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Envelope curves and equidistant sets

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Given two sets of points A and B in the plane (called the *focal sets*), the *equidistant set* (or *midset*) of A and B is the locus of points equidistant from A and B. This article studies envelope curves as realizations of focal sets. We prove two results: First, given a closed convex focal set A that lies within the convex region bounded by the graph of a concave-up function h, there is a second focal set B (an envelope curve for a suitable family of circles) such that the graph of h lies in the midset of A and B. Second, given any function $y = h(t)$ with a continuous third derivative and bounded curvature, the envelope curves A and B associated to any family of circles of sufficiently small constant radius centered on the graph of h will define a midset containing this graph.

1. Introduction

Given two sets of points A and B in the plane (called the *focal sets*), the *equidistant set* (or *midset*) of A and B is the locus of points equidistant from A and B. For this definition to make sense, we need to know how to find the distance $d_A(p)$ from a point p to a set A. Intuitively, this is the smallest distance from p to a point of A; however, since the minimum distance may not exist if A is not closed, we define

$$d_A(p) = \inf\{d(a, p) \mid a \in A\}. \tag{1}$$

If there is a point $a \in A$ such that $d_A(p) = d(p, a)$, we call a a *foot point* of p with respect to A.

We now present a few examples of equidistant sets (see Figure 1).

Example 1.1. It is well known that the locus of points equidistant from two points in the plane is the perpendicular bisector of the line segment joining the two points.

Example 1.2. It is equally well known that the locus of points equidistant from a point (the focus) and a line not containing the point (the directrix) is a parabola.

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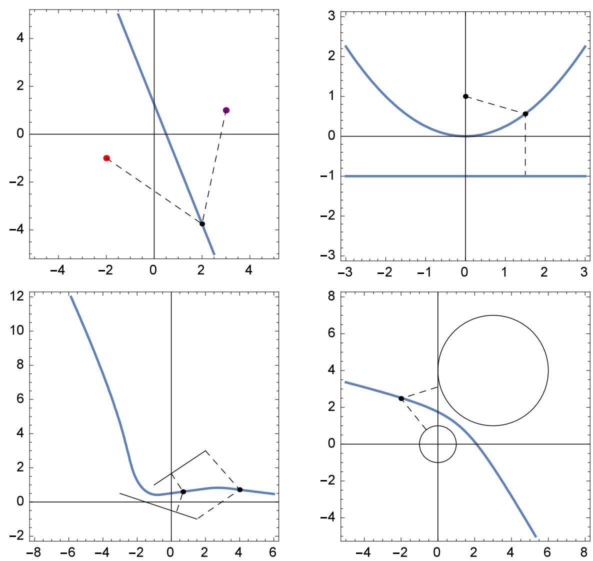

Figure 1. Clockwise from the upper left, the figures illustrate Examples 1.1, 1.2, 1.3, and 1.4.

Example 1.3. If two disjoint circles are taken as focal sets, the resulting midset is a conic section. Example 1.1 can be viewed as a special case in which the two circles have degenerated to two points, and the midset is a circle of infinite radius (the perpendicular bisector of the segment joining the two points).

Example 1.4. If the two focal sets are line segments, the midset is a curve pieced together from line segments and parabolic arcs.

The reader is encouraged to try to construct the midsets of other pairs of focal sets. It soon becomes clear that computing the exact midset of two focal sets can be a daunting problem even for relatively simple focal sets. In spite of this, certain general facts are known: For example, Ponce and Santibáñez [2014, Theorem 11, p. 28] show that under mild hypotheses on the focal sets, the equidistant sets vary continuously with the focal sets. They also suggest that the midsets associated to certain pairs of focal sets be viewed as generalized conics, by analogy with Examples 1.2 and 1.3.

In general, the results in the literature deal with the problem of describing the midset, given hypotheses on the focal sets. Our work arises from asking whether, given one focal set A and a proposed midset M, it is possible to find a second focal set B such that M is the midset (or a subset thereof) defined by A and B.

As an example, we can turn Example 1.2 on its head. Instead of being given a point (focus) and a line (directrix) and asked to find the parabolic midset they define, suppose that one is given a parabola and a point A lying in the convex region bounded by the parabola. Taking the parabola as the set M, one can ask if there

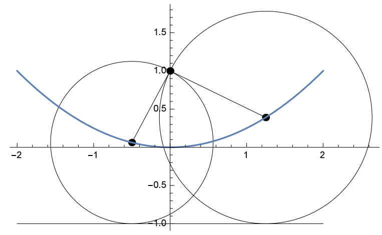

Figure 2. Another view of Example 1.2, showing how the family of circles with centers on the parabola and passing through the focus define the directrix (the line $y = -1$) as an envelope curve.

is a set B such that M is the midset of A and B. If such a set B were to exist, then for every point $m \in M$, the circle centered at m and passing through A would have to touch the boundary of B, since m must be equidistant from A and B (see Figure 2). This suggests that we can find B by constructing the envelope curves for the family of circles centered at points $m \in M$ and passing through A. In Section 4, we carry out this procedure for the more general situation in which M is the graph of a concave-up function $y = h(t)$ (that is, $h''(t) > 0$ for all t), and the point A is replaced by a closed convex set that lies "above" (in a sense to be made precise) the curve M. To set the stage, we first discuss convex sets A and the distance functions d_A they define in Section 2, and envelope curves of circles in Section 3.

In Section 5, we present our second result, which arose from asking what would happen if we started with a point A and a midset curve M that is not of uniform concavity, such as the graph of $h(t) = t^3$, and tried to find a second focal set B. We will see that the construction in Section 4 yields an envelope curve B such that the midset of A and B contains only a subset of M (not all of M). This leads to the question of how to realize the graph of $h(t) = t^3$ (or any sufficiently smooth graph M of varying concavity) as a midset. Our solution is to use the two envelope curves determined by a suitable family of circles of constant radius with centers on M: the key point is that the radius of the circles must be less than the reciprocal of the maximum curvature of M.

