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Flapan, Naimi and Pommersheim (2001) showed that every spatial embedding of K_{10}, the complete graph on ten vertices, contains a nonsplit three-component link; that is, K_{10} is *intrinsically triple-linked* in $\mathbb{R}^3$. The work of Bowlin and Foisy (2004) and Flapan, Foisy, Naimi, and Pommersheim (2001) extended the list of known intrinsically triple-linked graphs in $\mathbb{R}^3$ to include several other families of graphs. In this paper, we will show that while some of these graphs can be embedded 3-linklessly in $\mathbb{R}P^3$, the graph K_{10} is intrinsically triple-linked in $\mathbb{R}P^3$.

1. Introduction

There is a classic theory of knots and links in Euclidean 3-space (or the 3-sphere), and, as Manturov [2004] pointed out, there is a sympathetic theory of knots and links in $\mathbb{R}P^3$. Drobotukhina [1990] developed an analog of the Jones polynomial for the case of oriented links in $\mathbb{R}P^3$, and Mroczkowski [2003] described a method to unknot knots and links in $\mathbb{R}P^3$ through an analog of classical knot and link diagrams for knots in $\mathbb{R}^3$. Flapan, Howards, Lawrence, and Mellor [Flapan et al. 2006] investigated intrinsic linking and knotting in arbitrary 3-manifolds. Here, following Bustamente et al. [2009], we use a weaker notion of unlink than was used in [Flapan et al. 2006], and we examine the intrinsic linking properties of graphs embedded in $\mathbb{R}P^3$. In particular, we will examine graphs that contain a three-component nonsplit link in every embedding into $\mathbb{R}P^3$.

Real projective 3-space $\mathbb{R}P^3$ can be obtained from the 3-ball D^3 by identifying opposite points of its boundary; hence, a link in $\mathbb{R}P^3$ consists of a union of arcs and loops so that the endpoints of any arc lie on antipodal boundary points of the 3-ball. We may use ambient isotopy to move all arcs so that their endpoints lie on a fixed great circle, the "equator" of the ball. Therefore, a link may be represented

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in $\mathbb{R}P^2$ by its projection onto a 2-disk, D^2, whose boundary is the equator, with antipodal points identified.

Projective space has a nontrivial first homology group, $H_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$. Let g, the cycle consisting of a line in D^3 running between the north and south poles, be the generator of this group. Using crossing changes and ambient isotopy on an $\mathbb{R}P^2$ projection of a knot, Mroczkowski [2003] showed that every knot in $\mathbb{R}P^3$ can be transformed into either the trivial cycle or g. Thus, there are two nonequivalent unknots in $\mathbb{R}P^3$. Cycles that can be unknotted into a cycle homologous to g will be referred to as *1-homologous cycles*. Cycles that can be unknotted into a trivial cycle will be referred to as *0-homologous cycles*.

Following [Bustamante et al. 2009], we say a link in $\mathbb{R}P^3$ is *splittable* if one component can be contained within a 3-ball embedded in $\mathbb{R}P^3$, while the other component lies in the complement of the 3-ball. Otherwise, a link in $\mathbb{R}P^3$ is said to be *nonsplit*. A nonsplit link may be formed in one of three ways in $\mathbb{R}P^3$: by two 0-homologous cycles, by a 0-homologous cycle and a 1-homologous cycle, and by two 1-homologous cycles. Moreover, since a 1-homologous cycle cannot be contained within a ball embedded in $\mathbb{R}P^3$, two disjoint 1-homologous cycles will always form a nonsplit link. In this paper, we will refer to nonsplit linked cycles as *linked cycles* and to an embedded graph as *linked* if it contains a nonsplit link.

A graph H is a *minor* of G if H can be obtained from G through a series of vertex removals, edge removals, or edge contractions. A graph G is said to be *minor-minimal* with respect to property P if G has property P, but no minor of G has property P. The complete set of minor-minimal intrinsically linked graphs in $\mathbb{R}^3$ is given by the Petersen family graphs, including K_6 and the graphs obtained from K_6 by $\Delta-Y$ and $Y-\Delta$ exchanges [Conway and Gordon 1983; Robertson et al. 1995; Sachs 1984]. However, all Petersen family graphs except $K_{4,4}-\{e\}$, where e is an edge, embed linklessly in $\mathbb{R}P^3$, as shown in [Bustamante et al. 2009], a paper which also exhibits 597 graphs that are minor-minimal intrinsically linked in $\mathbb{R}P^3$. The complete set of minor-minimal intrinsically linked graphs in $\mathbb{R}P^3$ is finite [Robertson and Seymour 2004], and remains to be found.

A *nonsplit triple link* is a nonsplit link of three components, which, in an abuse of language, will be referred to as a *triple link* in this paper. An embedding of a graph is *triple-linked* if it contains a nonsplit link of three components, and a graph is *intrinsically triple-linked in* X, a topological space, if every embedding of the graph into X contains a nonsplit triple link.

Conway and Gordon [1983] and Sachs [1983; 1984] proved that K_6 is intrinsically linked in $\mathbb{R}^3$. In contrast, K_6 can be linklessly embedded in $\mathbb{R}P^3$ (see Figure 3). Bustamante et al. [2009] showed that 7 is the smallest n for which K_n is intrinsically linked in $\mathbb{R}P^3$. Flapan, Naimi, and Pommersheim [Flapan et al. 2001a] proved 10 is the smallest n for which K_n is intrinsically triple-linked in $\mathbb{R}^3$. In Section 3, we show that 10 is also the smallest n for which K_n is intrinsically triple-linked in $\mathbb{R}P^3$. It remains to show whether K_{10} is minor-minimal with respect to triple-linking in $\mathbb{R}P^3$.

In Section 4, we show that two intrinsically triple-linked graphs in $\mathbb{R}^3$ can be embedded 3-linklessly in $\mathbb{R}P^3$ and exhibit two other minor-minimal intrinsically triple-linked graphs in $\mathbb{R}P^3$. A complete set of minor-minimal intrinsically triple-linked graphs in both $\mathbb{R}^3$ and $\mathbb{R}P^3$ remains to be found. Such sets are finite due to the result in [Robertson and Seymour 2004].

2. Definitions and preliminary lemmas

We begin with some elementary definitions and notation. A *graph*, $G = (V, E)$, is a set of *vertices*, $V(G)$, and *edges*, $E(G)$, where an edge is an unordered pair (v_1, v_2) with $v_1, v_2 \in V(G)$. If G is a graph with $v_1, \ldots, v_n \in V(G)$ and $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1) \in E(G)$, with $v_i \neq v_j$ for all $i \neq j$, then the sequences of vertices $v_1, \ldots, v_n$ and edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)$ is an *n-cycle* of G, denoted $(v_1, \ldots, v_n)$. In this paper, we also refer to the image of a cycle under an embedding as an n-cycle.

