^p,\bar{f}_1^{p+2})$ *and* $(h_3^p,\bar{f}_1^{(p^2-2p-1)/2}h_2^{(p^2+1)/2})$.

Proof. Use the subduction given in Lemma 6.4 to construct an invariant h_4 with lead term z^{p^4}. Define $\mathcal{B}':=\{x,\hat{f}_1,h_2,h_3,h_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-têtes for $\mathcal{B}'$ are

$$(h_2^p,\bar{f}_1^{p+2})\quad\text{and}\quad(h_3^p,\bar{f}_1^{(p^2-2p-1)/2}h_2^{(p^2+1)/2}).$$

Using Lemmas 6.3 and 6.4, these tête-à-têtes subduct to zero, proving that $\mathcal{B}'$ is a SAGBI basis for A. By Lemma 6.2, we have $\mathbb{F}[V_M]^E[x^{-1}]=\mathbb{F}[x,\hat{f}_1,h_2][x^{-1}]$.

Using Theorem 1.1, $A=\mathbb{F}[V_M]^E$. Clearly $\mathrm{LT}(N_M(z))=z^{p^k}$ for $k\le 4$. Since $\mathcal{B}'$ is a SAGBI basis for $\mathbb{F}[V_E]^E$, this forces $k=4$, giving $\mathrm{LM}(\mathcal{B})=\mathrm{LM}(\mathcal{B}')$. □

7. The $\gamma_{1234}=0$, $\gamma_{1235}=0$, $\gamma_{1357}\neq 0$ strata

In this section we consider representations V_M for $M\in\mathbb{F}^{2\times 4}$ for which $\gamma_{1235}(M)=0$, $\gamma_{1234}(M)=0$ and $\gamma_{1357}(M)\neq 0$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$.

We first consider the case $\bar{\gamma}_{1257}=0$. Let r_i denote row i of the matrix $\Gamma(M)$. Since $\gamma_{1357}(M)\neq 0$, the set $\{r_1,r_3,r_5,r_7\}$ is linearly independent. Thus r_2 is a linear combination of r_1, r_5 and r_7. Since $\bar{\gamma}_{1235}=0$, we know that r_2 is a linear combination of r_1, r_3 and r_5. Using the $(1,2,3)(3,4,5,7,9)$ Plücker relation, $\bar{\gamma}_{1237}=0$. Thus r_2 is a linear combination of r_1, r_3 and r_7. Combining these observations, we see that r_2 is a scalar multiple of r_1. Using a change of coordinates (see Section 4 of [Campbell et al. 2013]), we may assume that r_2 is zero. If the second row of M is zero, then V_M is a symmetric square representation and the invariants are generated by x, δ, $N_M(y)$ and $N_M(z)$. Since $\bar{\gamma}_{1357}\neq 0$, we have that $N_M(y)$ and $N_M(z)$ are both of degree p^4 and there is a single relation in degree $2p^4$ which can be constructed by subducting the tête-à-tête $(\delta^{p^4}, N_M(y)^2)$ (see Theorem 3.3 of [loc. cit.]).

For the rest of this section, we assume $\bar{\gamma}_{1257}\neq 0$. Evaluating coefficients gives the invariant $\bar{f}_{12357}$. Using the $(1,2,3)(3,4,5,7,9)$ Plücker relation, $\bar{\gamma}_{1237}^{p+1}=0$. Thus $\bar{\gamma}_{1237}=0$, and we have $\bar{f}_{12357}=\bar{\gamma}_{1257}y^px^{p^3-p}+\bar{\gamma}_{1357}\delta x^{p^3-2}+\bar{\gamma}_{2357}yx^{p^3-1}$. Divide by $\bar{\gamma}_{1257}x^{p^3-p}$ to get

$$h_1:=y^p+\frac{\bar{\gamma}_{1357}}{\bar{\gamma}_{1257}}\delta x^{p-2}+\frac{\bar{\gamma}_{2357}}{\bar{\gamma}_{1257}}yx^{p-1}.$$

Observe that $N_M(y)=\bar{f}_{13579}/\bar{\gamma}_{1357}$. Subducting $N_M(y)$ gives

$$\begin{aligned}\tilde{h}_2=N_M(y)-h_1^{p^3}+\alpha^{p^3}h_1^{2p^2}x^{p^4-2p^3}-2\alpha^{p^3+p^2}h_1^{p^2+2p}x^{p^4-p^3-2p^2}\\ +4\alpha^{p^3+p^2+p}h_1^{p^2+p+2}x^{p^4-p^3-p^2-2p},\end{aligned}$$

with $\alpha:=\bar{\gamma}_{1357}/\bar{\gamma}_{1257}$.

Lemma 7.1. $\mathrm{LT}(\tilde{h}_2)=8\alpha^{p^3+p^2+p+1}y^{p^3+p^2+p+2}x^{p^4-p^3-p^2-p-2}$.

