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Arranging kings k-dependently on hexagonal chessboards

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(Communicated by Arthur T. Benjamin)

Tessellate the plane into rows of hexagons. Consider a subset of $2n$ rows of these hexagons, each row containing $2n$ hexagons, forming a rhombus-shaped chessboard of $4n^2$ spaces. Two kings placed on the board are said to "attack" each other if their spaces share a side or corner. Placing kings in alternating spaces of every other row results in an arrangement where no two of the n^2 kings are attacking each other. According to our specific distance metric, n^2 is in fact the largest number of kings that can be placed on such a board with no two kings attacking one another, for a maximum "density" of $\frac{1}{4}$. We consider a generalization of this maximum density problem, instead requiring that no king attacks more than k other kings for $0 \le k \le 12$. For instance when $k = 2$ the density is at most $\frac{1}{3}$. For each k we give constructive lower bounds on the density, and use systems of inequalities and discharging arguments to yield upper bounds, where the bounds match in most cases.

1. Introduction

Consider the task of arranging as many king pieces as possible on a standard 8×8 chessboard so that no two squares containing kings share a side or corner. Note that the board partitions into sixteen 2×2 patches, and at most one king can reside in any patch. Yet placing a king in the upper left corner of each patch satisfies our requirements. So, in an optimal placement we have sixteen kings occupying $\frac{1}{4}$ of the board. We can generalize this problem as follows. Consider whole numbers k and n. What is the maximum number of kings that can be placed on an $n \times n$ board such that each king-occupied space shares at most k edges and corners with other king-occupied spaces? Note that in the previous example, $k = 0$ and $n = 8$. In fact, in [Ionascu et al. 2008] the following question is investigated:

MSC2010: 90C05, 90C27.
Keywords: k-dependence, combinatorial chessboard, optimization, discharging, linear programming.

> Given a whole number $k \leq 8$ (8 being the maximum number of squares a king can attack), what is the maximum number s of kings that can be placed on an $m \times n$ board so that no king attacks more than k other kings? When m and n are large, how large can the density $s/(mn)$ be?

Similar problems have also been studied on alternate boards. For instance, [Bode et al. 2003] studies a similar problem on triangular boards with $k = 0$, and [Bode and Harborth 2003] looks at a similar problem for knights on both triangular and hexagonal boards with $k = 0$. Other interesting articles on combinatorial chessboard problems include [Fricke et al. 1995; Haynes et al. 1998; Hedetniemi et al. 1998; Watkins 2004]. In this paper, we consider a board in which the spaces are regular hexagons arranged into an $n \times n$ rhombus as in Figure 1. As in Władysław Gliński's hexagonal chess [Bodlaender 1996], a *king* occupying a hexagon will be said to *attack* the 12 other hexagons "nearest" its hexagon (as shown in the left half of Figure 2). In particular, we study the following:

> Given a whole number $k \leq 12$, what is the maximum number s of kings that can be placed on an $n \times n$ hexagonal board so that no king attacks more than k other kings? When n is large, how large can the density s/n^2 be?

For most values of k, we find tight bounds on the optimal density s/n^2 when n is large. For those values of k where our bounds are not tight, the gaps between our upper and lower bounds are reasonably close, and we conjecture that a limiting density exists.

2. Notation and terminology

We establish some definitions and notation since we are not using the normal $n \times n$ chessboard. Consider a tiling of the plane by regular hexagons, each hexagon having two vertical sides. We call a finite collection of hexagons a *hexagonal board* and call each hexagon in that tiling a *space*. For convenience we consider only hexagonal boards $\mathcal{B}_n$ where the hexagons form an $n \times n$ rhombus in the plane as in Figure 1, where $n \geq 5$. We label the spaces on our board (as in Figure 1)

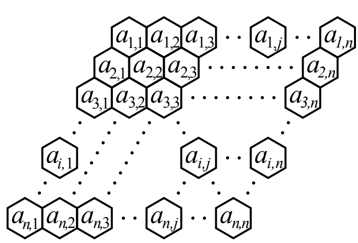


Figure 1. An $n \times n$ hexagonal board in the shape of a rhombus.

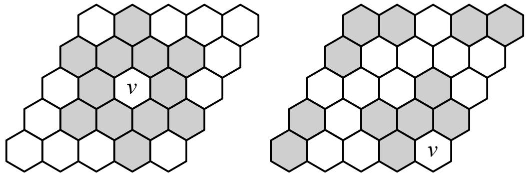


Figure 2. Some examples of $\mathcal{R}(v)$ shaded on a 5×5 board.

from top to bottom, left to right, as $a_{i,j}$, where i denotes the row the space is in and j denotes the diagonal column the space is in. We make this assumption since any finite set of spaces in the tiling will be contained in a suitably large rhombus of hexagons. We use scripted capital letters (such as $\mathcal{A}$) to name subsets of a board $\mathcal{B}_n$ and nonscripted letters (such as a or A) to label individual spaces of a board $\mathcal{B}_n$. To avoid distinguishing between a space and a king occupying that space, we introduce the following terminology. When a subset $\mathcal{A}$ of $\mathcal{B}_n$ is specified, we refer to each space $v \in \mathcal{B}_n$ as being a *king* if $v \in \mathcal{A}$. We define the *realm* (or *neighborhood*) of a space $v \in \mathcal{B}_n$, denoted $\mathcal{R}(v)$, to be the set of spaces a king in space v attacks. In particular, we define the spaces in $\mathcal{R}(v)$ to create a "wrap-around" property on the board as follows: Given a space $a_{i,j} \in \mathcal{B}_n$ we define $g,\ g',\ g'',\ g''',\ h,\ h',\ h'',\ h''' \in (1, 2, \ldots, n)$ as

$$\begin{aligned} g &\equiv i-2 \pmod n, & h &\equiv j-2 \pmod n,\\ g' &\equiv i-1 \pmod n, & h' &\equiv j-1 \pmod n,\\ g'' &\equiv i+1 \pmod n, & h'' &\equiv j+1 \pmod n,\\ g''' &\equiv i+2 \pmod n, & h''' &\equiv j+2 \pmod n. \end{aligned}$$