If G is a graph and $v_1, \ldots, v_n \in V(G)$, define the *induced subgraph*, $G[v_1, \ldots, v_n]$, to be the subgraph of G with

$$V(G[v_1, \ldots, v_n]) = \{v_1, \ldots, v_n\},$$
$$E(G[v_1, \ldots, v_n]) = \big\{(v_i, v_j) \in E(G) \mid v_i, v_j \in \{v_1, \ldots, v_n\}\big\}.$$

The classical notion of linking number extends to links embedded in $\mathbb{R}P^3$. Suppose L and K are two loops embedded in $\mathbb{R}P^3$; orient L and K. At each crossing, assign $+1$ or -1, as drawn in Figure 1. Then the *mod* 2 *linking number of L and K*, $\mathrm{lk}(L, K)$, is the sum of the numbers, $+1$ or -1, at each crossing in the embedding of L and K divided by 2, taken modulo 2. In $\mathbb{R}P^3$, there are five generalized Reidemeister moves [Manturov 2004], which are drawn in Figure 2. As in $\mathbb{R}^3$, one can use Reidemeister moves to justify that mod 2 linking number is well-defined in $\mathbb{R}P^3$. In particular, the mod 2 linking number of a splittable two-component link is 0. However, in $\mathbb{R}P^3$, the mod 2 linking number need not be an integer; for example, two disjoint 1-homologous cycles can have mod 2 linking number $\pm\frac{1}{2}$.

The following lemmas provide us information about carefully chosen induced subgraphs of the graphs we study.

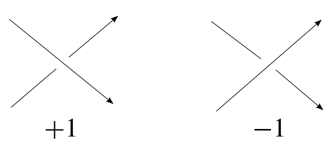

Figure 1. Link crossings.

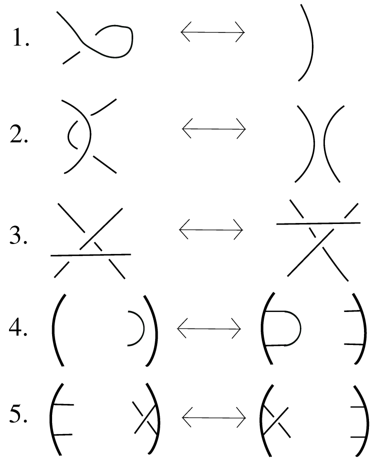

Figure 2. Generalized Reidemeister moves in $\mathbb{R}P^3$.

Lemma 1 [Bustamante et al. 2009]. *The graphs obtained by removing two edges from K_7 and removing one edge from $K_{4,4}$ are intrinsically linked in $\mathbb{R}P^3$.*

Lemma 2 [Bustamante et al. 2009]. *Given a linkless embedding of K_6 in $\mathbb{R}P^3$, no K_4 subgraph can have all 0-homologous cycles.*

In addition, we use the following elementary observation.

Lemma 3. *For every embedding into $\mathbb{R}P^3$, the graph K_4 has an even number of 1-homologous 3-cycles.*

The next two lemmas were shown true in $\mathbb{R}^3$ by [Flapan et al. 2001a] and [Bowlin and Foisy 2004], respectively. In each case, the proof holds analogously in $\mathbb{R}P^3$.

Lemma 4. *Let G be a graph embedded in $\mathbb{R}P^3$ that contains cycles C_1, C_2, C_3 and C_4. Suppose C_1 and C_4 are disjoint from each other and from C_2 and C_3 and suppose $C_2 \cap C_3$ is a simple path. If* $\mathrm{lk}(C_1, C_2) \neq 0$ *and* $\mathrm{lk}(C_3, C_4) \neq 0$, *then G contains a nonsplit three-component link.*

Lemma 5. *In an embedded graph with mutually disjoint simple closed curves, C_1, C_2, C_3, and C_4, and two disjoint paths x_1 and x_2 such that x_1 and x_2 begin in C_2 and end in C_3, if* $\mathrm{lk}(C_1, C_2) \neq 0$ *and* $\mathrm{lk}(C_3, C_4) \neq 0$, *then the embedded graph contains a nonsplit three-component link.*

3. Intrinsically triple-linked complete graphs on *n* vertices

The proposition below, that K_{11} is intrinsically triple-linked in $\mathbb{R}P^3$, is not the main result of this paper. In fact, our main result, that K_{10} is intrinsically triple-linked in $\mathbb{R}P^3$, implies this proposition, by a result of [Nešetřil and Thomas 1985]. However, the proof is included because it is (relatively) concise and follows from examining four carefully chosen subgraphs of K_{11} and applying Lemmas 4 and 5.

Proposition 6. *The graph K_{11} is intrinsically triple-linked in $\mathbb{R}P^3$.*

Proof. Let G be a complete graph isomorphic to K_{11} with vertex set $\{1, 2, \dots, 11\}$. Embed G in $\mathbb{R}P^3$.

Since K_7 is intrinsically linked in $\mathbb{R}P^3$, the graph $G[1, 2, 3, 4, 5, 6, 7] \cong K_7$ contains a pair of linked cycles. Without loss of generality, suppose the linked cycles are $C_1 = (1, 2, 3)$ and $C_2' = (4, 5, 6, 7)$. Homologically, the cycle $(4, 5, 6, 7)$ is the sum of the cycles $(4, 5, 6)$ and $(4, 6, 7)$. Thus,

$$\mathrm{lk}\big((1, 2, 3), (4, 5, 6, 7)\big) = \mathrm{lk}\big((1, 2, 3), (4, 5, 6)\big) + \mathrm{lk}\big((1, 2, 3), (4, 6, 7)\big).$$

Since the numbers on the right-hand side cannot both equal 0, without loss of generality, $C_1 = (1, 2, 3)$ links with $C_2 = (4, 5, 6)$.

Again, since K_7 is intrinsically linked in $\mathbb{R}P^3$, it follows that the subgraph $G[5, 6, 7, 8, 9, 10, 11] \cong K_7$ contains a pair of linked cycles. In the manner described above, this pair of cycles may be reduced to two linked 3-cycles. If it is not the case that one cycle contains $\{5\}$ and one cycle contains $\{6\}$, then Lemma 4 applies, and G is triple-linked. To handle the other case, suppose, without loss of generality, that $C_3 = (5, 7, 9)$ and $C_4 = (6, 8, 10)$ are the pair of linked cycles in $G[5, 6, 7, 8, 9, 10, 11]$.

To obtain two collections of disjoint 1-homologous cycles, consider two subgraphs isomorphic to K_6. First, if $G[1, 2, 3, 4, 6, 11] \cong K_6$ contains a pair of linked cycles, then one cycle shares vertex $\{6\}$ with C_4 and both are disjoint from C_3, so Lemma 4 applies and G is triple-linked. Otherwise, by Lemma 2, the set

$$A = \{(1, 2, 3), (1, 2, 11), (1, 3, 11), (2, 3, 11)\}$$

contains a 1-homologous cycle, C_5.

Similarly, if $G[6, 7, 8, 9, 10, 11] \cong K_6$ contains a pair of linked cycles, then one cycle shares vertex $\{6\}$ with C_2 and both are disjoint from C_1. So, Lemma 4 applies and G is triple-linked. Otherwise, by Lemma 2, the set

$$B = \{(7, 8, 9), (7, 8, 10), (7, 9, 10), (8, 9, 10)\}$$

contains a 1-homologous cycle, C_6.

Since $A \cap B = \varnothing$, the cycles $C_5 \in A$ and $C_6 \in B$ are disjoint and 1-homologous and hence are linked. So, C_2 and C_6 are disjoint from each other and from C_1 and C_5. In the case that $C_1 = C_5$, the cycles C_1, C_2, and C_6 form a triple link. Otherwise, $C_1 \cap C_5$ is a simple path, so G contains a triple link by Lemma 4. $\square$

To prove that K_{10} is intrinsically triple-linked in $\mathbb{R}P^3$, we first describe how its subgraphs isomorphic to K_6 must be embedded.