2. Convex sets

Recall that a set $A \subseteq \mathbb{R}^n$ is *convex* if and only if for $p, q \in A$ the segment pq lies in A. In the following discussion, we restrict attention to closed convex sets $A \subseteq \mathbb{R}^2$.

Lemma 2.1. *For any point p external to the (closed and convex) set A, there is a unique foot point $f(p)$ in A (in fact, in the boundary of A).*

Proof. Replacing A by the nonempty compact set

$$A' = A \cap \{p' \mid d(p, p') \le d_A(p) + 1\},$$

we see that there exists at least one point $q \in A' \subseteq A$ such that $d_A(p) = d(p, q)$, so a foot point exists. If there were two distinct foot points q_1 and q_2 for p, then the segment $q_1 q_2$ would be a subset of A, since A is convex. From this it follows immediately that there would be a point of A lying closer to p than either foot point, which is a contradiction. Finally, it is clear that a foot point for p cannot lie in the interior of A. □

A line L is a *support line* of a plane set A if and only if L contains at least one boundary point of A and is such that the entire set A is contained in one of the two half-planes determined by L.

Lemma 2.2. *Let p be a point external to A, and $f(p) = q$ the associated foot point. Let L be the line through q that is orthogonal to the segment pq. Then L is a support line of A such that A lies in the half-plane of L that does not contain p.*

Proof. Let d be the distance from p to q. If all the points of A lie in the half-plane of L that does not contain p, we are done. If not, then there is a point q' in the (open) half-plane of L containing p. Since L is tangent at q to the circle C of radius d centered at p, the line joining q and q' intersects C at two points, and therefore the segment qq' lies partially in the interior of C. Since A is convex, qq' lies in A; this implies that there are points of A that lie closer to p than d. This is a contradiction, whence the result. □

Lemma 2.3. *The map $p \mapsto f(p) = q$, sending each point not in A to its unique foot point, is continuous.*

Proof. Let (p_n) be a sequence of points external to A that converges to p, and let (q_n) be the associated sequence of foot points. By discarding the first $N \geq 1$ terms of this sequence, we may assume that $d(p_n, p) < 1$ for all n. The triangle inequality then yields that

$$d(p_n, q) \leq d(p_n, p) + d(p, q) < d(p, q) + 1 \quad \text{for all } n.$$

We claim that $q_n \to q$. To see this, we first observe that since the distance from p_n to A is minimized at q_n,

$$d(p, q_n) \leq d(p, p_n) + d(p_n, q_n) \leq d(p, p_n) + d(p_n, q) < d(p, q) + 2.$$

In other words, the sequence (q_n) lies in the (compact) intersection of A and the disk of radius $d(p, q) + 2$ centered at p, whence the sequence has a convergent subsequence (q_{n_k}) with limit q^*. We then have

$$d(p, q^*) \leq d(p, p_{n_k}) + d(p_{n_k}, q_{n_k}) + d(q_{n_k}, q^*) \qquad \text{for all } n_k;$$

since $(p_{n_k}) \to p$ and $(q_{n_k}) \to q^*$, we obtain $d(p, q^*) \leq d(p, q)$. The uniqueness of foot points (Lemma 2.1) then yields $q^* = q$. Indeed, the same argument shows that any subsequential limit of (q_n) must equal q; it follows that (q_n) converges to q. □

Corollary 2.4. *Let $A \subseteq \mathbb{R}^2$ be closed and convex. Then the distance function d_A (restricted to points $p \notin A$) is continuous.*

Proof. Let $(p_n) \to p$ be a sequence external to A, and q_n the associated sequence of foot points. Then by the preceding lemma,

$$d_A(p_n) = d(p_n, q_n) \to d(p, q) = d_A(p),$$

which yields the result. □

The situation is even better than this; indeed, we have the following (see [Giaquinta and Modica 2012, Theorem 2.21 (Motzkin), p. 75]):

Lemma 2.5. *Let $A \subseteq \mathbb{R}^2$ be closed and convex. Then the distance function d_A (restricted to points $p \notin A$) is differentiable.* □

Before proceeding to the next section, we need one more result connected to the convexity of the set A.

Lemma 2.6. *Let $p_1 \neq p_2$ be two points external to A, and let their foot points be q_1, q_2, respectively. Then $d(q_1, q_2) \leq d(p_1, p_2)$.*

Proof. Consider the segments p_1q_1 and p_2q_2. All the points on p_iq_i $(i = 1, 2)$, except for q_i, are external to A; otherwise q_i would not be the foot point of p_i. We claim that exactly one of the following cases holds:

<u>Case 1:</u> One of the segments contains the other, in which case $q_1 = q_2$.

<u>Case 2:</u> The two segments intersect (only) at $q_1 = q_2$.

<u>Case 3:</u> The two segments are disjoint.

To prove the claim, we consider the various possibilities: If the two segments are disjoint, we are in Case 3. If they meet, then they either overlap or they meet in exactly one point. If they overlap, then a moment's reflection shows that we are in Case 1. If they meet in one point p^*, then it is possible that $p^* = q_1 = q_2$, which is Case 2. If we had (say) $p^* = q_1 \neq q_2$, then $p^* \in A$ would be closer to p_2 than q_2, contradicting that q_2 is the foot point of p_2. Since we are assuming $p_1 \neq p_2$, we see that if the segments meet in a single point p^* and we are not in Case 2, then p^* must be in the interior of both segments. Supposing this to be the case, we now complete the proof of the claim by deriving a contradiction: Let L denote the line perpendicular to p_1q_1 at q_1, which by Lemma 2.2 is a support line of A. Since $q_2 \in A$, we know that q_2 must lie in the half-plane of L that does not contain p_1. There are now two subcases (see Figure 3):

<u>Subcase 1:</u> The point p_2 lies in the same (open) half-plane of L as does p_1. In this case, we see that the angle $p_2q_1q_2$ must be obtuse, which implies that $d(p_2, q_1) < d(p_2, q_2)$, contradicting that q_2 is the foot point of p_2.