Finally we define $\mathcal{R}(a_{i,j})$ to be the set of spaces

$$\{a_{g,h'},\ a_{g',h},\ a_{g',h'},\ a_{g',j},\ a_{g',h''},\ a_{i,h'},\ a_{i,h''},\ a_{g'',h'},\ a_{g'',j},\ a_{g'',h''},\ a_{g'',h'''},\ a_{g''',h''}\}.$$

Note that if the board is less than 4×4, the realm of a king overlaps itself. To simplify matters, the theorems in this paper only address $n \times n$ boards where $n \geq 4$, so a king's realm always contains 12 spaces. (This is because realm spaces that are otherwise prevented by the boundary of the board are instead extended past the boundary and associated with the opposite side. Additionally when $n \geq 4$ it is impossible for the realm to overlap itself.) As an example,

$$\mathcal{R}(a_{1,1}) = \{a_{(n-1),n},\ a_{n,(n-1)},\ a_{n,n},\ a_{n,1},\ a_{n,2},\ a_{1,n},\ a_{1,2},\ a_{2,n},\ a_{2,1},\ a_{2,2},\ a_{2,3},\ a_{3,2}\}.$$

A *placement of kings* on a hexagonal board $\mathcal{B}_n$ is a subset $\mathcal{A}$ of $\mathcal{B}_n$, the members of which we call *kings*. If u, v are kings, u is said to *attack* v if $v \in \mathcal{R}(u)$. A placement of kings $\mathcal{A}$ is *k-dependent* if $\mathcal{R}(v)$ contains at most k kings for all $v \in \mathcal{A}$. We denote the collection of all k-dependent placements of kings on $\mathcal{B}_n$ by $\mathcal{B}_n(k)$.

Our notation of a k-dependent arrangement of kings relates to the following graph-theoretic terminology. For a graph $G = (V, E)$ and a vertex $v \in V$, the set of vertices directly joined by an edge to v is denoted by $N(v)$. A subset $\mathcal{A}$ of V is called *independent* if $N(a) \cap \mathcal{A} = \varnothing$ for each vertex $a \in \mathcal{A}$, and more generally, $\mathcal{A}$ is called *k-dependent* (for a specified constant k) if $|N(a) \cap A| \leq k$ for each $a \in \mathcal{A}$. The *k-dependence number* of $G = (V, E)$ is the maximum cardinality among k-dependent subsets of V.

The *maximum k-count* on $\mathcal{B}_n$, denoted by $\mathrm{MNK}(k, n)$, is the maximum size of a k-dependent arrangement or placement of kings on $\mathcal{B}_n$. That is,

$$\mathrm{MNK}(k, n) = \max\{|\mathcal{A}| : \mathcal{A} \in \mathcal{B}_n(k)\}$$

when $n \geq 4$. We let $\mu(k)$ denote the least upper bound of the density of kings in k-dependent placements. That is,

$$\mu(k) = \sup\left\{\frac{|A|}{n^2} : \mathcal{A} \in \mathcal{B}_n(k),\ n \geq 4\right\}.$$

Alternatively, $\mu(k) = \sup\{\mathrm{MNK}(k, n)/n^2 : n \geq 4\}$.

3. Initial upper bounds

Following tradition, we refer to the following technique as *linear programming*, even though we are not optimizing some objective function subject to constraints in the standard sense. For each k, we produce a simple system of linear inequalities valid for all $\mathcal{A} \in \mathcal{B}_n(k)$. Summing these inequalities will yield upper bounds for $\mu(k)$. Let *T(k)* be the maximum number of kings in $\mathcal{R}(v)$ for $v \in \mathcal{A}^c = \mathcal{B}_n \setminus \mathcal{A}$ over all $\mathcal{A} \in \mathcal{B}_n(k)$ for $n \geq 4$. That is, if v is a non-king with respect to some k-dependent placement, then the largest number of kings possible in $\mathcal{R}(v)$ is $T(k)$. Given a placement $\mathcal{A} \subseteq \mathcal{B}_n$ the *indicator function* of $\mathcal{A}$ over $\mathcal{B}_n$ is

$$\chi_{\mathcal{A}}(w) = \begin{cases} 1 & \text{if } w \in \mathcal{A}, \\ 0 & \text{if } w \in \mathcal{A}^c. \end{cases}$$

Theorem 3.1. *Given $n \geq 4$, for all $\mathcal{A} \in \mathcal{B}_n(k)$ we have*

$$|\mathcal{A}| \leq \frac{T(k)}{T(k) - k + 12} n^2.$$

Proof. Let $\mathcal{A} \in \mathcal{B}_n(k)$. Then for each $v \in \mathcal{B}_n$, we have that $\sum_{w \in \mathcal{R}(v)} \chi_{\mathcal{A}}(w)$ is the number of kings that are in the realm of v. We consider the inequality

$$(T(k) - k)\,\chi_{\mathcal{A}}(v) + \sum_{w \in \mathcal{R}(v)} \chi_{\mathcal{A}}(w) \leq T(k). \tag{1}$$