Proof. It will be convenient to work modulo the ideal $\langle x^{p^4-p^3}, x^{p^4-p^3-p^2-p-1}y\rangle$, so that $N_M(y)\equiv y^{p^4}$ and $h_1^{p^3}\equiv y^{p^4}+\alpha^{p^3}\delta^{p^3}x^{p^4-2p^3}$. Thus $N_M(y)-h_1^{p^3}\equiv -\alpha^{p^3}\delta^{p^3}x^{p^4-2p^3}$. Expanding gives

$$x^{p^4-2p^3}(h_1^{p^2})^2\equiv x^{p^4-2p^3}y^{p^3}(y^{p^3}+2\alpha^{p^2}\delta^{p^2}x^{p^3-2p^2}).$$

Thus

$$N_M(y)-h_1^{p^3}+\alpha^{p^3}h_1^{2p^2}x^{p^4-2p^3}\equiv 2\alpha^{p^3+p^2}y^{p^3}\delta^{p^2}x^{p^4-p^3-2p^2}.$$

Again expanding gives

$$h_1^{p^2+2p}x^{p^4-p^3-2p^2} \equiv x^{p^4-p^3-2p^2}y^{p^3+p^2}(y^{p^2}+2\alpha^p\delta^p x^{p^2-2p}).$$

Hence

$$N_M(y)-h_1^{p^3}+\alpha^{p^3}h_1^{2p^2}x^{p^4-2p^3}-2\alpha^{p^3+p^2}h_1^{p^2+2p}x^{p^4-p^3-2p^2} \equiv -4\alpha^{p^3+p^2+p}\delta^p y^{p^3+p^2}x^{p^4-p^3-p^2-2p}.$$

Since $h_1^{p^2+p+2}x^{p^4-p^3-p^2-2p} \equiv x^{p^4-p^3-p^2-2p}y^{p^3+p^2+p}(y^p+2\alpha\delta x^{p-2})$, we have

$$\tilde{h}_2 \equiv 8\alpha^{p^3+p^2+p+1}y^{p^3+p^2+p+2}x^{p^4-p^3-p^2-p-2}$$

and the result follows. □

Define $h_2 := \tilde{h}_2/(8\alpha^{p^3+p^2+p+1}x^{p^4-p^3-p^2-p-2})$ so that $\mathrm{LT}(h_2)=y^{p^3+p^2+p+2}$.

Lemma 7.2. *Subducting the tête-à-tête $(h_2^p, h_1^{p^3+p^2+p+2})$ gives an invariant with lead term*

$$\left(\frac{\bar{\gamma}_{1257}}{2\bar{\gamma}_{1357}}\right)^{p^3+p^2+p} z^{p^4}x^{p^3+p^2+2p}.$$

Proof. For $p=3$, this is a Magma calculation. For $p>3$, the subduction is given by

$$\begin{aligned}
&h_2^p - h_1^{p^3+p^2+p+2} + 2\alpha h_2 h_1^{p^3}x^{p-2}\\
&\quad+\frac{1}{4\alpha^{p^3+p^2+p}}\big(\beta_1 h_1^{p^3+p^2}x^{p^2+2p} - \beta_1\alpha^{p^2}h_1^{p^3+2p}x^{p^3-p^2+2p}\\
&\qquad+2\beta_1\alpha^{p^2+p}h_1^{p^3+p+2}x^{p^3} - 4\beta_1\alpha^{p^2+p+1}h_2h_1^{p^3-p^2}x^{p^3+p-2}\\
&\qquad-\beta_2x^{p^3}(h_1^{p^3+p}x^{2p} - \alpha^p h_1^{p^3+2}x^{p^2} + 2\alpha^{p+1}h_2h_1^{p^3-p^2-p}x^{p^2+p-2})\\
&\qquad+\beta_3x^{p^3+p^2+p}(h_1^{p^3+1} - \alpha h_2 h_2^{p^3-p^2-p-1}x^{p-2})\\
&\qquad-\beta_4 h_2^{(p+1)/2}h_1^{(p^2+p+1)(p-3)/2}x^{p^3+p^2+2p-1}\big),
\end{aligned}$$

with

$$\alpha := \frac{\bar{\gamma}_{1357}}{\bar{\gamma}_{1257}},\quad \beta_1 := \frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}},\quad \beta_2 := \frac{\bar{\gamma}_{1379}}{\bar{\gamma}_{1357}},\quad \beta_3 := \frac{\bar{\gamma}_{1579}}{\bar{\gamma}_{1357}},\quad \beta_4 := \bar{\gamma}_{1357}^{p-1}.$$

To calculate the lead term, work modulo the ideal generated by $x^{p^3+p^2+2p+1}$ and $x^{p^3+p^2+2p}y$. □