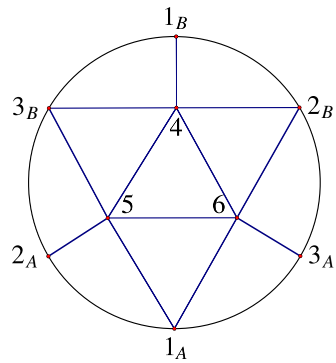

Figure 3. A projection of a linkless embedding of K_6 in $\mathbb{R}P^3$.

Proposition 7. *If G is isomorphic to K_6 and embedded in $\mathbb{R}P^3$ and G contains two disjoint* 0*-homologous cycles, then G contains a nonsplit link.*

Proof. Let G be isomorphic to K_6 and suppose G is embedded so that it contains two disjoint 0-homologous cycles and no nonsplit link. Without loss of generality, let $(1, 2, 3)$ and $(4, 5, 6)$ be 0-homologous cycles in G. Consider $G[1, 2, 3, 4]$. Since G is not linked, by Lemmas 2 and 3, the subgraph $G[1, 2, 3, 4]$ contains two 1-homologous cycles. Without loss of generality, let $(1, 2, 4)$ and $(1, 3, 4)$ be 1-homologous cycles.

Similarly, $G[2, 4, 5, 6]$ contains two 1-homologous cycles. The cycle $(4, 5, 6)$ is 0-homologous by assumption and since $(2, 5, 6)$ is disjoint from $(1, 3, 4)$, which is 1-homologous, $(2, 5, 6)$ is 0-homologous since G is assumed to have no nonsplit link. Thus, $(2, 4, 5)$ and $(2, 4, 6)$ are 1-homologous cycles.

In addition, $G[1, 2, 3, 6]$ contains two 1-homologous cycles. Since $(1, 2, 3)$ is 0-homologous by assumption and $(1, 3, 6)$ is disjoint from $(2, 4, 5)$, which is 1-homologous, $(1, 2, 6)$ and $(2, 3, 6)$ are 1-homologous.

Finally, $G[1, 3, 5, 6]$ contains two 1-homologous cycles. But, $(2, 4, 6)$, $(2, 4, 5)$, and $(1, 2, 4)$ are 1-homologous and disjoint from $(1, 3, 5)$, $(1, 3, 6)$, and $(3, 5, 6)$ respectively, a contradiction, since $G[1, 3, 5, 6]$ must contain two 1-homologous cycles. □

Proposition 8. *Up to ambient isotopy and crossing changes, Figure 3 describes the only way to linklessly embed K_6 in $\mathbb{R}P^3$.*

Proof. Let G be a complete graph on vertex set $\{1, 2, 3, 4, 5, 6\}$. Embed G in $\mathbb{R}P^3$ linklessly. The graph G contains a 0-homologous 3-cycle, since otherwise G contains two disjoint 1-homologous cycles and is linked. Without loss of generality, let $(4, 5, 6)$ be a 0-homologous 3-cycle. Proposition 7 implies that the cycle $(1, 2, 3)$ is 1-homologous as it is disjoint from $(4, 5, 6)$.

Mroczkowski [2003] showed that every cycle can be made, via crossing changes and ambient isotopy, into an unknotted 0-cycle or the 1-homologous cycle g as explained in the Introduction. Apply crossing changes and ambient isotopy so that the embedding has a projection with vertices as drawn in Figure 3. A priori, the edges between vertices $\{1, 2, 3\}$ and $\{4, 5, 6\}$ may be more complicated than as drawn in the figure.

Vertices $\{1, 2, 3\}$ and the 3-cycle $(1, 2, 3)$ lie on the boundary. In the projection, we label the pair of antipodal identified vertices by $\{v_A, v_B\}$ for $v \in \{1, 2, 3\}$.

Consider the edge E between 1 and 4. Together with the path $(1_B, 4)$ pictured in Figure 3, it forms either a 0-homologous or a 1-homologous cycle. If the cycle formed is 0-homologous, then by Mroczkowski's result, $E \cup (1_B, 4)$ can be made into the unknot by crossing changes, and then deformed so that E is within a small neighborhood of the path $(1_B, 4)$. That is, the cycle does not cross the boundary of D^2. If $E \cup (1_B, 4)$ forms a 1-homologous cycle, then E and the path formed by connecting 4 to 1_A by a straight line segment form a 0-homologous cycle. By similar reasoning, the edge E can be deformed, by crossing changes and ambient isotopy, to be within a small neighborhood of $(1_A, 4)$; that is to say, it does not cross the boundary of D^2. By similar reasoning, all edges between vertices $\{1, 2, 3\}$ and $\{4, 5, 6\}$ may be drawn in the projection onto $\mathbb{R}P^2$ without crossing the boundary.

We now describe how vertices $\{1, 2, 3\}$ connect to vertices $\{4, 5, 6\}$. We use that G does not contain two disjoint 1-homologous cycles or a 0-homologous K_4 by Lemma 2.

Let $v \in \{1, 2, 3\}$. Then v connects to one of $\{4, 5, 6\}$ from v_A and connects to one of $\{4, 5, 6\}$ from v_B; otherwise, G has a 0-homologous K_4 subgraph. Without loss of generality, suppose that $\{2_A\}$ connects to $\{5\}$ and $\{2_B\}$ connects to $\{4\}$ and $\{6\}$.

If $\{1_A\}$ or $\{1_B\}$ connects to both $\{4\}$ and $\{6\}$, then $G[1, 2, 4, 6]$ is a 0-homologous K_4. Thus, without loss of generality, let $\{1_B\}$ connect to $\{4\}$ and $\{1_A\}$ connect to $\{6\}$.

Vertex $\{1_A\}$ connects to $\{5\}$ since otherwise, any arrangement of edges connecting vertex $\{3\}$ to vertices $\{4, 5, 6\}$ induces either two disjoint 1-homologous cycles or a 0-homologous K_4 subgraph, as shown in the table below.

vertices $\{3_A\}$ connects to	vertices $\{3_B\}$ connects to	1-homologous cycles or 0-homologous K_4
$\{4\}$	$\{5\}, \{6\}$	$(1, 3, 6), (2, 4, 5)$
$\{5\}$	$\{4\}, \{6\}$	$G[2, 3, 4, 6]$
$\{6\}$	$\{4\}, \{5\}$	$G[1, 3, 4, 5]$
$\{4\}, \{5\}$	$\{6\}$	$(1, 3, 6), (2, 4, 5)$
$\{4\}, \{6\}$	$\{5\}$	$G[2, 3, 4, 6]$
$\{5\}, \{6\}$	$\{4\}$	$(1, 2, 5), (3, 4, 6)$

Finally, the following table shows that vertex $\{3_A\}$ connects to $\{6\}$ and vertex $\{3_B\}$ connects to 4 and 5. Indeed, all other arrangements lead to either two disjoint 1-homologous cycles or a 0-homologous K_4 subgraph.

vertices $\{3_A\}$ connects to	vertices $\{3_B\}$ connects to	1-homologous cycles or 0-homologous K_4
$\{4\}, \{5\}$	$\{6\}$	$(1, 3, 6), (2, 4, 5)$
$\{4\}, \{6\}$	$\{5\}$	$G[2, 3, 4, 6]$
$\{5\}, \{6\}$	$\{4\}$	$G[1, 3, 5, 6]$
$\{4\}$	$\{5\}, \{6\}$	$G[1, 3, 4, 6]$
$\{5\}$	$\{4\}, \{6\}$	$G[2, 3, 4, 6]$

Thus, up to crossing changes and ambient isotopy, Figure 3 depicts the only way K_6 may be linklessly embedded in $\mathbb{R}P^3$. □

Introduced by Harary [1953], *signed graphs* are graphs with each edge assigned a $+$ or a $-$ sign, and constitute the final tool in our proof that K_{10} is intrinsically triple-linked in $\mathbb{R}P^3$. An embedding of a graph G into $\mathbb{R}P^3$ induces a signed graph as follows: deform the embedding to that no vertices touch the bounding sphere in the model of $\mathbb{R}P^3$ with $\partial(D^3) \cong S^2$ and so that all intersections of edges with the bounding sphere are transverse. Define an edge of G to be a $^+$edge if the edge intersects the boundary an even number of times and to be a $^-$edge if the edge intersects the boundary an odd number of times. An example is drawn in Figure 4. Note that a cycle with an odd number of $^-$edges is 1-homologous.