If $\chi_{\mathcal{A}}(v) = 0$, then (1) holds by the definition of $T(k)$. Suppose $\chi_{\mathcal{A}}(v) = 1$. Then $\sum_{w\in\mathcal{R}(v)} \chi_{\mathcal{A}}(w) \le k$ because $\mathcal{A}$ is a k-dependent placement of kings. Therefore, since $(T(k) - k)\cdot 1 + k \le T(k)$, inequality (1) holds whether or not $\chi_{\mathcal{A}}(v) = 0$. Now, we sum the inequalities of form (1), over all choices of $v \in \mathcal{B}_n$, and simplify the result:

$$\sum_{v\in\mathcal{B}_n}\Big((T(k)-k)\chi_{\mathcal{A}}(v) + \sum_{w\in\mathcal{R}(v)}\chi_{\mathcal{A}}(w)\Big) \le \sum_{v\in\mathcal{B}_n} T(k),$$

$$\Big((T(k)-k)\sum_{v\in\mathcal{B}_n}\chi_{\mathcal{A}}(v)\Big) + 12\sum_{v\in\mathcal{B}_n}\chi_{\mathcal{A}}(v) \le T(k)n^2,$$

$$(T(k)-k+12)\sum_{v\in\mathcal{B}_n}\chi_{\mathcal{A}}(v) \le T(k)n^2.$$

Hence,

$$|\mathcal{A}| \le \frac{T(k)}{T(k)-k+12}n^2. \qquad \square$$

Therefore, Theorem 3.1 establishes that

$$\frac{|\mathcal{A}|}{n^2} \le \frac{T(k)}{T(k)-k+12}$$

for all $k \in \{0, 1, 2, \dots, 12\}$ and $n \ge 4$.

Finding T-values. For each $k \in \{0, 1, 2, \dots, 12\}$, Theorem 3.1 gives us an upper bound for $|\mathcal{A}|/n^2$, the fraction of $\mathcal{B}_n$ that can be occupied by a k-dependent set of kings. Since these upper bounds depend upon k and $T(k)$, we must find the exact values of $T(k)$ for each separate choice of k. We refer to values of $T(k)$ as *T-values*.

The following illustrates our process for finding the T-values for each k. To see why $T(0) = 4$, suppose $\mathcal{A} \in \mathcal{B}_n(0)$ for some $n \ge 4$ and consider some $v \in \mathcal{A}^c$. We label the spaces in $\mathcal{R}(v)$ as in Figure 3. Partition $\mathcal{R}(v)$ into the sets $\{A, B, C\}$, $\{D, E, F\}$, $\{G, H, I\}$, and $\{J, K, L\}$. Since $\mathcal{A}$ is 0-dependent, the maximum number of kings in each of $\{A, B, C\}$, $\{D, E, F\}$, $\{G, H, I\}$, and $\{J, K, L\}$ is 1. Therefore, $T(0) \le 4$. Also, since $\{A, D, G, J\}$ is a 0-dependent placement, we have $T(0) \ge 4$. Therefore $T(0) = 4$.

Figure 4 demonstrates lower bounds for all remaining T-values. The reader can check by hand that $T(k)$ equals the lower bound given below each picture.

Therefore, Theorem 3.1 establishes upper bounds for $\mu(k)$ as seen below:

k	0	1	2	3	4	5	6	7	8	9	10	11	12
Upper bound for $\mu(k)$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{8}{17}$	$\frac{5}{9}$	$\frac{10}{17}$	$\frac{2}{3}$	$\frac{12}{17}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{6}{7}$	$\frac{12}{13}$	1

A board consisting entirely of kings would be 12-dependent. Therefore, $\mu(12) = 1$.

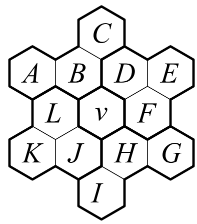


Figure 3. $\mathcal{R}(v)$ partitioned into four subsets.

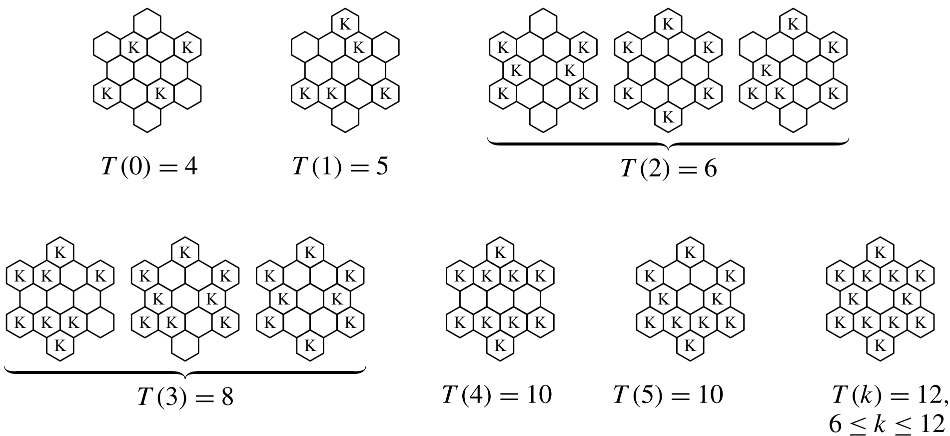


Figure 4. Lower bounds for the T-values for $0 \leq k \leq 12$.

4. Lower bounds

Ideally, we wish to find matching upper and lower bounds for $\mu(k)$ for each k, so as to determine $\mu(k)$ exactly. In this section we give constructive lower bounds for each such $\mu(k)$. To establish lower bounds for $\mu(k)$ we create patterns via "puzzle pieces". We use them to construct arbitrarily large k-dependent placements with calculable density. To construct these placements, we "stamp" the puzzle piece

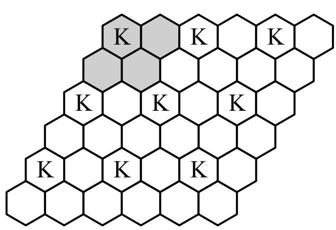
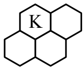
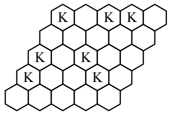


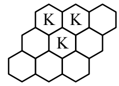
Figure 5. The shaded region represents the puzzle piece $P(0)$, which has been stamped nine times to create a 0-dependent placement on a 6×6 board.