Theorem 7.3. *If $\gamma_{1234}(M)=0$, $\gamma_{1235}(M)=0$, $\gamma_{1357}(M)=0$ and $\gamma_{1257}(M)\neq 0$, then the set $\mathcal{B} := \{x, h_1, h_2, N_M(z)\}$ is a SAGBI basis for $\mathbb{F}[V_M]^E$. Furthermore, $\mathbb{F}[V_M]^E$ is a hypersurface with the relation coming from the subduction of the tête-à-tête $(h_2^p, h_1^{p^3+p^2+p+2})$.*

Proof. Use the subduction given in Lemma 7.2 to construct an invariant h_3 with lead term z^{p^4}. Define $\mathcal{B}' := \{x, h_1, h_2, h_3\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-tête for $\mathcal{B}'$ is $(h_2^p, h_1^{p^3+p^2+p+2})$, which subducts to zero using the definition of h_3. Thus $\mathcal{B}'$ is a SAGBI basis for A. Since h_1 is of degree one in z with coefficient $-\alpha x^{p-1}$, it follows from [Campbell and Chuai 2007] that $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, h_1, N_M(y)][x^{-1}]$. Since $N_M(y) \in \mathbb{F}[x, h_1, h_2]$, we have $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, h_1, h_2][x^{-1}]$. Using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Clearly $\mathrm{LT}(N_M(z)) = z^{p^k}$ for $k \le 4$. Since $\mathcal{B}'$ is a SAGBI basis for $\mathbb{F}[V_E]^E$, this forces $k = 4$, giving $\mathrm{LM}(\mathcal{B}) \subset \mathrm{LM}(\mathcal{B}')$. □

8. The $\gamma_{1234} = 0,\ \gamma_{1235} \neq 0,\ \gamma_{1357} = 0$ stratum

In this section we consider representations V_M with $\gamma_{1235}(M) \neq 0,\ \gamma_{1234}(M) = 0$ and $\gamma_{1357}(M) = 0$. The results of this section are valid for $p \ge 3$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$. Observe that $N_M(y) = \bar{f}_{12357}/\bar{\gamma}_{1235}$ (see Remark 1.2). Thus $N_M(y)$ has lead term y^{p^3}. Furthermore, $\bar{f}_{12345}$ has lead term $\bar{\gamma}_{1235} y^{2p} x^{p^2-2p}$. Define $h_1 := \bar{f}_{12345}/(\bar{\gamma}_{1235} x^{p^2-2p})$ so that $\mathrm{LT}(h_1) = y^{2p}$.

Lemma 8.1. $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, h_1, N_M(y)][x^{-1}]$.

Proof. We argue as in the proof of Theorem 4.4 of [Campbell et al. 2013]. Since $N_M(y)$ and h_1/x^p are algebraically independent elements of $\mathbb{F}[x, y, \delta/x]^E$ with $\deg(N_M(y))\deg(h_1/x^p) = p^4 = |E|$, applying Theorem 3.7.5 of [Derksen and Kemper 2002] gives $\mathbb{F}[x, y, \delta/x]^E = \mathbb{F}[x, N_M(y), h_1/x^p]$. The result then follows from the observation that

$$\mathbb{F}[x, y, z]^E[x^{-1}] = \mathbb{F}[x, y, \delta/x]^E[x^{-1}].$$ □

Subducting the tête-à-tête $(N_M(y)^2, h_1^{p^2})$ gives

$$\tilde{h}_2 := N_M(y)^2 - h_1^{p^2} + \frac{2}{\bar{\gamma}_{1235}}\left(\bar{\gamma}_{1237} x^{p^3-p^2} h_1^{(p^2+p)/2} - \bar{\gamma}_{1257} x^{p^3-p} h_1^{(p^2+1)/2}\right).$$

Lemma 8.2. $\mathrm{LT}(\tilde{h}_2) = 2\bar{\gamma}_{2357} y^{p^3+1} x^{p^3-1}/\bar{\gamma}_{1235}$.

Proof. We work modulo the ideal $\langle x^{p^3}\rangle$. Expand $N_M(y)^2$ and observe that $h_1^{p^2} \equiv y^{2p^3}$, $h_1^p x^{p^3-p^2} \equiv y^{2p^2} x^{p^3-p^2}$ and $h_1 x^{p^3-p} \equiv y^{2p} x^{p^3-p}$. □

Using the $(1,3,5)(2,3,4,5,7)$ Plücker relation, we have $\bar{\gamma}_{1345}\bar{\gamma}_{2357} = \bar{\gamma}_{1235}^{p+1}$. Thus $\bar{\gamma}_{2357} \neq 0$. Define $h_2 := \bar{\gamma}_{1235}\tilde{h}_2/(2\bar{\gamma}_{2357} x^{p^3-1})$ so that $\mathrm{LT}(h_2) = y^{p^3+1}$.

Lemma 8.3. $\mathrm{LM}(h_2^p - h_1^{(p^3+1)/2}) = z^{p^4} x^p$.

