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A generalization of the matrix transpose map and its relationship to the twist of the polynomial ring by an automorphism

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A generalization of the notion of symmetric matrix was introduced by Cassidy and Vancliff in 2010 and used by them in a construction that produces quadratic regular algebras of finite global dimension that are generalizations of graded Clifford algebras. In this article, we further their ideas by introducing a generalization of the matrix transpose map and use it to generalize the notion of skew-symmetric matrix. With these definitions, an analogue of the result that every $n \times n$ matrix is a sum of a symmetric matrix and a skew-symmetric matrix holds. We also prove an analogue of the result that the transpose map is an antiautomorphism of the algebra of $n \times n$ matrices, and show that the antiautomorphism property of our generalized transpose map is related to the notion of twisting the polynomial ring on n variables by an automorphism.

Introduction

In [Cassidy and Vancliff 2010], a generalization of the notion of symmetric matrix was introduced and used in a construction that produces quadratic regular algebras of finite global dimension that are generalizations of graded Clifford algebras. In the same paper, it was also shown that such a matrix corresponds to a noncommutative analogue of a quadratic form. In this article, we further these ideas by introducing a generalization of the matrix transpose map and use it to generalize the notion of skew-symmetric matrix. In particular, we prove in Theorem 2.5 an analogue of the result that every $n \times n$ matrix is a sum of a symmetric matrix and a skew-symmetric matrix. We also prove, in Proposition 2.6 and Corollary 2.16, an analogue of the result that the transpose map is an antiautomorphism of the algebra of $n \times n$ matrices. This latter property is shown in Corollary 2.16 to be related to the twist of the polynomial ring on n variables by an automorphism.

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The article is outlined as follows. In Section 1, we define generalizations of symmetric and skew-symmetric matrices together with a few other concepts that will be used in the subsequent section. Section 2 is in two parts: the first defines and explores a generalization of the transpose map, whereas the second ties the behavior of this transpose map to the notion of twisting a polynomial ring by an automorphism.

1. Definitions

In this section, we recall the generalizations of symmetric matrix and quadratic form that were introduced in [Cassidy and Vancliff 2010]. We also introduce a generalization of the notion of skew-symmetric matrix.

Throughout, $\Bbbk$ denotes a field. We use the notation $M(n, \Bbbk)$ to denote the vector space of $n \times n$ matrices with entries in $\Bbbk$ and $M(m, r, \Bbbk)$ to denote the vector space of $m \times r$ matrices with entries in $\Bbbk$. For any matrix $N \in M(m, r, \Bbbk)$, we let N_{ij} denote the ij-entry of N.

Definition 1.1. Let $\mu \in M(n, \Bbbk)$ be such that $\mu_{ij}\mu_{ji} = 1$ for all distinct i, j. A matrix $M \in M(n, \Bbbk)$ is said to be

(a) μ-symmetric if $M_{ij} = \mu_{ij} M_{ji}$ for all i, j [Cassidy and Vancliff 2010];

(b) skew-μ-symmetric if $M_{ij} = -\mu_{ij} M_{ji}$ for all i, j.

If $\mu_{ij} = 1$ for all i, j, then any μ-symmetric matrix is a symmetric matrix, and any skew-μ-symmetric matrix is a skew-symmetric matrix. Consequently, we generalize the notion of transpose in the next section and relate the notions of μ-symmetry and skew-μ-symmetry to that concept.

The notion of μ-symmetry was used in [Cassidy and Vancliff 2010] to produce algebras that may be viewed as quantized graded Clifford algebras. In other words, the main use of μ-symmetry is to "tie together" two or more matrices to a particular matrix μ, and to do so in a symmetrical manner.