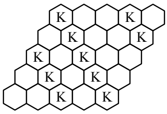
$P(0): K(0) = 1,$
$s(0) = 2, l(0) = 2$



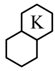
$P(1): K(1) = 7,$
$s(1) = 5, l(1) = 5$



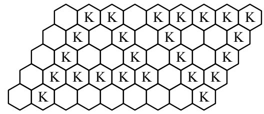
$P(2): K(2) = 3,$
$s(2) = 3, l(2) = 3$



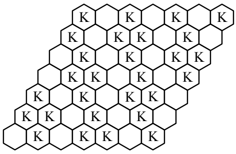
$P(3): K(3) = 10,$
$s(3) = 5, l(3) = 5$



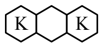
$P(4): K(4) = 1,$
$s(4) = 2, l(4) = 1$



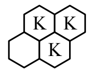
$P(5): K(5) = 28,$
$s(5) = 5, l(5) = 9$



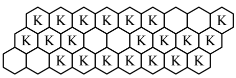
$P(6): K(6) = 28,$
$s(6) = 7, l(6) = 7$



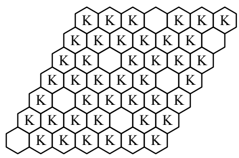
$P(7): K(7) = 2,$
$s(7) = 1, l(7) = 3$



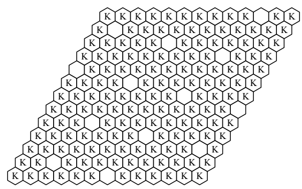
$P(8): K(8) = 3,$
$s(8) = 2, l(8) = 2$



$P(9): K(9) = 21,$
$s(9) = 3, l(9) = 9$



$P(10): K(10) = 42,$
$s(10) = 7, l(10) = 7$



$P(11): K(11) = 156,$
$s(11) = 13, l(11) = 13$

Figure 6. Puzzle pieces $P(k)$ that result in a k-dependent placement.

(using translated copies) as many times as needed to fill large $n \times n$ boards. For example, in Figure 5, the shaded region represents a 2×2 puzzle piece, which we stamped a total of nine times to create a 6×6 placement. This results in a 0-dependent placement. We call this method of obtaining a k-dependent placement the *stamping method*. For a given k, we define $P(k)$ (as in Figure 6) to be a particular $s \times l$ puzzle piece which results in a k-dependent placement. Additionally, for a given $P(k)$, we define $K(k)$ to be the number of kings in $P(k)$. We denote the number of rows in $P(k)$ by $s(k)$ and the number of diagonal columns in $P(k)$ by $l(k)$.

Theorem 4.1. *For a given k, puzzle piece $P(k)$, and $n \geq \max\{s, l, 4\}$, we have*

$$\mu(k) \geq \frac{K(k)}{s(k)l(k)}.$$

Proof. Let $n = q_1 s(k) + r_1 = q_2 l(k) + r_2$ for integers $q_1, q_2, r_1, r_2 \geq 0$, where $r_1 < s(k)$ and $r_2 < l(k)$. Consider a particular k and its corresponding puzzle piece $P(k)$ placed in the top-left of the $n \times n$ board. Let $\mathcal{Z}$ be the set of kings in the $n \times n$ board at this stage. We define

$$\mathcal{Z}' = \{a_{i+cs,j+dl} : a_{i,j} \in \mathcal{Z},\ c \in \{0, 1, 2, \dots, q_1 - 1\},\ d \in \{0, 1, 2, \dots, q_2 - 1\}\}.$$

Therefore, $\mathcal{Z}'$ is the placement of kings arising from a specific stamp and the process of stamping it in $\mathcal{B}_n$.

Since $q_1 q_2$ is the number of copies of the puzzle piece on the $n \times n$ board, $|\mathcal{Z}| = K(k) q_1 q_2$. Thus, since $\mathrm{MNK}(k, n)$ is the maximum number of kings on a k-dependent $n \times n$ board, we have $\mathrm{MNK}(k, n) \geq K(k) q_1 q_2$ for $n \geq \max\{s, l\}$. Also, note that $q_1 = (n - r_1)/s(k)$ and $q_2 = (n - r_2)/l(k)$. So it follows that

$$q_1 q_2 = \frac{(n - r_1)(n - r_2)}{s(k) l(k)}.$$

Thus,

$$\frac{\mathrm{MNK}(k, n)}{n^2} \geq \frac{K(k)(n - r_1)(n - r_2)}{s(k) l(k) n^2}.$$

This implies

$$\mu(k) \geq \sup\left\{ \frac{K(k)(n - r_1)(n - r_2)}{s(k) l(k) n^2} : n \geq \max\{s, l, 4\} \right\} = \frac{K(k)}{s(k) l(k)}. \qquad \square$$

Our choices of $P(k)$ for $0 \leq k \leq 12$ are shown in Figure 6. Each of these is crafted carefully to optimize the maximum proportion of kings, yielding the best lower bound we can manage. The following is a table of the lower bounds we constructed for $\mu(k)$ using the stamping method:

k	0	1	2	3	4	5	6	7	8	9	10	11	12
Lower bound for $\mu(k)$	$\frac{1}{4}$	$\frac{7}{25}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{7}{9}$	$\frac{6}{7}$	$\frac{12}{13}$	1

5. Tightening the upper bounds

Although the upper and lower bounds for $\mu(k)$ were matched for some k using the methods in Sections 3 and 4, we were unable to match others. In this section we use two additional methods to attempt to bring down the upper bound for $\mu(k)$ to match the lower bound.