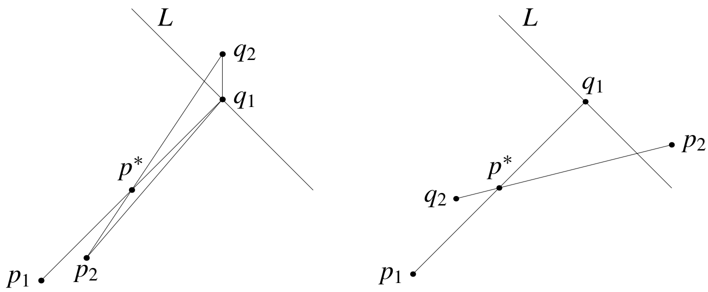

Figure 3. The two subcases of Lemma 2.6; Subcase 1 to the left and Subcase 2 to the right.

Subcase 2: The point p_2 lies in the (closed) half-plane of L that does not contain p_1. Then, since p^* is in the other half-plane of L, we see that the segment $p_2 p^*$ intersects L. From this it follows that q_2 lies in the open half-plane of L that contains p_1, which in turn implies that $q_2 \notin A$, and again we have a contradiction.

Since both cases lead to contradictions, the claim is proved.

In Cases 1 and 2, we have that $d(q_1, q_2) = 0$ and $d(p_1, p_2) > 0$, so the desired inequality is immediate. It remains to prove the inequality in Case 3. To this end, consider the segment $q_1 q_2$ and the angles that the segments $p_i q_i$ make with it for $i = 1, 2$ (see Figure 4). Since $q_1 q_2 \subseteq A$, the support line L perpendicular to $p_i q_i$ at q_i must be such that the segment $q_1 q_2$ lies in the (closed) half-plane not containing p_i; this implies that the angle θ_i between $p_i q_i$ and $q_1 q_2$ is either right or obtuse. It is then a routine geometric exercise to show that $d(q_1, q_2) \leq d(p_1, p_2)$. $\square$

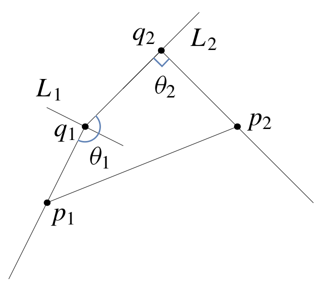

Figure 4. The angle θ_1 is drawn obtuse and the second angle θ_2 is right. One can see that if either of these angles was acute, then the segment $q_1 q_2$ would lie on the "wrong" side of the associated support line L_i. In the case illustrated, p_1 and p_2 lie on the same side of the line through q_1 and q_2; the reader can check that the desired conclusion still holds if one or both of p_1 and p_2 are either on this line or lie on opposite sides of it.

3. Envelopes of circles

In this section we recall the definition of the envelope of a family of circles, and show how to find parametric equations of the envelope curves. We will assume that the circles have centers on the graph of a differentiable function $y = h(t)$, and radii given by a differentiable function $r(t)$. Then the family of circles is described by

$$F(x, y, t) = (x-t)^2 + (y-h(t))^2 - r^2(t) = 0; \tag{2}$$

that is, a point (x, y) lies on one of the circles if and only if there is a value of t such that (2) holds. As explained in, e.g., [Bruce and Giblin 1992, Chapter 5, pp. 99–103] or [Cox et al. 1992, Chapter 3, Section 4, pp. 139–144], the point (x, y) lies on one of the envelope curves if and only if there is a t such that both (2) and the following equation hold:

$$\frac{\partial F(x, y, t)}{\partial t} = -2(x-t) - 2(y-h(t))h'(t) - 2r(t)r'(t) = 0. \tag{3}$$

Solving (3) for x, we obtain

$$x = t - (y-h(t))h'(t) - r(t)r'(t); \tag{4}$$

substituting this into (2) yields a quadratic equation for y:

$$\big(-(y-h(t))h'(t) - r(t)r'(t)\big)^2 + (y-h(t))^2 - r^2(t) = 0. \tag{5}$$

From the quadratic formula, we obtain parametric equations for the two envelope curves ($i = 1, 2$):

$$\begin{aligned} y_i(t) &= \frac{h(t)+h(t)h'(t)^2 - r(t)r'(t)h'(t) + (-1)^{i+1}\sqrt{r(t)^2(1+h'(t)^2-r'(t)^2)}}{1+h'(t)^2}, \\ x_i(t) &= t-(y_i(t)-h(t))h'(t)-r(t)r'(t). \end{aligned} \tag{6}$$

For example, when $h(t) = \sin t$ and $r(t) = 2/(2+t^2)$, the two envelope curves are shown in Figure 5. Also note that in order for the envelope curves to be defined as real curves, the expression under the radical must be nonnegative, which in turn requires that

$$1 + h'(t)^2 - r'(t)^2 \geq 0. \tag{7}$$

In other (less precise) words, the radius function cannot change too rapidly.