Two embeddings G_1 and G_2 of a graph G are *crossing-change equivalent* if G_1 can be obtained from G_2 by crossing changes and ambient isotopy. By Proposition 8, a linkless K_6 embedded in $\mathbb{R}P^3$ is crossing-change equivalent to the embedding drawn in Figure 4. That is, if G is a signed graph isomorphic to K_6 with vertex set $\{1, 2, 3, 4, 5, 6\}$, then G is crossing-change equivalent to a signed graph with $^-$edge set $S = \{(1, 2), (1, 3), (2, 3), (1, 4), (2, 5), (3, 6)\}$ and $^+$edge set $E(G) \backslash S$.

Our next result shows that if G is a graph isomorphic to K_{10}, then G is intrinsically triple-linked in $\mathbb{R}P^3$. We first sketch an outline. Using results of [Flapan et al. 2001a; Bowlin and Foisy 2004], we show that a 3-linkless embedding of G, if such an

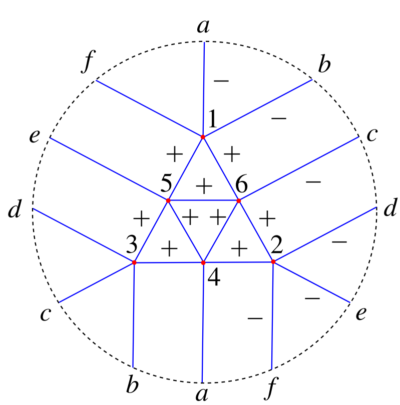

Figure 4. A signed linkless embedding of K_6 in $\mathbb{R}P^3$.

embedding exists, must contain a linkless K_6 subgraph. We prove the remaining four vertices must induce a 0-homologous K_4 subgraph or the embedded graph contains a nonsplit triple link. Finally, we determine the signs of the edges connecting the K_6 subgraph to the K_4 subgraph, eventually determining that any possible sign assignment results in a triple link. Thus, no 3-linkless embedding of G can exist.

Theorem 9. *The graph K_{10} is intrinsically triple-linked in $\mathbb{R}P^3$.*

Proof. Let G be a graph isomorphic to K_{10} with vertex set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Embed G in $\mathbb{R}P^3$ as a signed graph and assume, toward a contradiction, that G is 3-linkless.

If every subgraph of G isomorphic to K_6 is linked, then the proof in [Flapan et al. 2001a] that K_{10} is intrinsically linked in $\mathbb{R}^3$ nearly holds in $\mathbb{R}P^3$. However, at the end of their proof, they use that $K_{3,3,1}$ is intrinsically linked in $\mathbb{R}^3$, but this graph embeds linklessly in $\mathbb{R}P^3$. Bowlin and Foisy [2004] modified the proof in [Flapan et al. 2001a] to only use the fact that K_6 is intrinsically linked in $\mathbb{R}^3$. Thus, in the case that every subgraph of G isomorphic to K_6 is linked, G contains a triple link. So, we may assume that there exists a linkless K_6 subgraph of G. Without loss of generality, suppose that $G[1, 2, 3, 4, 5, 6]$ is linkless. By Proposition 8, the subgraph $G[1, 2, 3, 4, 5, 6]$ has an embedding that is crossing-change equivalent to that drawn in Figure 4. In particular, since crossing changes do not change the homology of cycles, we may assume $(1, 2, 3)$ is 1-homologous.

Claim. *The embedded induced subgraph $G[7, 8, 9, 10]$ is 0-homologous.*

Proof. Suppose $G[7, 8, 9, 10]$ has a 1-homologous cycle. Without loss of generality, suppose $(7, 8, 9)$ is 1-homologous. If $G[4, 5, 6, 10]$ is not 0-homologous, then two of $(4, 5, 10)$, $(4, 6, 10)$, and $(5, 6, 10)$ are 1-homologous by Lemma 3, since we have assumed $(4, 5, 6)$ is 0-homologous. Then $(1, 2, 3)$, $(7, 8, 9)$, and a cycle from $G[4, 5, 6, 10]$ comprise three disjoint 1-homologous cycles, so G is triple-linked. Thus, $G[4, 5, 6, 10]$ is 0-homologous and so $G[1, 2, 4, 5, 6, 10]$ has a pair of linked cycles by Lemma 2. Since $(7, 8, 9)$ is 1-homologous, and $(7, 8, 9)$ is disjoint from all the 1-homologous cycles in the second column of Table 1, Lemma 4 applies and G has a triple link. Thus, $G[7, 8, 9, 10]$ is 0-homologous. □

Since ambient isotopy and crossing changes do not change the homology of cycles, we may modify the embedding of G so that all edges in $G[7, 8, 9, 10]$ are $^+$edges and the edges in $G[1, 2, 3, 4, 5, 6]$ are $^+$edges and $^-$edges as defined in Figure 4. Many of the remaining arguments rely on linked K_6 subgraphs of G and use the argument highlighted in Table 1. In particular, though the K_6 subgraph of the modified embedding may contain a different pair of linked cycles than the original embedding, our argument relies only on the existence of linked cycles, not on the specific pair of linked cycles. Thus, we now consider the signs of the edges connecting $G[1, 2, 3, 4, 5, 6]$ to $G[7, 8, 9, 10]$.

possible linked cycles in $G[1,2,4,5,6,10]$	1-homologous cycle that shares an edge with a linked cycle
$(1,2,4)$, $(5,6,10)$	$(1,2,3)$
$(1,2,5)$, $(4,6,10)$	$(1,2,3)$
$(1,2,6)$, $(4,5,10)$	$(1,2,3)$
$(1,2,10)$, $(4,5,6)$	$(1,2,3)$
$(1,4,5)$, $(2,6,10)$	$(1,3,5)$
$(1,4,6)$, $(2,5,10)$	$(1,4,6)$
$(1,4,10)$, $(2,5,6)$	$(2,5,6)$
$(1,5,6)$, $(2,4,10)$	$(1,3,5)$
$(1,5,10)$, $(2,4,6)$	$(1,3,5)$
$(1,6,10)$, $(2,4,5)$	$(2,4,5)$

Table 1

Claim. *If* $v \in \{1,2,3\}$, *then edges from* v *to* $G[7,8,9,10]$ *have the same sign.*

Proof. Assume toward a contradiction that the edges from $\{1\}$ to $G[7,8,9,10]$ do not all have the same sign. Without loss of generality, let $(1,7)$ be a $^{+}$edge and $(1,8)$ a $^{-}$edge. Then $(1,7,8)$ is a 1-homologous cycle.