Taxation. We now use a standard discharging method, which we refer to as *taxation*, to improve the upper bound for $\mu(k)$ when $k \in \{1, 2, 3, 4\}$. In this method, we call any non-king space a *pawn*. So, for a placement of kings $\mathcal{A}$, the set of pawns is $\mathcal{A}^c$. To understand taxation, consider the following scenario. Initially, each pawn starts with X dollars. Then each pawn pays all its money to the kings adjacent

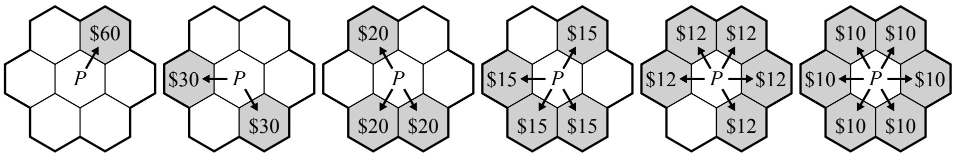


Figure 7. Examples of a taxation rule.

to it, following taxation rules. For example, each pawn could start with \$60 and distribute it evenly among all the kings adjacent to it. Some examples of a pawn following this taxation rule can be seen in Figure 7, where the shaded spaces are kings and the white spaces are pawns.

An *\$X taxation rule* for $\mathcal{A} \in \mathcal{B}_n(k)$ is a function $f : \mathcal{A}^c \times \mathcal{A} \to [0, X]$ such that $\sum_{K \in \mathcal{A}} f(P, K) = X$ for every $P \in \mathcal{A}^c$. Note that in an \$$X$ taxation rule, $\sum_{(P,K) \in \mathcal{A}^c \times \mathcal{A}} f(P, K) = X|\mathcal{A}^c|$. When f is understood, we denote the value of funds a king K receives by $F(K) = \sum_{P \in \mathcal{A}^c} f(P, K)$. In an \$$X$ taxation rule, we know that the total amount of taxes paid, and also received, is \$$X|\mathcal{A}^c|$. So if each king receives a known minimum amount of money, \$$Y$, we can calculate an upper bound for $|\mathcal{A}|$.

Theorem 5.1. *Consider a placement of kings $\mathcal{A} \in \mathcal{B}_n(k)$, where $n \geq 4$. If, following an \$$X$ taxation rule, $F(K) \geq Y$ for all $K \in \mathcal{A}$, then*

$$|\mathcal{A}| \leq \frac{X}{X+Y} n^2.$$

Proof. Consider a placement $\mathcal{A} \in \mathcal{B}_n(k)$. The number of kings is $|\mathcal{A}|$, and the number of pawns is $n^2 - |\mathcal{A}|$. Suppose for some \$$X$ taxation rule f, we have $F(K) \geq Y$ for each king $K \in \mathcal{A}$. Comparing the amounts paid by pawns and received by kings, we see that

$$Y|\mathcal{A}| \leq X(n^2 - |\mathcal{A}|) \implies (X+Y)|\mathcal{A}| \leq Xn^2 \implies |\mathcal{A}| \leq \frac{X}{X+Y} n^2. \qquad \square$$

For a given k, a *general \$X taxation plan* (*tax plan*) $G(\mathcal{A})$ is a function G that assigns an \$$X$ taxation rule f to each $\mathcal{A} \in \mathcal{B}_n(k)$. Given such a G, let

$$\mathcal{Y}_G = \min\{F(K) : K \in \mathcal{A} \text{ and } \mathcal{A} \in \mathcal{B}_n(k)\}.$$

Corollary 5.2. *Suppose that $\mathcal{Y}_G \geq Y$ for some \$X taxation rule G. Then*

$$\mu(k) \leq \frac{X}{X+Y}.$$

Proof. Suppose $\mathcal{Y}_G \geq Y$ for some \$$X$ taxation rule G. For all $\mathcal{A} \in \mathcal{B}_n(k)$,

$$|\mathcal{A}| \leq \frac{X}{X+\mathcal{Y}_G} n^2 \leq \frac{X}{X+Y} n^2,$$

where the first inequality follows from Theorem 5.1. So, $(X/(X+Y))n^2$ is an upper bound on $|\mathcal{A}|$ for all $\mathcal{A} \in \mathcal{B}_n(k)$. Thus, $X/(X+Y)$ is an upper bound for $\mu(k)$. □

For each $k \in \{1, 2, 3, 4\}$, we choose a convenient X and Y and show that there exists an \$$X$ tax plan G such that $\mathcal{Y}_G \geq Y$. Thus, by Corollary 5.2, $\mu(k) \leq X/(X+Y)$. To achieve this, it is helpful to partition each placement $\mathcal{A}$ into parts and analyze how much funding each part receives collectively. We define a *cluster* in a placement of kings $\mathcal{A}$ to be a nonempty set $\mathcal{K} \subseteq \mathcal{A}$ such that for each $v, w \in \mathcal{K}$, there exists a sequence $v = v_1, v_2, \ldots, v_s = w$, where v_i and v_{i+1} are adjacent for each $i \in \{1, 2, \ldots, s-1\}$. When a tax plan is understood, we denote the funds a cluster $\mathcal{K} \subseteq \mathcal{A}$ receives by $F(\mathcal{K}) = \sum_{K \in \mathcal{K}} F(K)$. Since each k-dependent placement of kings is partitioned into clusters in our arguments, it suffices to show that \$$Y|\mathcal{K}| \leq F(\mathcal{K})$ for each cluster $\mathcal{K} \in \mathcal{B}_n(k)$ we consider.