We conclude this section with one more lemma concerning the envelope curves of families of circles. We begin by defining the following vector-valued functions ($i = 1, 2$), viewing $\mathbb{R}^2$ as the xy-plane in $\mathbb{R}^3$:

$$\begin{aligned} \boldsymbol{v}(t) &= (1, h'(t), 0), \\ \boldsymbol{f}_i(t) &= \big(x_i(t)-t,\ y_i(t)-h(t),\ 0\big). \end{aligned} \tag{8}$$

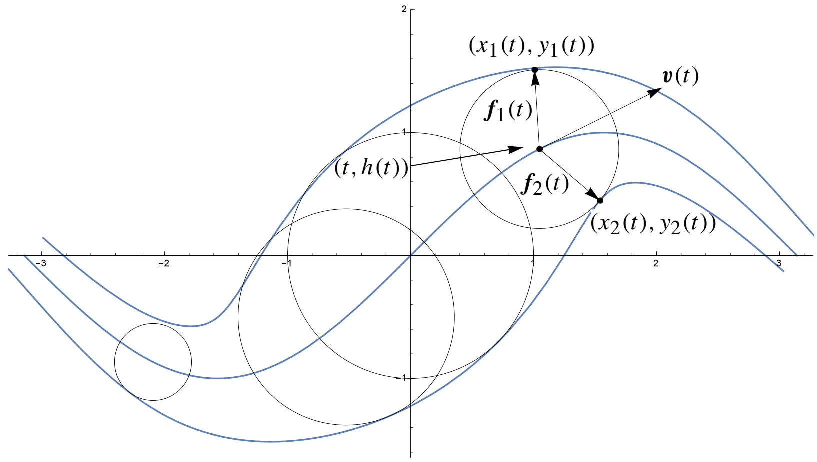

Figure 5. The envelope curves for the circles with centers on the curve $y = \sin x$ and radii given by $r(t) = 2/(2+t^2)$ are shown. The vectors $\boldsymbol{f}_i(t) = (x_i(t) - t,\ y_i(t) - h(t))$ make equal angles with the vector $\boldsymbol{v}(t) = (1, h'(t))$, as asserted by Lemma 3.1.

Lemma 3.1. *The angles between* $\boldsymbol{v}(t)$ *and the vectors* $\boldsymbol{f}_i(t)$, $i = 1, 2$, *are equal and lie on opposite sides of* $\boldsymbol{v}(t)$ *at* $(t, h(t))$ (*see* Figure 5). *Concomitantly*, *the cross products* $\boldsymbol{v}(t) \times \boldsymbol{f}_i(t)$ *have nonzero third components that are equal in magnitude and opposite in sign*, *with the sign being positive for* $i = 1$.

Proof. By direct computation, one obtains the following equations (note that the first is equivalent to (3)):

$$\begin{aligned} \boldsymbol{v}(t) \cdot \boldsymbol{f}_i(t) &= -r(t)r'(t), \\ \boldsymbol{v}(t) \times \boldsymbol{f}_i(t) &= (-1)^{i+1}\big(0, 0, \sqrt{r(t)^2(h'(t)^2 - r'(t)^2 + 1)}\big). \end{aligned} \tag{9}$$

Since $\boldsymbol{v}(t) \cdot \boldsymbol{f}_1(t) = \boldsymbol{v}(t) \cdot \boldsymbol{f}_2(t)$, we obtain the equality of the angles. Moreover, it is clear that the third components of the cross products $\boldsymbol{v}(t) \times \boldsymbol{f}_i(t)$, $i = 1, 2$, are equal in magnitude and opposite in sign, with the sign being positive for $i = 1$. □

We will say that the envelope curve $(x_i(t), y_i(t))$ lies *above* (resp. *below*) the curve $(t, h(t))$ if and only if the sign of the third component of $\boldsymbol{v}(t) \times \boldsymbol{f}_i(t)$ is positive (resp. negative) (see Figure 5).

4. Generating a focal set B from a midset M and a focal set A

Let $y = h(t)$ be a function satisfying $h''(t) > 0$ for all t on some open interval (a, b), so that the graph of h is concave up on (a, b). (The graph of h is our intended midset M.) Let A be a closed and convex set, and f the foot-point map taking each point $p \notin A$ to its unique foot point $f(p) = q \in A$. For points $p = (t, h(t))$ on the

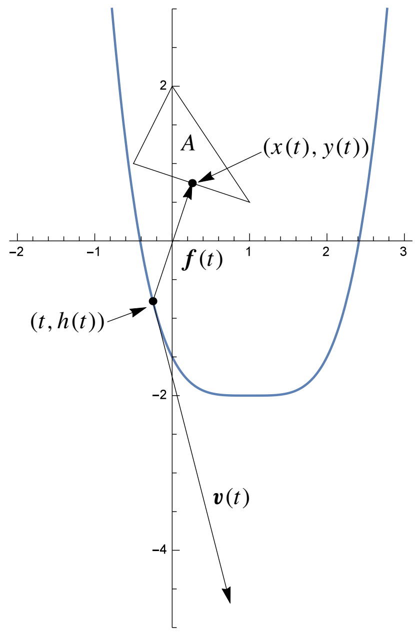

Figure 6. This picture illustrates the general situation we are considering: the closed and convex set A lies above a concave-up curve $y = h(t)$. The vectors $\boldsymbol{v}(t)$ and $\boldsymbol{f}(t)$ are shown.

graph of h, the points $f(t, h(t)) = (x(t), y(t))$ lie on a parametrized curve that we will call the *foot-point curve*.

Lemma 4.1. *The foot-point curve* $(x(t), y(t))$ *is continuous.*

Proof. Since the foot-point map f is continuous by Lemma 2.3, and the curve $(t, h(t))$ is continuous (in fact, twice differentiable) by hypothesis, we have that $(x(t), y(t))$ is the composition of continuous functions, and hence continuous. □

Let

$$\boldsymbol{f}(t) = (x(t) - t,\ y(t) - h(t),\ 0). \tag{10}$$

We assume that A lies above the graph of h in the sense that $\boldsymbol{v}(t) \times \boldsymbol{f}(t)$ is nonzero and points in the positive z-direction for all $t \in (a, b)$ (see Figure 6).

