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Gonality of random graphs

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The gonality of a graph is a discrete analogue of the similarly named geometric invariant of algebraic curves. Motivated by recent progress in Brill–Noether theory for graphs, we study the gonality of random graphs. In particular, we show that the gonality of a random graph is asymptotic to the number of vertices.

1. Introduction

In the moduli space of curves, the locus of Brill–Noether general curves is a dense open subset [Griffiths and Harris 1980]. In the moduli space of tropical curves, however, the Brill–Noether general locus is open [Lim et al. 2012; Len 2014] and nonempty [Cools et al. 2012], but it is not dense [Jensen 2014]. A natural question, therefore, is *how likely is it that a graph is Brill–Noether general?*

In this paper, we approach this question by studying the gonality of Erdős–Rényi random graphs. Recall that an Erdős–Rényi random graph $G(n, p)$ is obtained by fixing n vertices and, for each pair of vertices, introducing an edge between them with probability p. We will often refer to such graphs simply as random graphs, where the probability distribution is understood to be that of Erdős–Rényi. It is common to define the probability p as a function of n, and to consider the expected value of combinatorial invariants as n increases. Throughout, we use $\mathbb{P}$ and $\mathbb{E}$ to denote the probability and expected value, respectively.

The current article is a natural follow-up to other recent work on the divisor theory of random graphs. Most notably, Lorenzini [2008] asked about the distribution of divisor class groups of random graphs, and in [Clancy et al. 2015b] it is conjectured that they are distributed according to a variation of the Cohen–Lenstra heuristics. This conjecture is proved in [Wood 2014], expanding on the preliminary work of [Clancy et al. 2015a].

Before stating our main result, we briefly recall the basic theory of divisors on graphs. For a more detailed account, see [Baker and Norine 2007] and [Baker 2008]. A *divisor* on a simple graph G is an element of the free abelian group on the

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vertices of G, and a divisor $D=\sum_{v\in V(G)} a_v v$ is said to be *effective* if $a_v \geq 0$ for all v. Given a divisor $D=\sum_{v\in V(G)} a_v v$ and a vertex v', we may *fire* v' to obtain a new divisor $D'=\sum_{v\in V(G)} b_v v$, where

$$b_v = \begin{cases} a_v - \operatorname{val}(v) & \text{if } v = v', \\ a_v + 1 & \text{if } v \text{ is adjacent to } v', \\ a_v & \text{otherwise,} \end{cases}$$

where $\operatorname{val}(v)$ is the valence of the vertex v. Two divisors are *equivalent* if one can be obtained from the other by firing a sequence of vertices, and we say that a divisor D has positive rank if $D-v$ is equivalent to an effective divisor for all vertices v in G. The *gonality* $\operatorname{gon}(G)$ is the smallest degree of a divisor with positive rank. Our main result is the following.

Theorem 1.1. *Let $p(n)=c(n)/n$, and suppose that $\log(n) \ll c(n) \ll n$. Then*

$$\mathbb{E}(\operatorname{gon}(G(n,p))) \sim n.$$

Theorem 1.1 essentially says that the expected gonality of a random graph is as high as possible. We note, however, that the gonality of random graphs nevertheless falls short of that for a general curve, as the genus of a random graph is asymptotically $\frac{1}{2}c(n)n$, and if $c(n)$ is unbounded, this grows faster than n. From this perspective, it may be more natural to study the gonality of random *regular* graphs, as the genus of such graphs grows in proportion to the number of vertices. The case of 3-regular graphs would be particularly interesting, as such graphs correspond to top-dimensional strata of the moduli space of tropical curves.

Although Theorem 1.1 follows directly from the earlier work of de Bruyn and Gijswijt [2014] and Wang et al. [2011], it appears to be unknown to experts in tropical Brill–Noether theory. At the time of writing, we became aware of simultaneous work by Amini and Kool [2016], in which they use an improvement on the spectral methods of [Cornelissen et al. 2015] to show that the gonality of a random graph is bounded above and below by constant multiples of n. Our results are essentially a tightening of these bounds, so that both upper and lower bounds are asymptotic to n, which indeed is conjectured in [Amini and Kool 2016, Section 5.2]. Their techniques apply additionally to metric graphs, which we do not discuss here, and to the case of random regular graphs, which they show to have gonality bounded above and below by constant multiples of n as well.