Following [Vancliff and Veerapen 2013], we write $M^{\mu}(n, \Bbbk)$ for the set of μ-symmetric $n \times n$ matrices with entries in $\Bbbk$. Likewise, we write $M^{s\mu}(n, \Bbbk)$ for the set of skew-μ-symmetric $n \times n$ matrices with entries in $\Bbbk$. Clearly, $M^{\mu}(n, \Bbbk)$ and $M^{s\mu}(n, \Bbbk)$ are subspaces of $M(n, \Bbbk)$.

Mirroring the theory for symmetric matrices and following [Cassidy and Vancliff 2010], a μ-symmetric matrix corresponds to a noncommutative analogue of a quadratic form, provided $\mu_{ii} = 1$ for all i; this correspondence is summarized as follows.

Definition 1.2 [Cassidy and Vancliff 2010]. Let $\mu \in M(n, \Bbbk)$ be as in Definition 1.1, with the additional assumption that $\mu_{ii} = 1$ for all i. Let (S, μ) denote the quadratic $\Bbbk$-algebra on generators $z_1, \dots, z_n$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$ for all $i, j = 1, \dots, n$, and let S_2 denote the span of the homogeneous elements of (S, μ) of degree two. A (noncommutative) quadratic form is defined to be any element of S_2.

The algebra (S, μ) has no zero divisors and has the same Hilbert series as the polynomial ring on n variables. By [Cassidy and Vancliff 2010], if $\mu_{ii} = 1$ for all i, then $M^{\mu}(n, \Bbbk) \cong S_2$, as vector spaces, via the map $M \mapsto z^T M z \in S_2$, where $z = (z_1, \dots, z_n)^T$.

In the next section, the algebra (S, μ) will be considered in the special case where $\mu_{ij} = \mu_{ik}\mu_{kj}$ for all $i, j, k = 1, \dots, n$. By [Nafari and Vancliff 2015, Lemma 2.2], (S, μ) is a twist (see Definition 1.3 below) of the polynomial ring R on n variables by a graded automorphism of R of degree zero if and only if this condition on μ holds.

Definition 1.3 [Artin et al. 1991, §8]. Let $A = \bigoplus_{k \geq 0} A_k$ be a graded $\Bbbk$-algebra and let ϕ be a graded degree-zero automorphism of A. The twist A' of A by ϕ is a graded $\Bbbk$-algebra that is the vector space $\bigoplus_{k \geq 0} A_k$ with a new multiplication $*$ defined as follows: if $a' \in A'_i = A_i$ and $b' \in A'_j = A_j$, then $a' * b' = (a\phi^i(b))'$, where the right-hand side is computed using the original multiplication in A and a, b are the images of a', b', respectively, in A.

Clearly, the twist of a quadratic algebra is again a quadratic algebra. Moreover, this notion of twist is reflexive and symmetric.

2. Main results

In this section, we define a generalization of the notion of transpose of a matrix and explore properties of this new concept. Our main results are given in Theorem 2.5, Proposition 2.6, Theorem 2.15 and Corollary 2.16.

2A. *The transpose map.*

Definition 2.1. If $\nu \in M(r, m, \Bbbk)$ and $N \in M(m, r, \Bbbk)$, we define the ν-transpose of N, denoted $N^{\nu T}$, to be the $r \times m$ matrix with ij-entry given by $\nu_{ij} N_{ji}$ for all i, j.

Clearly, if $\nu_{ij} = 1$ for all i, j, then the ν-transpose map is the transpose map. Alternatively, we may view the ν-transpose as a composition of maps; for this purpose, let $\hat{\nu} : M(r, m, \Bbbk) \to M(r, m, \Bbbk)$ be defined by $\hat{\nu}(K) = (\nu_{ij} k_{ij})$, where $K = (k_{ij}) \in M(r, m, \Bbbk)$.