Proof. A careful calculation shows that

$$\mathrm{LT}(h_2^p - h_1^{(p^3+1)/2}) = \frac{\bar{\gamma}_{1235}^p}{2\bar{\gamma}_{2357}^p} x^p z^{p^4}.$$ □

Theorem 8.4. *If $\gamma_{1234}(M)=0$, $\gamma_{1235}(M)\neq 0$ and $\gamma_{1357}(M)=0$, then the set $\mathcal{B}:=\{x, h_1, h_2, N_M(y), N_M(z)\}$ is a SAGBI basis for $\mathbb{F}[V_M]^E$. Furthermore, $\mathbb{F}[V_M]^E$ is a complete intersection with relations coming from the subduction of the tête-à-têtes $(N_M(y)^2, h_1^{p^2})$ and $(h_2^p, h_1^{(p^3+1)/2})$.*

Proof. Use the subduction from Lemma 8.3 to construct an invariant h_3 with lead term z^{p^4}. Define $\mathcal{B}':=\{x, N_M(y), h_1, h_2, h_3\}$ and let A denote the algebra generated by $\mathcal{B}'$. The nontrivial tête-à-têtes for $\mathcal{B}'$ subduct to zero using Lemmas 8.2 and 8.3. Thus $\mathcal{B}'$ is a SAGBI basis for A. From Lemma 8.1, $\mathbb{F}[V_M]^E[x^{-1}]=\mathbb{F}[x, h_1, N_M(y)][x^{-1}]$. Thus, using Theorem 1.1, $A=\mathbb{F}[V_M]^E$. Clearly $\mathrm{LT}(N_M(z))=z^{p^k}$ for $k\leq 4$. Since $\mathcal{B}'$ is a SAGBI basis for $\mathbb{F}[V_E]^E$, this forces $k=4$, giving $\mathrm{LM}(\mathcal{B})=\mathrm{LM}(\mathcal{B}')$. □

9. The $\gamma_{1234}\neq 0$, $\gamma_{1235}=0$, $\gamma_{1357}=0$ strata

In this section we consider representations V_M with $\gamma_{1235}(M)=0$, $\gamma_{1234}(M)\neq 0$ and $\gamma_{1357}(M)=0$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$. Using the $(1,3,5)(3,4,5,6,7)$ Plücker relation, $\bar{\gamma}_{1345}=0$. Thus

$$\bar{f}_1=\bar{\gamma}_{1234}y^{p^2}+\bar{\gamma}_{1245}y^p x^{p^2-p}+\bar{\gamma}_{2345}yx^{p^2-1}\in\mathbb{F}[x,y].$$

Since $\bar{\gamma}_{1234}\neq 0$, the orbit of y contains at least p^2 elements. Thus $N_M(y)=\bar{f}_1/\bar{\gamma}_{1234}$ (see Remark 1.2).

Lemma 9.1. $\mathbb{F}[V_M]^E[x^{-1}]=\mathbb{F}[x, N_M(y), \bar{f}_{12346}][x^{-1}]$.

Proof. We argue as in the proof of Lemma 8.1 (and Theorem 4.4 of [Campbell et al. 2013]). Since $N_M(y)$ and $\bar{f}_{12346}/x^{p^2}$ are algebraically independent elements of $\mathbb{F}[x,y,\delta/x]^E$ with $\deg(N_M(y))\deg(\bar{f}_{12346}/x^{p^2})=p^4=|E|$, applying Theorem 3.7.5 of [Derksen and Kemper 2002] gives

$$\mathbb{F}[x,y,\delta/x]^E=\mathbb{F}[x, N_M(y), \bar{f}_{12346}/x^{p^2}].$$

The result then follows from the observation that

$$\mathbb{F}[x,y,z]^E[x^{-1}]=\mathbb{F}[x,y,\delta/x]^E[x^{-1}].$$ □

We first consider the case $\bar{\gamma}_{1245}\neq 0$. Define $\hat{f}_2:=\bar{f}_2/(\bar{\gamma}_{1234}\bar{\gamma}_{1245}x^p)$ so that $\mathrm{LT}(\hat{f}_2)=y^{p^2+p}$. Subduct the tête-à-tête $(\hat{f}_2^p, N_M(y)^{p+1})$ to get

$$\tilde{h}_3:=N_M(y)^{p+1}-\hat{f}_2^p-\left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1234}}-\frac{\bar{\gamma}_{2345}^p}{\bar{\gamma}_{1245}^p}\right)\hat{f}_2 N_M(y)^{p-1}x^{p^2-p}.$$

Lemma 9.2. $$\mathrm{LT}(\tilde{h}_3)=\left(\frac{\bar{\gamma}_{2345}^{p+1}}{\bar{\gamma}_{1245}^{p+1}}\right)x^{p^2-1}y^{p^3+1}.$$

Proof. Expand and reduce modulo the ideal $\langle x^{p^2}\rangle$. □

Define

$$h_3 := \frac{\bar{\gamma}_{1245}^{p+1}}{x^{p^2-1}\bar{\gamma}_{2345}^{p+1}}\tilde{h}_3$$

so that $\mathrm{LT}(h_3) = y^{p^3+1}$.