Corollary 5.3. *Consider an* \$$X$ *tax plan for a given* k *and* $n \geq 4$. *Suppose for each* $\mathcal{A} \in \mathcal{B}_n(k)$ *that* $\mathcal{A}$ *partitions into clusters where* \$$Y|\mathcal{K}| \leq F(\mathcal{K})$ *for each cluster* $\mathcal{K}$ *in that partition. Then* $\mu(k) \leq X/(X+Y)$.

In all of our tax plans, any pawn with no adjacent kings is assumed to divide its money equally between all the kings in $\mathcal{A}$. Since we are looking for a maximum number of kings on $\mathcal{B}_n(k)$ for some k with $n \geq 4$, we need not worry about the case $\mathcal{A} = \varnothing$, so there will always be at least one king in any relevant arrangement. In the remainder of this subsection the following notation is used in the diagrams. A *white* hexagon denotes a pawn. A *shaded* hexagon denotes a king. We omit proofs of the following four propositions, as they took many pages of case analysis.

Tax plan for $k = 1$. We show that $\mu(1) \leq \frac{7}{25}$ using a taxation argument. We employ a tax plan where each pawn starts with \$14 and distributes it evenly among kings adjacent to it unless the pawn and king arrangement is one shown in Figure 8. When the arrangement is as in Figure 8(a), the pawn P_1 pays \$6 to K_1 and \$8 to K_2. When the arrangement is as in Figure 8(b)–(c) the pawn P_2 pays \$8 to K_1 and \$6 to K_2. When the arrangement is as in Figure 8(d)–(e) the pawn P_3 pays \$8 to K_1 and \$6 to K_2. Note that the orientation of these figures is arbitrary, with respect to rotation and reflection.

Proposition 5.4. *Every king in a* 1*-dependent arrangement on* $\mathcal{B}_n$ *receives at least* \$36 *using the aforementioned tax plan.*

By Proposition 5.4, $\mu(1) \leq \frac{7}{25}$.

Tax plan for $k = 2$. We show that $\mu(2) \leq \frac{1}{3}$ using a taxation argument. We employ a tax plan where each pawn starts with \$6 and distributes it evenly among kings adjacent to it unless the pawn and king arrangement is as shown in Figure 9. When the arrangement is as in Figure 9, the pawn P_1 pays \$3 to K_1 and \$1.50 each to K_2 and K_3.

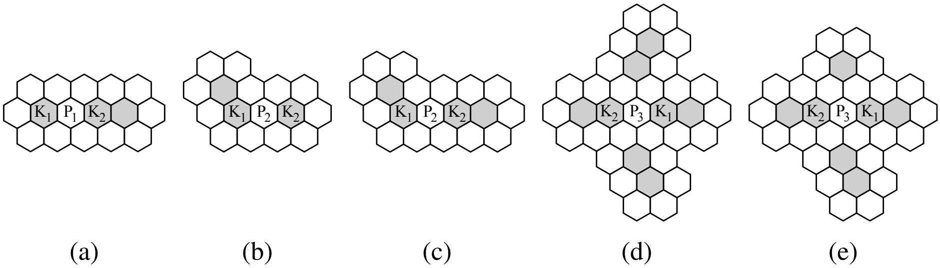


Figure 8. Pawn and king arrangements for the tax plan for $k = 1$.

Proposition 5.5. *Every king in a* 2*-dependent arrangement on* $\mathcal{B}_n$ *receives at least* \$12 *using the aforementioned tax plan.*

By Proposition 5.5, $\mu(2) \leq \frac{1}{3}$.

Tax plan for $k = 3$. We show that $\mu(3) \leq \frac{2}{5}$ using a taxation argument. We employ a tax plan where each pawn starts with \$4 and distributes it evenly among kings adjacent to it unless the pawn and king arrangement is as shown in Figure 9. When the arrangement is as in Figure 9, the pawn P_1 pays \$2 to K_3 and \$1 each to K_1 and K_2.

Proposition 5.6. *Every king in a* 3*-dependent arrangement on* $\mathcal{B}_n$ *receives at least* \$6 *using the aforementioned tax plan.*

By Proposition 5.6, $\mu(3) \leq \frac{2}{5}$.

Tax plan for $k = 4$. We show that $\mu(4) \leq \frac{1}{2}$ using a taxation argument. We employ a tax plan where each pawn starts with \$60 and distributes it evenly amongst all kings adjacent to it.

Proposition 5.7. *Every king in a* 4*-dependent arrangement on* $\mathcal{B}_n$ *receives at least* \$60 *using the aforementioned tax plan.*

By Proposition 5.7, $\mu(4) \leq \frac{1}{2}$.

Further linear programming. We improve our bounds using a more general form of linear programming we call *weighting patterns*. In this method, we seek more complicated linear inequality constraints that are valid for all k-dependent sets.

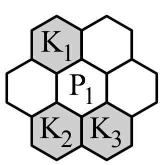


Figure 9. The special case for $k = 2$ and $k = 3$.