Consider $G[3,4,6,9]$. Since $(3,4,6)$ is 1-homologous, $G[3,4,6,9]$ contains another 1-homologous cycle by Lemma 3. If $(3,4,9)$ or $(3,6,9)$ is 1-homologous then the set $\{(1,7,8),(2,5,6),(3,4,9)\}$ or $\{(1,7,8),(2,4,5),(3,6,9)\}$ contains three disjoint 1-homologous cycles, respectively, and so G is triple-linked. Thus, $(4,6,9)$ is the second 1-homologous cycle in $G[3,4,6,9]$.

Since $(2,3,4)$ is 1-homologous, the induced subgraph $G[2,3,4,9]$ contains a second 1-homologous cycle by Lemma 3. As shown above, $(3,4,9)$ is 0-homologous. If $(2,4,9)$ is 1-homologous, then $(1,7,8)$, $(2,4,9)$, and $(3,5,6)$ form three disjoint 1-homologous cycles, so G is triple-linked. So, $(2,3,9)$ is the second 1-homologous cycle in $G[2,3,4,9]$.

Similarly, since $(3,5,6)$ is 1-homologous, $G[3,5,6,9]$ contains a second 1-homologous cycle by Lemma 3. As shown above, $(3,6,9)$ is 0-homologous. Additionally, $(5,6,9)$ is 0-homologous; otherwise $(1,7,8)$, $(2,3,4)$, and $(5,6,9)$ form three disjoint 1-homologous cycles and G is triple-linked. Thus, $(3,5,9)$ is a 1-homologous cycle.

As $(1,7,8)$ and $(4,6,9)$ are 1-homologous, $G[2,3,5,10]$ is a 0-homologous K_4, since, otherwise, G contains three disjoint 1-homologous cycles. By Lemma 2, $G[2,3,4,5,6,10]$ contains a pair of linked cycles. Since $(1,7,8)$ is 1-homologous and disjoint from all of the 1-homologous cycles in the second column of Table 2, Lemma 4 applies and G contains a triple-link, a contradiction.

Thus, $\{1\}$ connects to $G[7,8,9,10]$ via all $^{+}$edges or all $^{-}$edges, and similar reasoning applies to vertices $\{2\}$ and $\{3\}$. $\square$

possible linked cycles in $G[2,3,4,5,6,10]$	1-homologous cycle that shares an edge with a linked cycle
$(2,3,4)$, $(5,6,10)$	$(2,3,4)$
$(2,3,5)$, $(4,6,10)$	$(4,6,9)$
$(2,3,6)$, $(4,5,10)$	$(2,3,9)$
$(2,3,10)$, $(4,5,6)$	$(2,3,9)$
$(2,4,5)$, $(3,6,10)$	$(2,4,5)$
$(2,4,6)$, $(3,5,10)$	$(4,6,9)$
$(2,4,10)$, $(3,5,6)$	$(3,5,9)$
$(2,5,6)$, $(3,4,10)$	$(2,5,6)$
$(2,5,10)$, $(3,4,6)$	$(4,6,9)$
$(2,6,10)$, $(3,4,5)$	$(3,5,9)$

Table 2

A similar argument, using different induced subgraphs, shows the edges between each of the remaining vertices of $G[1,2,3,4,5,6]$ and $G[7,8,9,10]$ also have the same sign.

Claim. *If $v \in \{4,5,6\}$, then all edges from v to $G[7,8,9,10]$ have the same sign.*

Proof. Towards a contradiction, suppose not all the edges from $\{4\}$ to $G[7,8,9,10]$ have the same sign. Without loss of generality, let $(4,7)$ be a $^{+}$edge and $(4,8)$ be a $^{-}$edge. Then $(4,7,8)$ is a 1-homologous cycle.

Since $(1,2,3)$ is a 1-homologous cycle, the subgraph $G[1,2,3,9]$ contains a second 1-homologous cycle by Lemma 3. If $(1,3,9)$ or $(1,2,9)$ is 1-homologous, then the set $\{(1,3,9),(2,5,6),(4,7,8)\}$ or $\{(1,2,9),(3,5,6),(4,7,8)\}$ contains three disjoint 1-homologous cycles, respectively. So, $(2,3,9)$ is the second 1-homologous cycle in $G[1,2,3,9]$.

Since $(2,3,9)$ and $(4,7,8)$ are 1-homologous, the subgraph $G[1,5,6,10]$ is a 0-homologous K_4; otherwise, G contains three disjoint 1-homologous cycles. By Lemma 2, $G[1,2,3,5,6,10]$ contains a pair of linked cycles. Since $(4,7,8)$ is 1-homologous and disjoint from all 1-homologous cycles in the second column of Table 3, Lemma 4 applies and G contains a triple link.

Thus, all edges from $\{4\}$ to $G[7,8,9,10]$ have the same sign. A similar argument shows that all edges from vertices $\{5\}$ and $\{6\}$ to $G[7,8,9,10]$ have the same sign. □

The previous two claims show that the edges from each vertex in $G[1,2,3,4,5,6]$ to the vertices of $G[7,8,9,10]$ have the same sign. As we have assigned signs to the edges of $G[1,2,3,4,5,6]$ and $G[7,8,9,10]$, there remain 2^6 possible embedding classes. We consider all cases. If all edges from vertex $v \in \{1,2,3,4,5,6\}$ to $G[7,8,9,10]$ are $^{+}$edges, we write v_+, and otherwise v_-. For v_x with $x \in \{+,-\}$, we say "the sign of vertex v is x".

possible linked cycles in $G[1, 2, 3, 5, 6, 10]$	1-homologous cycle that shares an edge with a linked cycle
$(1, 2, 3)$, $(5, 6, 10)$	$(1, 2, 3)$
$(1, 2, 5)$, $(3, 6, 10)$	$(2, 5, 9)$
$(1, 2, 6)$, $(3, 5, 10)$	$(1, 2, 6)$
$(1, 2, 10)$, $(3, 5, 6)$	$(3, 5, 6)$
$(1, 3, 5)$, $(2, 6, 10)$	$(1, 3, 5)$
$(1, 3, 6)$, $(2, 5, 10)$	$(2, 5, 9)$
$(1, 3, 10)$, $(2, 5, 6)$	$(2, 5, 9)$
$(1, 5, 6)$, $(2, 3, 10)$	$(2, 3, 9)$
$(1, 5, 10)$, $(2, 3, 6)$	$(2, 3, 9)$
$(1, 6, 10)$, $(2, 3, 5)$	$(2, 3, 9)$

Table 3

Claim. *The two vertices in each of the pairs* $\{1, 4\}$, $\{2, 5\}$, *and* $\{3, 6\}$ *have different signs.*

Proof. Suppose toward a contradiction that $\{1\}$ and $\{4\}$ are both $^+$edges. Then $(1, 4, 7)$ is a 1-homologous cycle.

Since both $(2, 5)$ and $(3, 6)$ are $^-$edges, if both pairs of vertices $\{2, 5\}$ and $\{3, 6\}$ share the same sign (e.g., $2_+, 5_+, 3_-, 6_-$), then $(2, 5, 8)$ and $(3, 6, 9)$ are 1-homologous cycles. Thus, the cycles $(1, 4, 7)$, $(2, 5, 8)$, and $(3, 6, 9)$ are disjoint and 1-homologous, so G is triple-linked.

Since both $(2, 6)$ and $(3, 5)$ are $^+$edges, if both pairs of vertices $\{2, 6\}$ and $\{3, 5\}$ have different signs (e.g., $2_+, 6_-, 3_+, 5_-$), then $(2, 6, 8)$ and $(3, 5, 9)$ are 1-homologous cycles. Thus, we know $(1, 4, 7)$, $(2, 6, 8)$, and $(3, 5, 9)$ are disjoint 1-homologous cycles, so G is triple-linked.