Lemma 9.3. *Subducting the tête-à-tête $(h_3^p, N_M(y)^{p^2-1}\hat{f}_2)$ gives an invariant with lead monomial $x^p z^{p^4}$.*

Proof. Work modulo the ideal $\langle x^{p+1}, x^p y\rangle$ and expand to get

$$h_3^p - \hat{f}_2 N_M(y)^{p^2-1} + \frac{\bar{\gamma}_{2345}}{\bar{\gamma}_{1234}} x^{p-1} h_3 N_M(y)^{p^2-p} \equiv \left(\frac{\bar{\gamma}_{1234}^{p^2}\bar{\gamma}_{1245}^{p}}{\bar{\gamma}_{2345}^{p^2+p}}\right) z^{p^4} x^p. \qquad \square$$

Theorem 9.4. *If $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) = \gamma_{1357}(M) = 0$ and $\gamma_{1245}(M) \neq 0$, then the set $\mathcal{B} := \{x, N_M(y), \hat{f}_2, h_3, N_M(z)\}$ is a SAGBI basis for $\mathbb{F}[V_M]^E$. Furthermore, $\mathbb{F}[V_M]^E$ is a complete intersection with relations coming from the subduction of the tête-à-têtes $(\hat{f}_2^p, N_M(y)^{p+1})$ and $(h_3^p, N_M(y)^{p^2-1}\hat{f}_2)$.*

Proof. Use the subduction given in Lemma 9.3 to construct an invariant h_4 with lead term z^{p^4}. Define $\mathcal{B}' := \{x, N_M(y), \hat{f}_2, h_3, h_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The nontrivial tête-à-têtes for $\mathcal{B}'$ subduct to zero using Lemmas 9.2 and 9.3. Thus $\mathcal{B}'$ is a SAGBI basis for A. From Lemma 9.1, $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), \bar{f}_{12346}][x^{-1}]$. However, since $f_2 = (f_1^2 + \gamma_{1234} f_{12346})/(2x^{p^2-2p})$, we see that

$$\mathbb{F}[x, N_M(y), \bar{f}_{12346}][x^{-1}] = \mathbb{F}[x, N_M(y), \hat{f}_2][x^{-1}].$$

Thus, using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Clearly $\mathrm{LT}(N_M(z)) = z^{p^k}$ for $k \leq 4$. Since $\mathcal{B}'$ is a SAGBI basis for $\mathbb{F}[V_E]^E$, this forces $k = 4$, giving $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. $\square$

Suppose $\bar{\gamma}_{1245} = 0$ and let r_i denote row i of the matrix $\Gamma(M)$. Since $\bar{\gamma}_{1234} \neq 0$, we see that $\{r_1, r_2, r_3, r_4\}$ is linearly independent. Using the assumptions that $\bar{\gamma}_{1235} = \bar{\gamma}_{1245} = 0$, we see that $r_5 \in \mathrm{Span}(r_1, r_2, r_3) \cap \mathrm{Span}(r_1, r_2, r_4)$. Therefore $r_5 \in \mathrm{Span}(r_1, r_2)$. However, since $\bar{\gamma}_{1357} = 0$, using a change of coordinates (see [Campbell et al. 2013, §4]) and the $\mathrm{GL}_4(\mathbb{F}_p)$-action, we may assume

$$M := \begin{pmatrix} 1 & c_{12} & c_{13} & 0 \\ 0 & c_{22} & c_{23} & c_{24} \end{pmatrix}$$

with $c_{24} \neq 0$. Since $r_5 = r_1^{p^2}$, we conclude that $r_5 = r_1$. Thus $\bar{\gamma}_{2345} = -\bar{\gamma}_{1234}$. Hence $N_M(y) = \bar{f}_1/\bar{\gamma}_{1234} = y^{p^2} - yx^{p^2-1}$. Define $\hat{h}_2 := -\bar{f}_2/(\bar{\gamma}_{1234}^2 x^{2p-1})$ so that $\mathrm{LT}(\hat{h}_2) = y^{p^2+1}$.