Given $n \geq 4$ and a *weighting function* $\varpi : \mathcal{B}_n \to [0, \infty)$, not everywhere zero, let $W(\varpi)$ denote the *total weight*, $\sum_{v \in \mathcal{B}_n} \varpi(v)$, of ϖ. Given k and ϖ on $\mathcal{B}_n$, let $M_k(n, \varpi)$ denote the maximum value of $\sum_{v \in \mathcal{B}_n} \varpi(v) \chi_{\mathcal{A}}(v)$ over all $\mathcal{A} \in \mathcal{B}_n(k)$. So, for any such ϖ and any $\mathcal{A} \in \mathcal{B}_n(k)$, we always have

$$\sum_{v \in \mathcal{B}_n} \varpi(v) \chi_{\mathcal{A}}(v) \leq M_k(n, \varpi).$$

To define a weighting function ϖ we refer to a figure with mapped values in their corresponding spaces. For example, if we defined ϖ by Figure 10(a), then $\varpi(a_{5,4}) = 10740$. Given a weighting function ϖ, we define the *shifted function* $\varpi_{x,y} : \mathcal{B}_n \to [0, \infty)$ by $\varpi_{x,y}(a_{i,j}) = \varpi(a_{(i+x) \bmod n, (j+y) \bmod n})$. Note that $\varpi_{0,0} = \varpi$.

Theorem 5.8. *Consider any weighting function ϖ on $\mathcal{B}_n$. Then*

$$\mu(k) \leq \frac{M_k(n, \varpi)}{W(\varpi)}$$

whenever $n \geq n_0 \geq 4$, where n_0 is the minimum value such that the weighting pattern ϖ fits within an $n_0 \times n_0$ rhombus.

Proof. Given $n \geq n_0 \geq 4$, let $\mathcal{A} \in \mathcal{B}_n(k)$, and let $\Gamma = \{0, 1, 2, \ldots, n-1\}$. For any $x, y \in \Gamma$, we know that $\varpi_{x,y}$ is also a weighting function on $\mathcal{B}_n$. Thus we have

$$\sum_{a_{i,j} \in \mathcal{B}_n} \varpi_{x,y}(a_{i,j}) \chi_{\mathcal{A}}(a_{i,j}) \leq M_k(n, \varpi).$$

Therefore

$$\sum_{x,y \in \Gamma} \Bigg(\sum_{a_{i,j} \in \mathcal{B}_n} \varpi_{x,y}(a_{i,j}) \chi_{\mathcal{A}}(a_{i,j}) \Bigg) \leq \sum_{x,y \in \Gamma} M_k(n, \varpi),$$

$$\sum_{a_{i,j} \in \mathcal{B}_n} \Bigg(\chi_{\mathcal{A}(a_{i,j})} \Bigg(\sum_{x,y \in \Gamma} \varpi_{x,y}(a_{i,j}) \Bigg) \Bigg) \leq n^2 M_k(n, \varpi),$$

$$\sum_{a_{i,j} \in \mathcal{B}_n} \Bigg(\chi_{\mathcal{A}(a_{i,j})} \Bigg(\sum_{x,y \in \Gamma} \varpi(a_{(i+x) \bmod n, (j+y) \bmod n}) \Bigg) \Bigg) \leq n^2 M_k(n, \varpi),$$

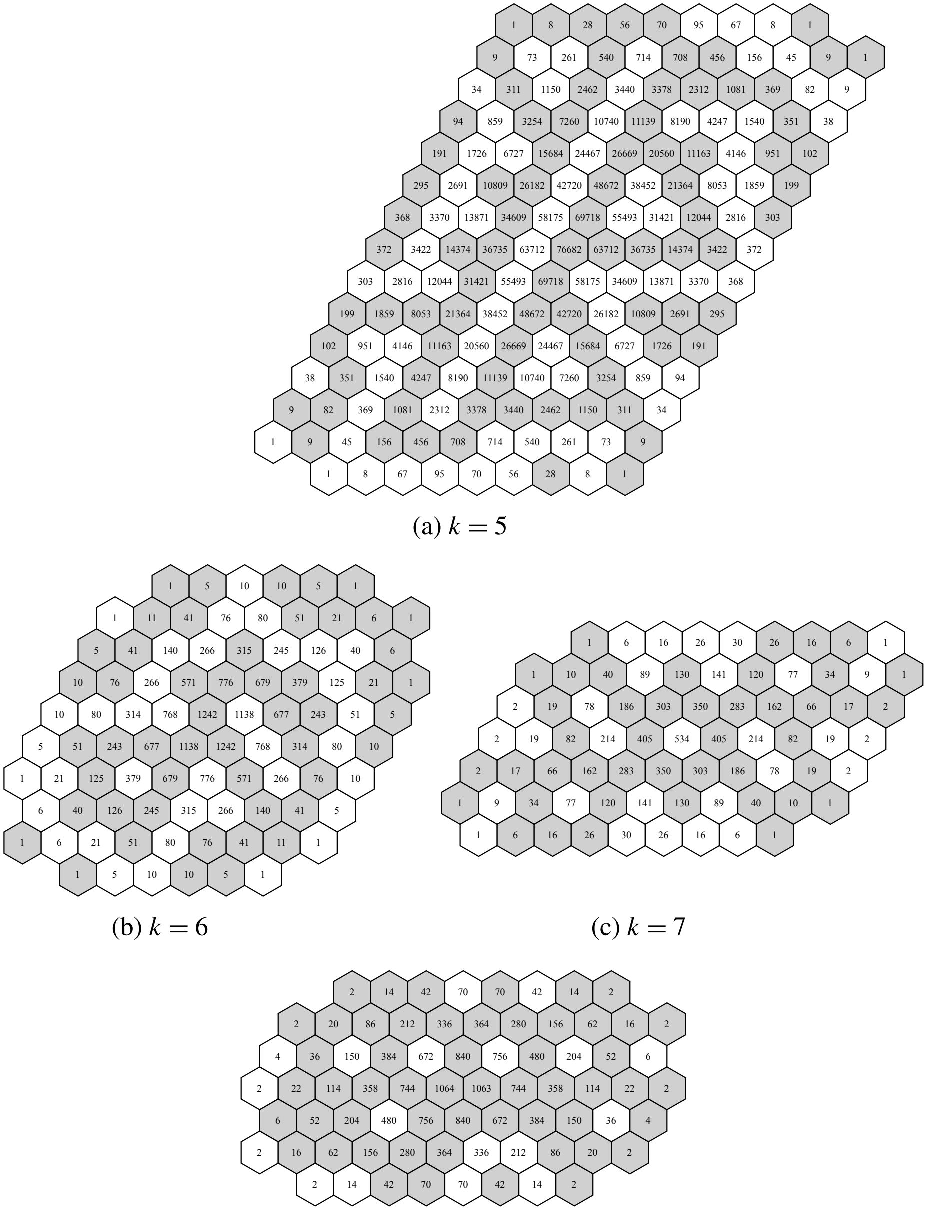
$$\sum_{a_{i,j} \in \mathcal{B}_n} (\chi_{\mathcal{A}}(a_{i,j}) W(\varpi)) \leq n^2 M_k(n, \varpi),$$