Also of note is the bound that we provide on the error term $n-\mathbb{E}(\operatorname{gon}(G(n,p)))$ (see Theorem 3.3). In the future, it would be interesting to explore with what precision we can bound this term.

A more complete study of the Brill–Noether theory of random graphs would involve divisors of rank greater than one. A natural generalization of the current line of inquiry would be to study the Clifford index of random graphs, defined as

$$\mathrm{Cliff}(G) := \min_{D\in \mathrm{Jac}(G)} \{\deg(D) - 2r(D) \mid r(D) > 0 \ \text{and}\ r(K_G - D) > 0\}.$$

Note that if the minimum in this expression is obtained by a divisor of rank one, then $\mathrm{Cliff}(G) = \mathrm{gon}(G) - 2$. The Clifford index of an algebraic curve C is known to always be either $\mathrm{gon}(C) - 2$ or $\mathrm{gon}(C) - 3$ [Coppens and Martens 1991]. The corresponding statement remains open for graphs, but if true, it would imply that the Clifford index of a random graph is asymptotic to the number of vertices as well.

2. A lower bound

In this section, we obtain a lower bound on the expected gonality of a random graph. The first step is to identify a lower bound for the gonality of an arbitrary graph. This is done in [de Bruyn and Gijswijt 2014], where it is shown that the *treewidth* of a graph is a lower bound for the gonality.

Definition 2.1. A *tree decomposition* of a graph G is a tree T whose nodes are subsets of the vertices of G, satisfying the following properties:

(1) Each vertex of G is contained in at least one node of T.

(2) If two nodes of T both contain a given vertex v, then all nodes of the tree in the unique path between these two nodes must contain v as well.

(3) If two vertices v and w are adjacent in G, then there is a node of T that contains both v and w.

The *width* of a tree decomposition is one less than the number of vertices in its largest node. The *treewidth* $\mathrm{tw}(G)$ of a graph G is the minimum width among all possible tree decompositions of G.

Proposition 2.2 [de Bruyn and Gijswijt 2014]. *Let G be a simple connected graph. Then*

$$\mathrm{gon}(G) \geq \mathrm{tw}(G).$$

Although we will not use it, we note the following simple consequence.

Corollary 2.2.1. *For a simple connected graph G,*

$$\mathrm{gon}(G) \geq \min\{\mathrm{val}(v) \mid v \in V(G)\}.$$

Proof. The result follows immediately from Proposition 2.2 and the fact that $\mathrm{tw}(G) \geq \min\{\mathrm{val}(v) \mid v \in V(G)\}$ (see [Bodlaender and Koster 2011]). □

The treewidth of random graphs has been studied extensively in [Wang et al. 2011; Gao 2012].

Lemma 2.3 [Wang et al. 2011]. *Let $p(n) = c(n)/n$, and suppose that $c(n) \ll n$ is unbounded. Then*

$$\lim_{n\to\infty} \mathbb{P}\big(\mathrm{tw}(G(n, p)) \geq n - o(n)\big) = 1.$$

Theorem 2.4. *Let $p(n) = c(n)/n$, and suppose that* $\log(n) \ll c(n) \ll n$. *Then*

$$\lim_{n\to\infty} \mathbb{P}\big(\mathrm{gon}(G(n, p)) \geq n - o(n)\big) = 1.$$

Proof. A random graph is always simple, and by a well-known result of Erdős and Rényi [1959], the assumption $c(n) \gg \log(n)$ implies that such a graph is connected with probability approaching 1. It follows that

$$\lim_{n\to\infty} \mathbb{P}\big(\mathrm{gon}(G(n, p)) \geq n - o(n) \text{ and } G(n, p) \text{ is connected}\big) = \lim_{n\to\infty} \mathbb{P}\big(\mathrm{gon}(G(n, p)) \geq n - o(n)\big).$$