We define the radius function r as

$$r(t) = d\big((t, h(t)), (x(t), y(t))\big) = \|\boldsymbol{f}(t)\|. \tag{11}$$

Lemma 2.5 yields the following result.

Lemma 4.2. *The radius function* $r(t)$ *is* (*continuous and*) *differentiable.*

Proof. This is an immediate consequence of the fact that $r(t) = (d_A \circ h)(t)$ is a composition of differentiable functions. □

We now consider the envelope curves (6) for the family of circles with centers on the curve $y = h(t)$ and radii given by $r(t)$ for $t \in (a, b)$. Note that condition (7) holds in this case, since the rate at which $r(t)$ changes cannot possibly be greater than the speed with which the point $(t, h(t))$ moves, which is equivalent to (7). Our first main result is the following:

Proposition 4.3. *The first envelope curve* $(x_1(t), y_1(t))$ *is equal to the foot-point curve* $(x(t), y(t))$.

Proof. Assume for the moment that we know that $(x(t), y(t))$ is one of the two envelope curves. Then, since $\boldsymbol{v}(t) \times \boldsymbol{f}(t)$ has positive third component by hypothesis, Lemma 3.1 implies that $(x(t), y(t)) = (x_1(t), y_1(t))$, as desired. So we are reduced to showing that the foot-point curve $(x(t), y(t))$ satisfies conditions (2) and (3) defining the envelope curves.

It is clear that $(x(t), y(t))$ satisfies (2), since $(x(t), y(t))$ lies at distance $r(t)$ from $(t, h(t))$, by definition. We proceed to show that $(x(t), y(t))$ satisfies (3), which (as noted in the proof of Lemma 3.1) can be rearranged to read

$$r(t)r'(t) = -(1, h'(t)) \cdot (x(t) - t,\ y(t) - h(t)) = -\boldsymbol{v}(t) \cdot \boldsymbol{f}(t). \tag{12}$$

It is in fact straightforward to prove (12) at any point t at which the foot-point curve $(x(t), y(t))$ is differentiable. We simply differentiate the (differentiable, by Lemma 4.2) function $r(t)^2 = (x(t) - t)^2 + (y(t) - h(t))^2$ at t using the chain rule:

$$2r(t)r'(t) = 2(x(t) - t)(x'(t) - 1) + 2(y(t) - h(t))(y'(t) - h'(t)),$$

$$r(t)r'(t) = (x(t) - t,\ y(t) - h(t)) \cdot (x'(t), y'(t)) - (x(t) - t,\ y(t) - h(t)) \cdot (1, h'(t)).$$

We see that (3) will hold at t provided that the first term on the right-hand side of the last equation vanishes, but this follows immediately from the fact that the function

$$\boldsymbol{dd}(u) = d\big((t, h(t)), (x(u), y(u))\big)^2 = (x(u) - t)^2 + (y(u) - h(t))^2$$

has a global minimum at the point $u = t$, so that

$$\frac{d}{du}(\boldsymbol{dd})(t) = \big((x(t) - t,\ y(t) - h(t)) \cdot (x'(t), y'(t)\big) = 0. \tag{13}$$

The proof of the proposition is now complete for any case in which the foot-point curve $(x(t), y(t))$ is differentiable everywhere on its domain. (Such cases include A being a point or a disk.) Unfortunately, $(x(t), y(t))$ is not in general differentiable everywhere; however, we claim that it is always differentiable almost everywhere (that is, off of a set of measure 0). Indeed, we already know from Lemma 4.1 that $(x(t), y(t))$ is continuous, so that each component function is continuous. We will presently show that each component function has bounded variation on any closed interval $[c, d]$ in its domain, from which it follows that each component function is differentiable almost everywhere on $[c, d]$ (see, e.g., [Bressoud 2008, Theorem 7.4, p. 213]). The foregoing then implies that $(x(t), y(t))$ and the envelope curve $(x_1(t), y_1(t))$ agree almost everywhere, whence, since both of these curves are continuous, it follows that they are equal.

It remains to show that the functions $x(t)$ and $y(t)$ have bounded variation on any closed interval $[c, d]$ in the domain of h. The argument for $x(t)$ proceeds as follows; the argument for $y(t)$ is similar. By definition, we must show that there is a real number $B > 0$ such that, for every partition $P = \{t_0, t_1, \dots, t_n\}$ of $[c, d]$,

$$\sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| \le B.$$

However,

$$\begin{aligned}\sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| &\le \sum_{i=1}^{n} d\big((x(t_i), y(t_i)), (x(t_{i-1}), y(t_{i-1}))\big) \\ &\le \sum_{i=1}^{n} d\big((t_i, h(t_i)), (t_{i-1}, h(t_{i-1}))\big) \\ &\le \int_c^d \sqrt{1 + h'(t)^2} = \text{length of curve } (t, h(t)) \text{ on } [c, d].\end{aligned}$$

The first inequality is obvious, the second follows from Lemma 2.6, and the third is due to the fact that $(t, h(t))$ is concave up, so that every polygonal approximation to its length obtained from a partition is an underestimate. □

We now want to show that the envelope curve $(x_2(t), y_2(t))$ gives us a second focal set B such that the midset of A and B contains all the points $(t, h(t))$ for $t \in (a, b)$. To do this, we must show that the distance from $(t, h(t))$ to the curve B is equal to $r(t) = d\big((t, h(t)), (x_2(t), y_2(t))\big)$, or, in other words, that the point $(x_2(t), y_2(t))$ in B is a foot point for the point $(t, h(t))$. Proposition 4.3 tells us that $(x_1(t), y_1(t)) = (x(t), y(t))$ is the foot point of $(t, h(t))$ in A, which implies that the following inequality holds for all $u, t \in (a, b)$:

$$(x_1(u) - t)^2 + (y_1(u) - h(t))^2 - \big((x_1(t) - t)^2 + (y_1(t) - h(t))^2\big) \ge 0. \qquad (14)$$