and thus

$$|\mathcal{A}| \cdot W(\varpi) \leq n^2 M_k(n, \varpi) \quad \Longrightarrow \quad \frac{|\mathcal{A}|}{n^2} \leq \frac{M_k(n, \varpi)}{W(\varpi)}.$$

Since $n \geq 4$ and $\mathcal{A} \in \mathcal{B}_n(k)$ were arbitrary, we have

$$\mu(k) \leq \frac{M_k(n, \varpi)}{W(\varpi)}. \qquad \square$$



(a) $k = 5$

(b) $k = 6$

(c) $k = 7$

(d) $k = 9$

Figure 10. Weighting functions ϖ for $k \in \{5, 6, 7, 9\}$.

Using Theorem 5.8, we improve our upper bounds for some $\mu(k)$ values by conveniently choosing the weighting patterns in Figure 10 to define ϖ for $k \in \{5, 6, 7, 9\}$. The shaded hexagons represent a k-dependent placement of kings resulting in the largest $M_k(n, \varpi)$.

Applying four weighting functions and employing a tedious case analysis yields the following upper bounds for $\mu(k)$:

k	5	6	7	9
Upper bound for $\mu(k)$	$\frac{902416}{1636944}$	$\frac{11099}{17872}$	$\frac{4520}{6474}$	$\frac{12287}{15359}$

6. Conclusion

When the upper and lower bounds for $\mu(k)$ match, we know

$$\lim_{n\to\infty}\left(\frac{\text{MNK}(k,n)}{n^2}\right)=\mu(k).$$

In the cases where the upper and lower bounds for $\mu(k)$ do not match, it remains to determine the values of $\lim_{n\to\infty}(\text{MNK}(k,n)/n^2)$, if indeed these limits exist. In summary, combining results from various methods, the best results we found concerning the values of $\mu(k)$ are

$$\mu(0)=\tfrac{1}{4},\quad \mu(1)=\tfrac{7}{25},\quad \mu(2)=\tfrac{1}{3},\quad \mu(3)=\tfrac{2}{5},\quad \mu(4)=\tfrac{1}{2},$$
$$\tfrac{8}{15}\le\mu(5)\le\tfrac{902416}{1636944},\quad \tfrac{3}{5}\le\mu(6)\le\tfrac{11099}{17872},\quad \tfrac{2}{3}\le\mu(7)\le\tfrac{4520}{6474},\quad \mu(8)=\tfrac{3}{4},$$
$$\tfrac{7}{9}\le\mu(9)\le\tfrac{12287}{15359},\quad \mu(10)=\tfrac{6}{7},\quad \mu(11)=\tfrac{12}{13},\quad \mu(12)=1.$$

Although our bounds for $\mu(k)$ are tight for most values of k, we were unable to match the bounds for $k\in\{5,6,7,9\}$. The tightness of these bounds can perhaps be improved through computer programming and other mathematical methods. For example, one could try a "reverse taxation" process to lower the upper bound for $\mu(9)$. In such a process the kings would be given money to distribute to the pawns, potentially limiting the number of cases in the case analysis. Additionally, a computer could be used to test larger weighting patterns, although we have reason to believe that it is unlikely they will lead to a tight bound. Finally, by computer search or by hand, one could search for patterns which raise the lower bound.

To create a new problem, one can investigate $\mu(k)$ on similar boards with a change in the realm of a king. Moreover, one can consider boards consisting of other shapes, or in higher dimensions. For example, one could examine the k-dependence of kings on an $n\times n$ rhombus tiled with equilateral triangles, a parallelepiped whose surface is tiled with equilateral triangles, or a parallelepiped internally tiled with regular tetrahedra. A similar type of problem one might investigate is the domination number as in [Haynes et al. 1998] of an $n\times n$ hexagonal board on a torus using our king's realm. One could also examine the domination number of a board with respect to different realms or of a different board altogether. We encourage the investigation of these problems, because they seem to us ideal problems for upper-level undergraduates with backgrounds or interests in discrete mathematics.

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doughtrl@miamioh.edu — *Miami University, Oxford, OH 45162, United States*

jlg131@zips.uakron.edu — *The University of Akron, Akron, OH 44304, United States*

adriana.morales@upr.edu — *University of Puerto Rico, San Juan, 00931, Puerto Rico*

berkeleyreiswig@gmail.com — *Anderson University, Anderson, SC 29621, United States*

jreiswig@math.sc.edu — *University of South Carolina, Columbia, SC 29208, United States*

kat.slyman@gmail.com — *Wake Forest University, Winston-Salem, NC 27106, United States*

pritikd@miamioh.edu — *Department of Mathematics, Miami University, 123 Bachelor Hall, 301 S. Patterson Ave., Oxford, OH 45056, United States*

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