The edge $(2, 3)$ is a $^-$edge and $(5, 6)$ is a $^+$edge, so if $\{2\}$ and $\{3\}$ share the same sign and $\{5\}$ and $\{6\}$ have different signs, (e.g., $2_+, 3_+, 5_+, 6_-$), then $(2, 3, 8)$ and $(5, 6, 9)$ are 1-homologous cycles. So, $(1, 4, 7)$, $(2, 3, 8)$, and $(5, 6, 9)$ form disjoint 1-homologous cycles, so G is triple-linked.

If G is embedded with $\{2_-, 3_+, 5_+, 6_-\}$ or $\{2_+, 3_-, 5_-, 6_+\}$, then $(1, 4, 7)$ and $(5, 6, 8)$ are disjoint 1-homologous cycles, so $G[2, 3, 9, 10]$ is a 0-homologous K_4, or G has a triple link. So, by Lemma 2, $G[1, 2, 3, 4, 9, 10]$ has a pair of linked cycles. Since $(5, 6, 8)$ is 1-homologous and is disjoint from all of the 1-homologous cycles in the second column of Table 4, G has a triple link by Lemma 4.

Finally, if G is embedded with one of the remaining configurations,

$$\{2_-, 3_+, 5_+, 6_+\}, \quad \{2_-, 3_+, 5_-, 6_-\}, \quad \{2_+, 3_-, 5_-, 6_-\}, \quad \{2_-, 3_+, 5_+, 6_+\},$$

then one of $\{(2, 5, 6), (3, 5, 6)\}$ must be 1-homologous. Since $G[7, 8, 9, 10]$ is a 0-homologous K_4, the subgraph $G[1, 4, 7, 8, 9, 10]$ contains a pair of linked cycles

possible linked cycles in $G[1, 2, 3, 4, 9, 10]$	1-homologous cycle that shares an edge with a linked cycle
(1, 2, 3), (4, 9, 10)	(1, 2, 3)
(1, 2, 4), (3, 9, 10)	(1, 4, 7)
(1, 2, 9), (3, 4, 10)	(1, 2, 7)
(1, 2, 10), (3, 4, 9)	(1, 2, 7)
(1, 3, 4), (2, 9, 10)	(1, 4, 7)
(1, 3, 9), (2, 4, 10)	(1, 3, 7)
(1, 3, 10), (2, 4, 9)	(1, 3, 7)
(1, 4, 9), (2, 3, 10)	(1, 4, 7)
(1, 4, 10), (2, 3, 9)	(1, 4, 7)
(1, 9, 10), (2, 3, 4)	(2, 3, 4)

Table 4

by Lemma 2. Both (2, 5, 6) and (3, 5, 6) are disjoint from one 1-homologous cycle in each row of the second column of Table 5. Thus, by Lemma 4, G is triple-linked.

So, in each embedding of G with 1_+ and 4_+, the graph G contains a triple link. A similar argument holds in the case that G is embedded with 1_- and 4_- and for the other vertex pairs {2, 5} and {3, 6}. □

We now suppose G is embedded with 1_+ and 4_-. By the last claim, the vertices in each of the pairs {2, 5} and {3, 6} have different signs. So, there are four cases to consider:

$$\{2_+, 3_+, 5_-, 6_-\}, \quad \{2_+, 3_-, 5_-, 6_+\}, \quad \{2_-, 3_+, 5_+, 6_-\}, \quad \{2_-, 3_-, 5_+, 6_+\}.$$

possible linked cycles in $G[1, 4, 7, 8, 9, 10]$	1-homologous cycles that share an edge with a linked cycle
(1, 4, 7), (8, 9, 10)	(1, 4, 7)
(1, 4, 8), (7, 9, 10)	(1, 4, 8)
(1, 4, 9), (7, 8, 10)	(1, 4, 9)
(1, 4, 10), (7, 8, 9)	(1, 4, 10)
(1, 7, 8), (4, 9, 10)	(1, 2, 7), (1, 3, 7)
(1, 7, 9), (4, 8, 10)	(1, 2, 7), (1, 3, 7)
(1, 7, 10), (4, 8, 9)	(1, 2, 7), (1, 3, 7)
(1, 8, 9), (4, 7, 10)	(1, 2, 8), (1, 3, 8)
(1, 8, 10), (4, 7, 9)	(1, 2, 8), (1, 3, 8)
(1, 9, 10), (4, 7, 8)	(1, 2, 9), (1, 3, 9)

Table 5

possible linked cycles in $G[4, 6, 7, 8, 9, 10]$	1-homologous cycle that shares an edge with a linked cycle
(4, 6, 7), (8, 9, 10)	(4, 6, 7)
(4, 6, 8), (7, 9, 10)	(4, 6, 8)
(4, 6, 9), (7, 8, 10)	(4, 6, 9)
(4, 6, 10), (7, 8, 9)	(4, 6, 10)
(4, 7, 8), (6, 9, 10)	(5, 6, 9)
(4, 7, 9), (6, 8, 10)	(5, 6, 8)
(4, 7, 10), (6, 8, 9)	(5, 6, 8)
(4, 8, 9), (6, 7, 10)	(5, 6, 7)
(4, 8, 10), (6, 7, 9)	(5, 6, 7)
(4, 9, 10), (6, 7, 8)	(5, 6, 7)

Table 6

First, if the embedding has $\{2_+, 3_+, 5_-, 6_-\}$, then $(1, 6, 7)$, $(2, 4, 9)$, and $(3, 5, 8)$ form three disjoint 1-homologous cycles, so G is triple-linked. Second, suppose the embedding has $\{2_+, 3_-, 5_-, 6_+\}$ or $\{2_-, 3_+, 5_+, 6_-\}$. Then the second column of Table 6 contains 1-homologous cycles. Since $G[7, 8, 9, 10]$ is 0-homologous, $G[4, 6, 7, 8, 9, 10]$ has a pair of linked cycles by Lemma 2. Since $(1, 2, 3)$ is 1-homologous and disjoint from all 1-homologous cycles in the second column of Table 6, Lemma 4 applies and G contains a triple link.

Finally, if the embedding has $\{2_-, 3_-, 5_+, 6_+\}$, then the second column of Table 7 contains 1-homologous cycles. As above, since $(1, 2, 3)$ is 1-homologous and disjoint from all 1-homologous cycles in the second column of Table 7, Lemma 4 applies and G contains a triple link.

possible linked cycles in $G[4, 6, 7, 8, 9, 10]$	1-homologous cycle that shares an edge with a linked cycle
(4, 6, 7), (8, 9, 10)	(4, 6, 7)
(4, 6, 8), (7, 9, 10)	(4, 6, 8)
(4, 6, 9), (7, 8, 10)	(4, 6, 9)
(4, 6, 10), (7, 8, 9)	(4, 6, 10)
(4, 7, 8), (6, 9, 10)	(4, 5, 7)
(4, 7, 9), (6, 8, 10)	(4, 5, 7)
(4, 7, 10), (6, 8, 9)	(4, 5, 7)
(4, 8, 9), (6, 7, 10)	(4, 5, 8)
(4, 8, 10), (6, 7, 9)	(4, 5, 8)
(4, 9, 10), (6, 7, 8)	(4, 5, 9)

Table 7

The same argument holds if G is embedded with 1_- and 4_+. So, for any assignment of signs to the edges from {1} and {4} to $G[7, 8, 9, 10]$, G contains a triple link, a contradiction. Thus, every embedding of G into $\mathbb{R}P^3$ contains a triple link, so G is intrinsically triple-linked in $\mathbb{R}P^3$. ☐

Flapan et al. [2001a] show that K_9 can be embedded 3-linklessly in $\mathbb{R}^3$, and so K_9 can be embedded 3-linklessly in $\mathbb{R}P^3$ as well. Thus, 10 is the smallest n for which K_n is intrinsically triple-linked in $\mathbb{R}P^3$.