Lemma 2.2. *If ν, $\hat{\nu}$ and N are as above, then $N^{\nu T} = \hat{\nu}(N^T)$, where N^T denotes the transpose of N. In particular, the ν-transpose map is a linear transformation.* □

Lemma 2.3. *Let μ be as in Definition 1.1. A matrix $M \in M(n, \Bbbk)$ is μ-symmetric if and only if $M^{\mu T} = M$. Additionally, M is skew-μ-symmetric if and only if $M^{\mu T} = -M$.*

Proof. If $M \in M(n, \Bbbk)$ is μ-symmetric, then $M_{ij} = \mu_{ij} M_{ji}$ for all i, j, so $M = M^{\mu T}$; reversing the argument proves the converse. The proof of skew-μ-symmetric case is similar. □

Proposition 2.4. *Let $\mu \in M(n,\Bbbk)$ be such that $\mu_{ij}\mu_{ji} = 1$ for all i, j. If $M \in M(n,\Bbbk)$, then*

(a) $(M^{\mu T})^{\mu T} = M$,

(b) $M + M^{\mu T} \in M^{\mu}(n,\Bbbk)$,

(c) $M - M^{\mu T} \in M^{s\mu}(n,\Bbbk)$.

Proof. (a) We have $[M^{\mu T}]^{\mu T} = (\mu_{ij}M_{ji})^{\mu T} = (\mu_{ij}\mu_{ji}M_{ij}) = (M_{ij}) = M$.

(b)–(c) We have $M \pm M^{\mu T} = (M_{ij} \pm \mu_{ij}M_{ji}) = \big(\pm\mu_{ij}(M_{ji} \pm \mu_{ji}M_{ij})\big)$. Thus,

$$[M \pm M^{\mu T}]^{\mu T} = \big(\pm\mu_{ij}\mu_{ji}(M_{ij} \pm \mu_{ij}M_{ji})\big) = \big(\pm(M_{ij} \pm \mu_{ij}M_{ji})\big) = \pm[M \pm M^{\mu T}],$$

and so the result follows from Lemma 2.3. □

Theorem 2.5. *Suppose* $\operatorname{char}(\Bbbk) \neq 2$. *If $\mu \in M(n,\Bbbk)$ is such that $\mu_{ij}\mu_{ji} = 1$ for all i, j, then*

$$M(n,\Bbbk) = M^{\mu}(n,\Bbbk) \oplus M^{s\mu}(n,\Bbbk).$$

Proof. If $M \in M(n,\Bbbk)$, then $M = \frac{1}{2}(M + M^{\mu T}) + \frac{1}{2}(M - M^{\mu T})$, since $\operatorname{char}(\Bbbk) \neq 2$. It follows from Proposition 2.4 that $M(n,\Bbbk) = M^{\mu}(n,\Bbbk) + M^{s\mu}(n,\Bbbk)$. However, the assumption on the characteristic of $\Bbbk$ ensures that $M^{\mu}(n,\Bbbk) \cap M^{s\mu}(n,\Bbbk) = \{0\}$, which completes the proof. □

A well-known result for symmetric matrices is that if $X \in M(n,\Bbbk)$ is symmetric, then P^TXP is also symmetric for all $P \in M(n,\Bbbk)$. This result is a consequence of the fact that $[XY]^T = Y^TX^T$ for all $X, Y \in M(n,\Bbbk)$; that is, the transpose map is an antiautomorphism of $M(n,\Bbbk)$. However, the analogues of these results are false in general for μ-symmetry, unless μ satisfies certain conditions as follows.

Proposition 2.6. *If $\mu \in M(n,\Bbbk)$ is such that $\mu_{ij} = \mu_{ik}\mu_{kj}$ for all i, j, k, then $[XY]^{\mu T} = Y^{\mu T}X^{\mu T}$ for all $X, Y \in M(n,\Bbbk)$.*