By Proposition 2.2, if $G(n, p)$ is connected, then $\mathrm{gon}(G(n, p)) \geq \mathrm{tw}(G(n, p))$, and it follows that

$$\lim_{n\to\infty} \mathbb{P}\big(\mathrm{gon}(G(n, p)) \geq n - o(n)\big) \geq \lim_{n\to\infty} \mathbb{P}\big(\mathrm{tw}(G(n, p)) \geq n - o(n)\big) = 1,$$

where the final equality follows from Lemma 2.3. □

3. An upper bound

In this section, we obtain an upper bound on the expected gonality of a random graph. Together with the results of the previous section, this will imply that the gonality of a random graph is asymptotically equal to the number of vertices. We note that the number of vertices n is a very simple upper bound for the gonality of a graph and, together with Theorem 2.4, this would be enough to establish the main theorem. We actually go a bit further and obtain a bound on the expected value of $n - \mathrm{gon}(G(n, p))$. In the future, it would be interesting to explore this with higher precision.

Recall that an *independent set* in a graph is a set of vertices, no pair of which are connected by an edge. The *independence number* $\alpha(G)$ of a graph G is defined to be the maximal size of an independent set.

Proposition 3.1. *If G is a simple connected graph with n vertices, then* $\mathrm{gon}(G) \leq n - \alpha(G)$.

Proof. Let I be a maximal independent set, and let D be the sum of the vertices in the complement of I. We will show that D has positive rank. If $v \notin I$, then $D - v$ is effective by definition. On the other hand, if $v \in I$, then since all of the neighbors of v are not in I and the graph is simple, by firing all of the vertices other than v we obtain an effective divisor equivalent to D with at least one chip on v. It follows that D has rank at least one, hence $\mathrm{gon}(G) \leq \deg(D) = n - \alpha(G)$. □

Note that gonality $n-1$ is achieved by the complete graph K_n, so this bound is sharp. Note further that the complete graph is the only simple graph with n vertices whose gonality is $n-1$.

The expected independence number of a random graph has been studied in [Frieze 1990].

Lemma 3.2 [Frieze 1990]. *Let $p(n) = c(n)/n$, and suppose that $c(n) \ll n$ is unbounded. For any $\epsilon > 0$, we have*

$$\lim_{n\to\infty} \mathbb{P}\biggl(\Bigl|\alpha(G(n,p)) - \frac{2}{p(n)}(\log c(n) - \log\log c(n) - \log 2 + 1)\Bigr| \le \frac{\epsilon}{p(n)}\biggr) = 1.$$

From this, we can conclude the following.

Theorem 3.3. *Let $p(n) = c(n)/n$, and suppose that* $\log(n) \ll c(n) \ll n$. *Then*

$$\lim_{n\to\infty} \mathbb{P}\Bigl(\mathrm{gon}(G(n,p)) \le n - \frac{2}{p(n)}(\log c(n) - \log\log c(n) - \log 2 + 1)\Bigr) = 1.$$

Proof. By Lemma 3.2, for any $\epsilon > 0$, we have

$$\alpha(G(n,p)) > \frac{2}{p(n)}(\log c(n) - \log\log c(n) - \log 2 + 1 - \epsilon)$$

with probability 1 as n approaches infinity. By Proposition 3.1, the number $n - \alpha(G(n,p))$ is an upper bound for the gonality of $G(n,p)$. □

Proof of Theorem 1.1. Again, the assumption that $c(n) \gg \log(n)$ implies that the random graph is connected with high probability. By Theorem 3.3, the gonality of a random graph is bounded above by $n - o(n)$. Similarly, by Theorem 2.4, the gonality of a random graph is bounded below by $n - o(n)$. It follows that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\bigl(\mathrm{gon}(G(n,p))\bigr) = \lim_{n\to\infty} \frac{n - o(n)}{n} = 1. \qquad \square$$

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