Theorem 9.5. *If $\gamma_{1234}(M) \neq 0$ and $\gamma_{1235}(M) = \gamma_{1357}(M) = \gamma_{1245}(M) = 0$, then the set $\mathcal{B} := \{x, N_M(y), \hat{h}_2, N_M(z)\}$ is a SAGBI basis for $\mathbb{F}[V_M]^E$. Furthermore, $\mathbb{F}[V_M]^E$ is a hypersurface with the relation coming from the subduction of the tête-à-tête $(\hat{h}_2^{p^2}, N_M(y)^{p^2+1})$.*

Proof. Using the definition of $\hat{h}_2$ and the description given above of $N_M(y)$, we see that

$$\mathrm{LT}\big(\hat{h}_2^{p^2} - N_M(y)^{p^2+1} - \hat{h}_2(xN_M(y))^{p^2-1}\big) = -\tfrac{1}{2}z^{p^4}x^{p^2}.$$

Thus we can use the subduction of the tête-à-tête $(\hat{h}_2^{p^2}, N_M(y)^{p^2+1})$ to construct an invariant h_4 with lead term z^{p^4}. Define $\mathcal{B}' := \{x, N_M(y), \hat{h}_2, h_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-tête subducts to zero. Therefore $\mathcal{B}'$ is a SAGBI basis for A. From Lemma 9.1, $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), \bar{f}_{12346}][x^{-1}]$. However, it follows from the definition of $\hat{h}_2$ that $\mathbb{F}[x, N_M(y), \bar{f}_{12346}][x^{-1}] = \mathbb{F}[x, N_M(y), \hat{h}_2][x^{-1}]$. Thus, using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Clearly $\mathrm{LT}(N_M(z)) = z^{p^k}$ for $k \le 4$. Since $\mathcal{B}'$ is a SAGBI basis for $\mathbb{F}[V_E]^E$, this forces $k = 4$, giving $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

10. The $\gamma_{1234} = 0$, $\gamma_{1235} = 0$, $\gamma_{1357} = 0$ strata

In this section we consider representations V_M with $\gamma_{1235}(M) = 0$, $\gamma_{1234}(M) = 0$ and $\gamma_{1357}(M) = 0$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$. We assume that the first row of M is nonzero; otherwise, the representation is of type (2, 1) and the calculation of $\mathbb{F}[V_M]^E$ can be found in Section 4 of [Campbell et al. 2013]. Using a change of coordinates, see Proposition 4.3 of [loc. cit.], the $\mathrm{GL}_4(\mathbb{F}_p)$-action, and the hypothesis that $\bar{\gamma}_{1357} = 0$, we may take

$$M = \begin{pmatrix} 1 & c_{12} & c_{13} & 0 \\ 0 & c_{22} & c_{23} & c_{24} \end{pmatrix}.$$

Since $\bar{\gamma}_{1235} = 0$, either $c_{24} = 0$ or $\{1, c_{12}, c_{13}\}$ is linearly dependent over $\mathbb{F}_p$. We assume $c_{24} \neq 0$; otherwise the representation is not faithful and we can view V_M as a representation of a group of rank three. Using the $\mathrm{GL}_4(\mathbb{F}_p)$-action, we replace the third column by a linear combination of the first two columns to get

$$\begin{pmatrix} 1 & c_{12} & 0 & 0 \\ 0 & c_{22} & c_{23} & c_{24} \end{pmatrix}.$$

Expanding gives

$$\bar{\gamma}_{1234} = (c_{12} - c_{12}^p)\det\begin{pmatrix} c_{23} & c_{24} \\ c_{23}^p & c_{24}^p \end{pmatrix}.$$

Since $\bar{\gamma}_{1234} = 0$, either $c_{12} \in \mathbb{F}_p$ or $\{c_{23}, c_{24}\}$ is linearly dependent over $\mathbb{F}_p$. However, if $\{c_{23}, c_{24}\}$ is linearly dependent over $\mathbb{F}_p$, then the representation is not faithful. So we may assume $c_{12} \in \mathbb{F}_p$. Using the $\mathrm{GL}_4(\mathbb{F}_p)$-action to replace the second column with a linear combination of the first two columns gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{22} & c_{23} & c_{24} \end{pmatrix}.$$

If $\bar{\gamma}_{1246} = 0$, then $\{c_{22}, c_{23}, c_{24}\}$ is linearly dependent over $\mathbb{F}_p$, and again the representation is not faithful. Thus we may assume that $\bar{\gamma}_{1246} \neq 0$. Using the above form for M, it is clear that $\bar{\gamma}_{1236} = 0$, $\bar{\gamma}_{1346} = 0$ and $\bar{\gamma}_{1246} = -\bar{\gamma}_{2346}$. Thus

$$\bar{f}_{12346} = \bar{\gamma}_{1246}(y^p x^{2p^2-p} - yx^{2p^2-1}) \in \mathbb{F}[x, y]^E.$$

Since $\mathbb{F}[x, y]^E = \mathbb{F}[x, N_M(y)]$, we have

$$N_M(y) = \bar{f}_{12346}/(\bar{\gamma}_{1246}x^{2p^2-p}) = y^p - yx^{p-1}.$$

Lemma 10.1. $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), \bar{f}_{12468}][x^{-1}]$.