Theorem 4.4. *Let B be the set of points of the envelope curve $(x_2(t), y_2(t))$. Then the point $(x_2(t), y_2(t))$ is a foot point of $(t, h(t))$ in B; consequently, $(t, h(t))$ is in the midset of A and B.*

Proof. We will show that the inequality obtained from (14) by replacing $x_1(t)$, $y_1(t)$ by $x_2(t)$, $y_2(t)$, respectively, holds. This will imply that the distance from $(t, h(t))$ to B is minimized at the point $(x_2(t), y_2(t))$, from which the theorem follows at

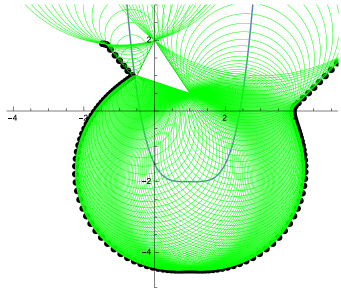

Figure 7. Another look at the example shown in Figure 6. The second envelope curve $(x_2(t), y_2(t))$ is indicated with large black dots. A subset of the circles forming the two envelope curves is shown; the result of Proposition 4.3 — that the first envelope curve $(x_1(t), y_1(t))$ is the same as the foot point curve $(x(t), y(t))$ — is also illustrated.

once. In fact, we will prove that

$$(x_2(u)-t)^2+(y_2(u)-h(t))^2-\big((x_2(t)-t)^2+(y_2(t)-h(t))^2\big) \\ \geq (x_1(u)-t)^2+(y_1(u)-h(t))^2-\big((x_1(t)-t)^2+(y_1(t)-h(t))^2\big)\geq 0.$$

Define $x_1(t)$, $x_2(t)$, $y_1(t)$, and $y_2(t)$ as in (6). Let

$$\boldsymbol{dd}_i(u)=(x_i(u)-t)^2+(y_i(u)-h(t))^2 \quad \text{for } i=1,2.$$

Using Mathematica to simplify $(\boldsymbol{dd}_2(u)-\boldsymbol{dd}_2(t))-(\boldsymbol{dd}_1(u)-\boldsymbol{dd}_1(t))$, we arrive at

$$-\frac{4\big(-h(t)+h(u)+(t-u)h'(u)\big)\sqrt{r(u)^2(1+h'(u)^2-r'(u)^2)}}{1+h'(u)^2}.$$

Recalling that the constraint (7) holds, the above Mathematica computation shows that the desired inequality will hold if and only if

$$-h(t)+h(u)+(t-u)h'(u)\leq 0.$$

Since this factor vanishes when $u=t$, we must show that it is nonpositive when $u\neq t$. However, this follows from our hypothesis that h is concave up ($h''>0$). Indeed, if $t<u$, then by the mean value theorem, there is a $v\in(t,u)$ such that

$$\frac{h(u)-h(t)}{u-t}=h'(v)<h'(u) \implies h'(u)(u-t)-h(u)+h(t)>0 \\ \implies h'(u)(t-u)+h(u)-h(t)<0,$$

as desired, and a similar proof applies if $u<t$. This completes the proof of the theorem. □

Figure 7 illustrates the theorem in the case previously shown in Figure 6.

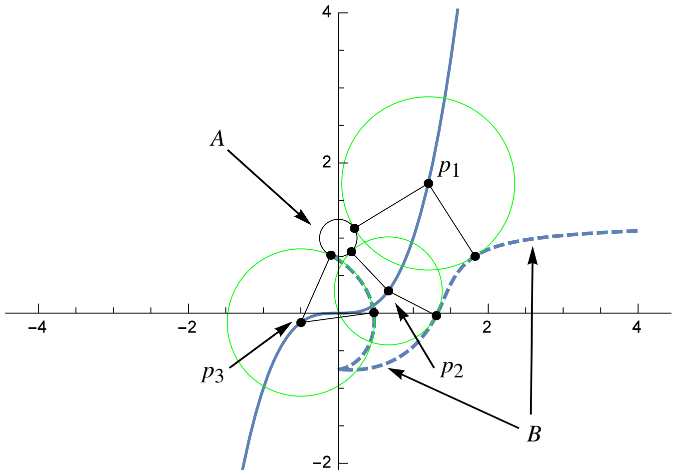

Figure 8. The first focal set A is the circle of radius $\frac{1}{4}$ centered at $(0, 1)$. The dashed curve is the second envelope curve B for the family of circles centered on the graph of $y = t^3$ and with radii $r(t)$ equal to the distance from (t, t^3) to A. The point p_1 appears to be equidistant from A and B, but this is clearly not the case for p_2 and p_3.

5. Envelopes of curves as focal sets

The investigation in this section was undertaken in response to the question of how to realize the graph of $y = h(t) = t^3$ as a midset. The idea of the preceding section doesn't work, as Figure 8 shows. Instead, we will explore the idea of using the envelope curves A and B defined by a family of circles of constant radius centered on the graph of h as the focal sets. Figure 9 shows the envelope curves corresponding to two different radii; it becomes apparent that for this idea to work, the radius cannot be chosen too large. (Note that if $r(t) = c$ is constant, then $r'(t) = 0$ and the constraint (7) is automatically satisfied.)

We now generalize to the case of a function $y = h(t)$ with at least a continuous third derivative defined on an open interval (a, b). We no longer assume h has constant concavity, as in the preceding section. We study the envelope curves (6) defined by the family of circles of constant radius $r(t) = c$ and centered on the graph of h. It is clear that under these conditions $x_i(t)$ and $y_i(t)$ have continuous second derivatives on (a, b).