4. Other intrinsically triple-linked graphs in $\mathbb{R}P^3$

In this section, we exhibit other intrinsically triple-linked graphs in $\mathbb{R}P^3$. We show that two graphs shown in [Bowlin and Foisy 2004] to be intrinsically triple-linked in $\mathbb{R}^3$ may be embedded 3-linklessly in $\mathbb{R}P^3$. Moreover, the graphs obtained by taking two disjoint copies of these graphs described in [Bowlin and Foisy 2004] give intrinsically triple-linked graphs in $\mathbb{R}P^3$. We begin by describing a family of intrinsically n-linked graphs in $\mathbb{R}P^3$.

Lemma 10. *If an embedded graph has all* 0*-homologous cycles, then it is crossing-change equivalent to a spatial embedding.*

Proof. Take a spanning tree in the embedded graph. Since a spanning tree is contractible, it can be deformed so that none of its edges touch the boundary of D^2. Order the edges that do not lie in the spanning tree. Now take the first edge not in the spanning tree. If this edge does not touch the boundary, move on to the next edge. Otherwise, the edge lies in a cycle that, by assumption, is 0-homologous. By Mroczkowski's result, the cycle can be made into an unknot by crossing changes. Since the unknot is 0-homologous, it bounds a disk. Deform the edge by pulling in the disk towards the edges of the cycle that lie in the spanning tree. Thus, the edge can be deformed so that it does not touch the boundary of D^2. Eventually, all of the edges not in the spanning tree can be deformed, if necessary, not to touch the boundary. The resulting embedding is equivalent to a spatial embedding. Thus, the original embedding was crossing-change equivalent to a spatial embedding. ☐

Proposition 11. *A graph composed of n disjoint copies of an intrinsically n-linked graph in $\mathbb{R}^3$ is intrinsically n-linked in $\mathbb{R}P^3$. In particular, three disjoint copies of intrinsically triple-linked graphs in $\mathbb{R}^3$ are intrinsically triple-linked in $\mathbb{R}P^3$.*

Proof. Let G be a graph that is intrinsically n-linked in $\mathbb{R}^3$, and let G_i be isomorphic to G for $i = 1, \ldots, n$. Let $\Gamma = \bigsqcup_{i=1}^{n} G_i$ be the disjoint union of n graphs isomorphic to G. If G_i contains all 0-homologous cycles for some i, then G_i is crossing-change equivalent to a spatial embedding by Lemma 10. Thus, G_i, and hence G, is n-linked in $\mathbb{R}P^3$.

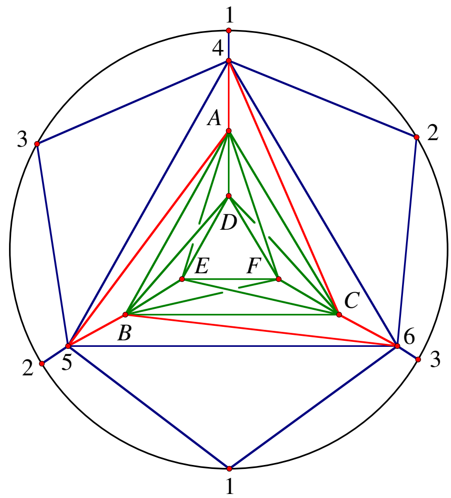

Figure 5. A 3-linkless embedding of K_6 connected to K_6 along a 6-cycle in $\mathbb{R}P^3$.

Otherwise, each G_i contains a 1-homologous cycle. Thus, Γ contains n disjoint 1-homologous cycles, and so contains an n-link. Therefore, Γ is intrinsically n-linked in $\mathbb{R}P^3$. □

The graph K_{10} is an example of a one-component graph that is intrinsically triple-linked in $\mathbb{R}P^3$. We now exhibit two intrinsically triple-linked graphs in $\mathbb{R}P^3$, each comprised of two components. In each case, the components are intrinsically triple-linked in $\mathbb{R}^3$. The question remains whether there exists a minor-minimal intrinsically triple-linked graph of three components in $\mathbb{R}P^3$.

Bowlin and Foisy prove the following graphs are intrinsically linked in $\mathbb{R}^3$.

Theorem 12 [Bowlin and Foisy 2004]. *Let G be a graph containing two disjoint graphs from the Petersen family, G_1 and G_2, as subgraphs. If there are edges between the two subgraphs G_1 and G_2 such that the edges form a 6-cycle with vertices that alternate between G_1 and G_2, then G is minor-minimal intrinsically triple-linked in $\mathbb{R}^3$.*

If G_1 and G_2, as in the theorem, are isomorphic to K_6, this result does not hold in $\mathbb{R}P^3$. A 3-linkless embedding of $G = G_1 \sqcup G_2$ is shown in Figure 5. We now show that the graph obtained from two disjoint copies of G is minor-minimal intrinsically triple-linked in $\mathbb{R}P^3$.

Theorem 13. *Let G_1 be a graph containing two disjoint copies of K_6 with edges between the two K_6 subgraphs that form a 6-cycle with vertices alternating between the two K_6 subgraphs. If G_2 is a graph isomorphic to G_1 and $G = G_1 \sqcup G_2$, then G is minor-minimal intrinsically triple-linked in $\mathbb{R}P^3$.*

Proof. Let $G = G_1 \sqcup G_2$ be as in the theorem, and embed G in $\mathbb{R}P^3$.

If either G_1 or G_2 contain all 0-homologous cycles, then that subgraph is crossing-change equivalent to a spatial embedding by Lemma 10, and hence triple-linked by Theorem 12. Thus, G contains a triple link. So, now suppose that both G_1 and G_2 contain a 1-homologous cycle.

In both G_1 and G_2, any cycle of length greater than 3 can be subdivided by an edge e into a "θ-graph": two cycles of smaller length, disjoint, except for edge e. That is, there exists an edge $e = (v_1, v_i)$ in $G[v_1, \dots, v_n]$ so that $c = (v_1, \dots, v_n)$ may be divided into $c_1 \cup c_2 = (v_1, \dots, v_i) \cup (v_i, \dots, v_n, v_1)$. If c is 1-homologous, then in any signed embedding of G, the cycle c has an odd number of $^-$edges. So, either c_1 or c_2 has an odd number of $^-$edges, and is thus 1-homologous. By iterating this procedure, we conclude that both G_1 and G_2 contain a 1-homologous 3-cycle.

Let the vertex set of G_1 be given by $\{1, 2, 3, 4, 5, 6, A, B, C, D, E, F\}$ so that $G[1, 2, 3, 4, 5, 6] \cong K_6$ and $G[A, B, C, D, E, F] \cong K_6$ are connected by edges $(4, A)$, $(4, C)$, $(5, A)$, $(5, B)$, $(6, B)$, and $(6, C)$. Up to isomorphism, there are five 3-cycle equivalence classes in G_1. The set

$$S = \{(1, 2, 3), (1, 2, 4), (1, 4, 5), (4, 5, 6), (4, 5, A)\}$$

contains one representative from each 3-cycle equivalence class. So, without loss of generality, we may suppose that S contains a 1-homologous 3-cycle.