Proof. The proof is similar to the proof of Theorem 4.4 of [Campbell et al. 2013] (and Lemmas 8.1 and 9.1). Since $N_M(y)$ and $\bar{f}_{12468}/x^{p^3}$ are algebraically independent elements of $\mathbb{F}[x, y, \delta/x]^E$ with $\deg(N_M(y)) \deg(\bar{f}_{12468}/x^{p^3}) = p^4 = |E|$, applying Theorem 3.7.5 of [Derksen and Kemper 2002] gives

$$\mathbb{F}[x, y, \delta/x]^E = \mathbb{F}[x, N_M(y), \bar{f}_{12468}/x^{p^3}].$$

The result then follows from the observation that

$$\mathbb{F}[x, y, z]^E[x^{-1}] = \mathbb{F}[x, y, \delta/x]^E[x^{-1}]. \qquad \square$$

Subducting $\bar{f}_{12468}$ gives

$$\tilde{h}_1 := \bar{f}_{12468} + \bar{\gamma}_{1246}\big(N_M(y)^{2p^2} + 2N_M(y)^{p^2+p}x^{p^3-p^2} + 2N_M(y)^{p^2+1}x^{p^3-p}\big).$$

Lemma 10.2. $\mathrm{LT}(\tilde{h}_1) = -2\bar{\gamma}_{1246}x^{p^3-1}y^{p^3+1}$.

Proof. We work modulo the ideal $\langle x^{p^3}\rangle$. Using the definition, $\bar{f}_{12468} \equiv -\bar{\gamma}_{1246}y^{2p^3}$. Since $N_M(y) = y^p - yx^{p-1}$, we have

$$N_M(y)^{2p^2} = y^{2p^3} - 2y^{p^3+p^2}x^{p^3-p^2} + y^{2p^2}x^{2p^3-2p^2} \equiv y^{2p^3} - 2y^{p^3+p^2}x^{p^3-p^2}.$$

Expanding and simplifying gives

$$N_M(y)^{p^2+p}x^{p^3-p^2} + N_M(y)^{p^2+1}x^{p^3-p} \equiv y^{p^3+p^2}x^{p^3-p^2} - y^{p^3+1}x^{p^3-1}.$$

Thus

$$\begin{aligned}\tilde{h}_1 &= \bar{f}_{12468} + \bar{\gamma}_{1246}\big(N_M(y)^{2p^2} + 2N_M(y)^{p^2+p}x^{p^3-p^2} + 2N_M(y)^{p^2+1}x^{p^3-p}\big)\\ &\equiv -2\bar{\gamma}_{1246}x^{p^3-1}y^{p^3+1}.\end{aligned} \qquad \square$$

Define $h_1 := -\tilde{h}_1/(2\bar{\gamma}_{1246}x^{p^3-1)}$ so that $\mathrm{LT}(h_1) = y^{p^3+1}$. Note that

$$\mathbb{F}[x, N_M(y), h_1][x^{-1}] = \mathbb{F}[x, N_M(y), \bar{f}_{12468}][x^{-1}].$$

Lemma 10.3. *Subducting the tête-à-tête $(h_1^p, N_M(y)^{p^3+1})$ gives an invariant with lead monomial $x^p z^{p^4}$.*

Proof. Refining the calculation in the proof of the previous lemma gives

$$\tilde{h}_1 \equiv_{\langle x^{p^3+1},\, x^{p^3} y\rangle} \bar{\gamma}_{1246}(-2y^{p^3+1}x^{p^3-1} + x^{p^3}z^{p^3}).$$

Thus

$$h_1 \equiv_{\langle x^2,\, xy\rangle} y^{p^3+1} - \tfrac{1}{2}z^{p^3}x \quad \text{and} \quad h_1^p \equiv_{\langle x^{p+1},\, x^p y\rangle} y^{p^4+p} - \tfrac{1}{2}z^{p^4}x^p.$$

Furthermore

$$N_M(y)^{p^3+1} \equiv_{\langle x^{p+1},\, x^p y\rangle} y^{p^4+p} - y^{p^4+1}x^{p-1}$$

and

$$h_1 N_M(y)^{p^3-p^2}x^{p-1} \equiv_{\langle x^{p+1},\, x^p y\rangle} y^{p^4+1}x^{p-1}.$$

Thus $\mathrm{LT}(h_1^p - N_M^{p^3+1} - h_1 N_M(y)^{p^3-p^2}) = -\frac{1}{2}x^p z^{p^4}$. □

Theorem 10.4. *If $\gamma_{1234}(M) = 0$, $\gamma_{1235}(M) = 0$, $\gamma_{1357}(M) = 0$ and $\gamma_{1246}(M) \neq 0$, then the set $\mathcal{B} := \{x, N_M(y), h_1, N_M(z)\}$ is a SAGBI basis for $\mathbb{F}[V_M]^E$. Furthermore, $\mathbb{F}[V_M]^E$ is a hypersurface with the relation coming from the subduction of the tête-à-tête $(h_1^p, N_M(y)^{p^3+1})$.*