Referring to Figure 9, it appears that one constraint to the envelope curves serving as focal sets is the presence of singularities. A parametrized curve $(x(t), y(t))$ with differentiable component functions has singularities only at points where $x'(t) = 0$ and $y'(t) = 0$. We proceed to find these points on the parametrized curves

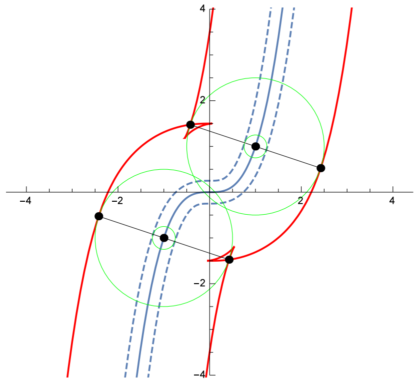

Figure 9. The envelope curves associated to the families of circles of constant radius $\frac{1}{4}$ and 1 and centered on the curve $y = t^3$. It is plausible that the dashed envelope curves have the central cubic as their midset, but this visibly cannot be the case for the solid envelope curves.

$(x_i(t), y_i(t))$; solving the system of equations $x_i'(t) = 0$ and $y_i'(t) = 0$ for c (using Mathematica), we obtain

$$c = \pm\frac{(1 + h'(t)^2)^{3/2}}{h''(t)}.$$

The keen observer will notice that the right-hand side is (up to sign) equal to the reciprocal of the curvature of the graph of h at $(t, h(t))$ (see, e.g., [Stewart 2012, Equation 11, p. 881]). To avoid singularities in the envelope curves, we should (if possible[1]) choose c such that

$$0 < c < \inf_{t\in(a,b)} \frac{(1 + h'(t)^2)^{3/2}}{|h''(t)|}. \tag{15}$$

We will refer to this upper bound for c (if it exists) as the *critical radius*; henceforth we will assume that h is such that the critical radius on (a, b) exists. Thus we have proved the following proposition.

Proposition 5.1. *If c is less than the critical radius, then the envelope curves $(x_i(t), y_i(t))$ will have no singularities.* □

For the proof of our main result, we need two more lemmas that also rely on c being less than the critical radius.

Lemma 5.2. *If c is less than the critical radius, then $x_i'(t) > 0$ for all t.*

[1]For example, the function $h(t) = \sin(1/t)$ does not have a critical radius on the interval $(0, 1)$, since the curvature at the extreme points $t_n = 1/(n\pi + \pi/2)$ is unbounded as $n \to \infty$.

Proof. Setting $r(t) = c$ (so $r'(t) = 0$), we find that

$$x_i'(t) = \frac{(-1)^i h''(t)\sqrt{c^2(h'(t)^2+1)} + h'(t)^4 + 2h'(t)^2 + 1}{(h'(t)^2+1)^2}.$$

The quantity will be positive so long as

$$|h''(t)|\sqrt{c^2(h'(t)^2+1)} \le (h'(t)^2+1)^2,$$

but this holds provided that

$$c \le \frac{(h'(t)^2+1)^{3/2}}{|h''(t)|},$$

which certainly holds for all t if c is less than the critical radius. □

Lemma 5.3. *For a fixed t, let $\boldsymbol{d}_i$ denote the function*

$$\boldsymbol{d}_i(u) = d\big((t, h(t)), (x_i(u), y_i(u))\big).$$

If c is less than the critical radius, then $\boldsymbol{d}_i(t) = c$ is a local minimum value of $\boldsymbol{d}_i$; indeed, $\boldsymbol{d}_i(u) > \boldsymbol{d}_i(t) = c$ for all u in some deleted neighborhood of t.

Proof. With $r(t) = c$, we again define the function

$$\boldsymbol{dd}_i(u) := (x_i(u) - t)^2 + (y_i(u) - h(t))^2,$$

that is, $\boldsymbol{dd}_i(u) = \boldsymbol{d}_i^2(u)$. Our hypothesis on h implies that $\boldsymbol{dd}_i$ has a continuous second derivative. We find that using Mathematica to simplify $\boldsymbol{dd}_i'(t)$ returns 0 and simplifying $\boldsymbol{dd}_i''(t)$ returns

$$\frac{2\big(1 + 2h'(t)^2 + h'(t)^4 + (-1)^i\sqrt{c^2(1+h'(t)^2)}\, h''(t)\big)}{1+h'(t)^2}.$$

An argument similar to that used in the proof of the preceding lemma shows that this quantity is positive if c is less than the critical radius, whence the first assertion follows from the second derivative test. The second assertion now follows from the fact that $\boldsymbol{dd}_i''$ is continuous and positive at $u = t$, and so $\boldsymbol{dd}_i$ is concave up on a neighborhood of t. □

With these lemmas in hand we now present the main theorem of this section.

Theorem 5.4. *Given any function $h(t)$ on an interval (a, b) with a continuous third derivative and having a critical radius, the minimum distance from the point $(t, h(t))$ to either envelope curve (for the family of circles of constant radius c less than the critical radius) is equal to c, so that $(x_i(t), y_i(t))$ is a foot point for $(t, h(t))$ on the i-th envelope curve.*

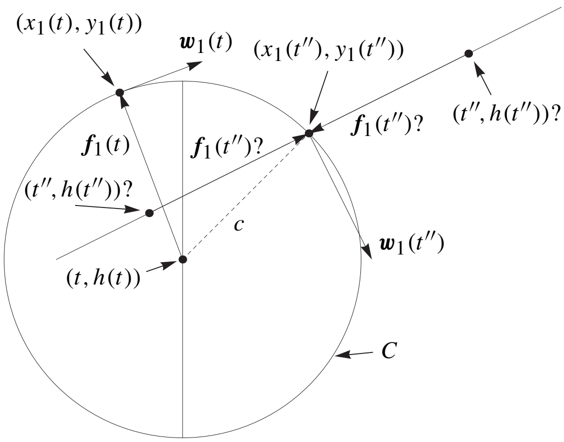

Figure 10. This figure illustrates the situation that arises in the proof of Theorem 5.4.