If $G[B, C, E, F] \cong K_4$ contains a 1-homologous cycle, then this cycle, the 1-homologous cycle in S and the 1-homologous cycle in G_2 form three disjoint 1-homologous cycles and so G contains a triple link. Now suppose $G[B, C, E, F]$ is 0-homologous, so that $G[A, B, C, D, E, F]$ contains a pair of linked cycles by Lemma 2.

First suppose that the 1-homologous cycle $c_1 \in S$ is not $(4, 5, A)$. By the pigeonhole principle, two vertices in $\{A, B, C\}$ are in one of the components, c_2, of the linked cycles in $G[A, B, C, D, E, F]$. Use the edges of the 6-cycle to join c_2 to c_1 along disjoint paths. By Lemma 5, G contains a triple link.

Now suppose that the 1-homologous cycle in S is $(4, 5, A)$. If there is a 1-homologous cycle in $G[1, 2, 3, 6]$ then this cycle will link with $(4, 5, A)$ and the 1-homologous cycle in G_2, so G contains a triple link. Else, $G[1, 2, 3, 4, 5, 6]$ has a pair of linked cycles by Lemma 2. By the pigeonhole principle, at least two vertices in the set $\{4, 5, 6\}$ are in a linked cycle, c_3, within $G[1, 2, 3, 4, 5, 6]$. Similarly, at least two vertices of $\{A, B, C\}$ are in a linked cycle, c_4, within $G[A, B, C, D, E, F]$. As a result of the 6-cycle connecting these two copies of K_6, there are two disjoint edges between c_3 and c_4. By Lemma 5, G is triple-linked.

To see G is minor-minimal with respect to intrinsic triple-linking in $\mathbb{R}P^3$, embed G so that G_1 is embedded as in the drawing in Figure 5 and G_2 is contained in a sphere that lies in the complement of G_1. Therefore, G_1 does not have any triple

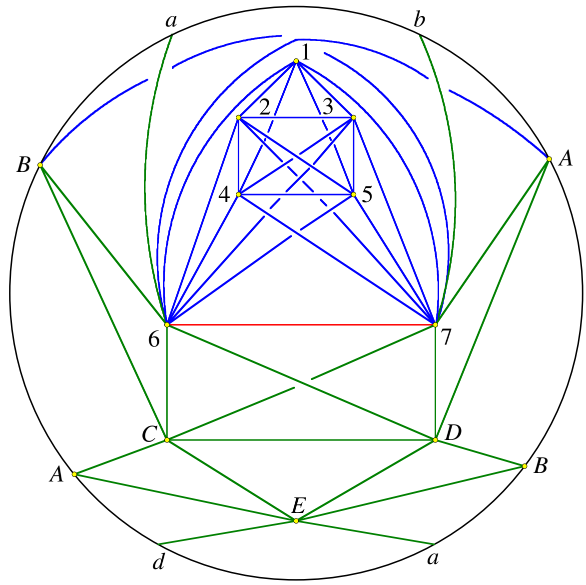

Figure 6. A 3-linkless embedding of K_7 connected to K_7 along an edge in $\mathbb{R}P^3$.

links and no cycle in G_1 is linked with a cycle in G_2. Without loss of generality, if we delete an edge, contract an edge or delete any vertex on G_2, it will have an affine linkless embedding. Thus, we can re-embed G_2 within the sphere in each case. Therefore, G is minor-minimal for intrinsic triple-linking. □

Bowlin and Foisy prove the following graph is intrinsically triple-linked in $\mathbb{R}^3$.

Theorem 14 [Bowlin and Foisy 2004]. *Let G be a graph formed by identifying an edge of K_7 with an edge from another copy of K_7. Then G is intrinsically triple-linked in $\mathbb{R}^3$.*

The graph G defined in Theorem 14 may be embedded 3-linklessly in $\mathbb{R}P^3$, as drawn in Figure 6. As in the previous result, the graph consisting of two disjoint copies of this graph is intrinsically linked in $\mathbb{R}P^3$.

Theorem 15. *Let G_1 be a graph formed by identifying an edge of K_7 with an edge from another copy of K_7. If G_2 is isomorphic to G_1 and $G = G_1 \sqcup G_2$ is the disjoint union of G_1 and G_2, then G is intrinsically linked in $\mathbb{R}P^3$.*

Proof. Let $G = G_1 \sqcup G_2$ be as above, and embed G in $\mathbb{R}P^3$. If either G_1 or G_2 contains all 0-homologous cycles, then that subgraph is crossing-change equivalent to a spatial embedding by Lemma 10, and hence triple-linked by Theorem 14. Thus, in this case, G has a triple link. Now suppose that both G_1 and G_2 contain a 1-homologous cycle.

Let the vertex set of G_1 be given by $\{1, 2, 3, 4, 5, 6, 7, A, B, C, D, E\}$ so that $G[1, 2, 3, 4, 5, 6, 7]$ and $G[6, 7, A, B, C, D, E]$ are isomorphic to K_7 and share edge $(6, 7)$. Up to isomorphism, there are three 3-cycle equivalence classes in G_1. The set $S = \{(1, 2, 3), (1, 2, 7), (1, 6, 7)\}$ contains one representative from each 3-cycle equivalence class. By the same argument given in Theorem 13, we may assume that S contains a 1-homologous cycle, c_1.

If $G[A, B, C, D]$ contains a 1-homologous cycle, then this cycle, c_1, and the 1-homologous cycle in G_2 form three disjoint 1-homologous cycles, so G contains a triple link. Otherwise, $G[A, B, C, D, E, 6]$ contains a pair of linked cycles by Lemma 2. Following the proof in Theorem 13, connect the linked cycle containing vertex $\{6\}$ to c_1 via two disjoint paths. By Lemma 5, G contains a triple link. □

The minor-minimality of the graph formed by identifying an edge of K_7 with an edge from another copy of K_7 with respect to intrinsic triple-linking is unknown in $\mathbb{R}^3$. If true, then the graph G defined in Theorem 15 is also minor-minimal with respect to intrinsic triple-linking; a similar argument to that in Theorem 13 holds in this case as well.

We also remark that the graph $G(n)$ as defined in [Flapan et al. 2001b] is a one-component minor-minimal intrinsically $(n+1)$-linked graph in $\mathbb{R}P^3$. The arguments given in [Flapan et al. 2001b] hold in $\mathbb{R}P^3$ since $K_{4,4} - \{e\}$, where e is an edge, is intrinsically linked in both $\mathbb{R}^3$ and $\mathbb{R}P^3$.

5. Graphs with linking number at least 1 in $\mathbb{R}P^3$

In $\mathbb{R}P^3$, there are intrinsically linked graphs for which there exists an embedding in which every pair of disjoint cycles has linking number less than 1, as a pair of linked cycles may have only one crossing. Work has been done in $\mathbb{R}^3$ [Flapan 2002] to find graphs containing disjoint cycles with large linking number in every spatial embedding. Using the fact that K_{10} is triple-linked in $\mathbb{R}^3$, Flapan [2002] showed that every spatial embedding of K_{10} contains a two-component link $L \cup J$ such that, for some orientation, $\mathrm{lk}(L, J) \geq 2$. A similar argument using Theorem 9 yields the following proposition.

Proposition 16. *Every projective embedding of K_{10} contains a two-component link $L \cup J$ such that, for some orientation,* $\mathrm{lk}(L, J) \geq 1$.

It remains an open question to determine whether 10 is the smallest number for which this property holds. At this point, we know the smallest n is such that $7 < n \leq 10$.

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