Proof. Use the subduction given in Lemma 10.3 to construct an invariant h_2 with lead term z^{p^4}. Define $\mathcal{B}' := \{x, N_M(y), h_1, h_2\}$ and let A denote the algebra generated by $\mathcal{B}'$. The single nontrivial tête-à-tête for $\mathcal{B}'$ subducts to zero using Lemma 10.3. Thus $\mathcal{B}'$ is a SAGBI basis for A. From Lemma 10.1, $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), h_1][x^{-1}]$. Thus, using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Clearly $\mathrm{LT}(N_M(z)) = z^{p^k}$ for $k \leq 4$. Since $\mathcal{B}'$ is a SAGBI basis for $\mathbb{F}[V_E]^E$, this forces $k = 4$, giving $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

References

[Adams and Loustaunau 1994] W. W. Adams and P. Loustaunau, *An introduction to Gröbner bases*, Graduate Studies in Mathematics **3**, American Mathematical Society, Providence, RI, 1994. MR 1287608 Zbl 0803.13015

[Benson 1993] D. J. Benson, *Polynomial invariants of finite groups*, London Mathematical Society Lecture Note Series **190**, Cambridge University Press, 1993. MR 1249931 Zbl 0864.13001

[Bosma et al. 1997] W. Bosma, J. J. Cannon, and C. Playoust, "The Magma algebra system, I: The user language", *J. Symbolic Comput.* **24**:3–4 (1997), 235–265. MR 1484478 Zbl 0898.68039

[Campbell and Chuai 2007] H. E. A. E. Campbell and J. Chuai, "Invariant fields and localized invariant rings of p-groups", *Q. J. Math.* **58**:2 (2007), 151–157. MR 2334859 Zbl 1134.13002

[Campbell and Wehlau 2011] H. E. A. E. Campbell and D. L. Wehlau, *Modular invariant theory*, Encyclopaedia of Mathematical Sciences **139**, Springer, Berlin, 2011. MR 2759466 Zbl 1216.14001

[Campbell et al. 2013] H. E. A. E. Campbell, R. J. Shank, and D. L. Wehlau, "Rings of invariants for modular representations of elementary abelian p-groups", *Transform. Groups* **18**:1 (2013), 1–22. MR 3022756 Zbl 1264.13009

[Derksen and Kemper 2002] H. Derksen and G. Kemper, *Computational invariant theory*, Encyclopaedia of Mathematical Sciences **130**, Springer, Berlin, 2002. MR 1918599 Zbl 1011.13003

[Kapur and Madlener 1989] D. Kapur and K. Madlener, "A completion procedure for computing a canonical basis for a k-subalgebra", pp. 1–11 in *Computers and mathematics* (Cambridge, MA, 1989), edited by E. Kaltofen and S. M. Watt, Springer, New York, NY, 1989. MR 1005954 Zbl 0692.13001

[Lakshmibai and Raghavan 2008] V. Lakshmibai and K. N. Raghavan, *Standard monomial theory: invariant theoretic approach*, Encyclopaedia of Mathematical Sciences **137**, Springer, Berlin, 2008. MR 2388163 Zbl 1137.14036

[Neusel and Smith 2002] M. D. Neusel and L. Smith, *Invariant theory of finite groups*, Mathematical Surveys and Monographs **94**, American Mathematical Society, Providence, RI, 2002. MR 1869812 Zbl 0999.13002

[Robbiano and Sweedler 1990] L. Robbiano and M. Sweedler, "Subalgebra bases", pp. 61–87 in *Commutative algebra* (Salvador, 1988), edited by W. Bruns and A. Simis, Lecture Notes in Math. **1430**, Springer, Berlin, 1990. MR 1068324 Zbl 0725.13013

[Sturmfels 1996] B. Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series **8**, American Mathematical Society, Providence, RI, 1996. MR 1363949 Zbl 0856.13020

[Wehlau 2013] D. L. Wehlau, "Invariants for the modular cyclic group of prime order via classical invariant theory", *J. Eur. Math. Soc.* **15**:3 (2013), 775–803. MR 3085091 Zbl 1297.13011

[Wilkerson 1983] C. W. Wilkerson, "A primer on the Dickson invariants", pp. 421–434 in *Proceedings of the Northwestern Homotopy Theory Conference* (Evanston, IL, 1982), edited by H. R. Miller and S. B. Priddy, Contemporary Mathematics **19**, American Mathematical Society, Providence, RI, 1983. MR 711066 Zbl 0525.55013

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theo.pierron@ens-rennes.fr *Département Mathématiques, ENS Rennes, 35170 BRUZ, France*

r.j.shank@kent.ac.uk *School of Mathematics, Statistics & Actuarial Science, University of Kent, Canterbury, CT2 7NF, United Kingdom*

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