Proof. We begin by defining the vector-valued functions

$$\boldsymbol{w}_i(t) = (x_i'(t),\ y_i'(t), 0) \quad \text{for } i = 1, 2. \tag{16}$$

Machine computation yields that

$$\begin{aligned}
&\boldsymbol{w}_i(t) \times \boldsymbol{f}_i(t) \\
&= \left(0,\ 0,\ (-1)^{i+1} \frac{(-1)^i c^2 h''(t) + h'(t)^2 \sqrt{c^2(h'(t)^2+1)} + \sqrt{c^2(h'(t)^2+1)}}{h'(t)^2+1}\right) \\
&= \left(0,\ 0,\ (-1)^{i+1} c \frac{(-1)^i c h''(t) + (1+h'(t)^2)^{3/2}}{h'(t)^2+1}\right),
\end{aligned} \tag{17}$$

where $\boldsymbol{f}_i(t)$ is defined in (8). Arguing as in the preceding lemmas, we see that whenever c is less than the critical radius, the sign $(-1)^{i+1}$ of the third component of these cross products is an invariant of the envelope curve (x_i, y_i) for $i = 1, 2$. We claim that $c = \|\boldsymbol{f}_i(t)\|$ is the global minimum distance from $(t, h(t))$ to the i-th envelope curve. We will prove this for $i = 1$, and invite the reader to check that a similar proof applies when $i = 2$.

We first observe that $\boldsymbol{w}_1(t)$ is perpendicular to $\boldsymbol{f}_1(t)$, since, by Lemma 5.3, the function $\boldsymbol{dd}_1(u)$ has a local minimum at $u = t$ (as in (13)). Hence, since $x_1'(t)$ is always positive, by Lemma 5.2, the vector $\boldsymbol{w}_1(t)$ is never parallel to the y-axis; therefore, the point $(x_1(t), y_1(t))$ lies in either the upper or lower (open) semicircle of the circle C of radius c centered at $(t, h(t))$. From (17) we learn that $\boldsymbol{w}_1(t) \times \boldsymbol{f}_1(t)$ has positive third component, whence $(x_1(t), y_1(t))$ must lie in the upper semicircle (see Figure 10). (Put more simply, $(x_1(t), y_1(t))$ is the upper envelope curve, and $(x_2(t), y_2(t))$ is the lower.)

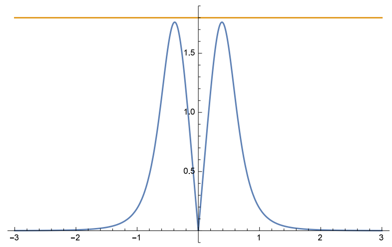

Figure 11. The graph of the curvature function $\kappa(t)$ for the function $h(t) = t^3$ is shown, along with the horizontal line $y = \frac{9}{5}$. This demonstrates that the critical radius on $(-\infty, \infty)$ exists and is slightly larger than $\frac{5}{9}$.

Arguing by contradiction, suppose that there is a point $(x_1(t'), y_1(t'))$ such that its distance from the point $(t, h(t))$ is less than c; we suppose that $t < t'$ and leave to the reader to check that a similar argument applies when $t > t'$. Since $x_1'(t)$ is always positive, we know that $(x_1(t'), y_1(t'))$ lies to the right of $(x_1(t), y_1(t))$ (indeed, this is so for any parameter value greater than t). Furthermore, by Lemma 5.3, we know that $\boldsymbol{d}_1(u) > c$ for some $u \in (t, t')$. We let

$$S = \{t^* \in (t, t') \mid \boldsymbol{d}_1(t^*) \geq c\},$$

and let $t'' = \text{lub}(S)$. By continuity, we know that $\boldsymbol{d}_1(t'') = c$, and we also know, since $\boldsymbol{d}_1(t^*) < c$ for $t^* \in (t'', t')$, that the tangent vector $\boldsymbol{w}_1(t'')$ must either be tangent to C or must enter C, as shown in Figure 10. The corresponding point $(t'', h(t''))$ must lie on the line perpendicular to the vector $\boldsymbol{w}_1(t'')$ at $(x_1(t''), y_1(t''))$, and at distance c from this point. There are consequently two possible positions for $(t'', h(t''))$, to the left or to the right of $(x_1(t''), y_1(t''))$. However, it is easy to verify that, as Figure 10 suggests, the left possibility implies that $(t'', h(t''))$ lies to the left of $(t, h(t))$ (or, more precisely, that $t'' \leq t$ holds), which contradicts $t < t''$, and that the right possibility implies that $\boldsymbol{w}_1(t'') \times \boldsymbol{f}_1(t'')$ has negative third component, which contradicts our earlier observation that this sign is always positive for (x_1, y_1). Since both possibilities lead to contradictions, we see that no point $(x_1(t'), y_1(t'))$ can lie closer than c to $(t, h(t))$, and we are done. □

Corollary 5.5. *Under the hypotheses of the theorem, the points $(t, h(t))$ all lie on the midset determined by the two envelope curves taken as focal sets.* □

Remark 5.6. For $h(t) = t^3$, the curvature function

$$\kappa(t) = \frac{|6t|}{(1 + (3t^2)^2)^{3/2}}$$

has the graph shown in Figure 11. We see that $\frac{9}{5}$ is a slight overestimate of the maximum curvature on $(-\infty, \infty)$, which implies that $\frac{5}{9}$ is a slight underestimate of the critical radius. Since $\frac{1}{4} < \frac{5}{9}$, we now have proved that the envelope curves associated to the family of circles centered on the graph of $h(t) = t^3$ and having constant radius $\frac{1}{4}$ are indeed a pair of focal sets for which the graph of h is their midset, as Figure 9 suggests.

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