Proof. Let $X, Y \in M(n,\Bbbk)$. We have

$$[XY]^{\mu T} = \left(\sum_{k=1}^{n} X_{ik}Y_{kj}\right)^{\mu T} = \left(\mu_{ij}\sum_{k=1}^{n} X_{jk}Y_{ki}\right),$$

whereas

$$Y^{\mu T}X^{\mu T} = (\mu_{ik}Y_{ki})(\mu_{kj}X_{jk}) = \left(\sum_{k=1}^{n} \mu_{ik}\mu_{kj}Y_{ki}X_{jk}\right) = \left(\mu_{ij}\sum_{k=1}^{n} X_{jk}Y_{ki}\right),$$

where the last equality is a consequence of the condition on μ. □

If $\mu \in M(n,\Bbbk)$ satisfies the hypotheses of Propositions 2.4 and 2.6, then $\mu_{ij} = \mu_{ik}\mu_{kj}$ for all i, j, k, and $\mu_{ii} = 1$ for all i; the converse also holds.

Corollary 2.7. *Let $\mu \in M(n, \Bbbk)$. If $\mu_{ij} = \mu_{ik}\mu_{kj}$ for all i, j, k, and if $\mu_{ii} = 1$ for all i, then $P^{\mu T} X P \in M^{\mu}(n, \Bbbk)$ for all $X \in M^{\mu}(n, \Bbbk)$ and for all $P \in M(n, \Bbbk)$.*

Proof. The conditions on μ imply $\mu_{ik}\mu_{ki} = \mu_{ii} = 1$ for all i, k, so that Lemma 2.3 and Propositions 2.4 and 2.6 may be applied to compute $[P^{\mu T} X P]^{\mu T}$; namely,

$$[P^{\mu T} X P]^{\mu T} = P^{\mu T}[P^{\mu T} X]^{\mu T} = P^{\mu T} X^{\mu T}[P^{\mu T}]^{\mu T} = P^{\mu T} X P$$

for all $X \in M^{\mu}(n, \Bbbk)$ and for all $P \in M(n, \Bbbk)$. The result follows from Lemma 2.3. □

The hypotheses on μ in the last result coincide with the hypotheses required for the skew polynomial ring (S, μ), defined in Definition 1.2, to be a twist (in the sense of Definition 1.3) of the polynomial ring R on n variables by a graded automorphism of R of degree zero. However, the above methods give no insight as to why this should be the case, so further analysis is required to explain this relationship and is the purpose of the next subsection.

2B. ***The transpose map and twisting the polynomial ring.*** The goal of this subsection is to show that the result of Corollary 2.7 is directly related to the algebra (S, μ) being a twist of the polynomial ring R as mentioned at the end of Section 2A. Our method will be to show that the result of Corollary 2.7 is directly related to a certain map $\bar{\mu} : M(n, \Bbbk) \to M(n, \Bbbk)$ (see Definition 2.14) being an automorphism, in which case $\bar{\mu}$ induces an automorphism of (S, μ) that twists (S, μ) to R.

Throughout this subsection, we assume that $\mu_{ii} = 1$ for all i and that $\mu_{ij}\mu_{ji} = 1$ for all i, j.

Let V denote the span of the homogeneous elements of (S, μ) of degree one. Since (S, μ) is a domain, for each $k = 1, \ldots, n$, we may define $\theta_k \in \mathrm{Aut}(S, \mu)$ via $s z_k = z_k \theta_k(s)$ for all $s \in (S, \mu)$. In particular, for every k, we have $\theta_k(z_i) = \mu_{ki} z_i$ for all i, so if we twist (S, μ) by θ_k, we obtain a quadratic algebra in which the image of z_k is central.

Let V^* denote the vector-space dual of V and let $\{z_1^*, \ldots, z_n^*\}$ in V^* denote the dual basis to the basis $\{z_1, \ldots, z_n\}$ of V. For each k, the linear transformation $\theta_k|_V : V \to V$ induces a linear map $\theta_k^* : V^* \to V^*$, where $\theta_k^*(z_i^*) = \mu_{ik} z_i^*$ for all i. Hence θ_k induces a linear map $\bar{\theta}_k : V \otimes_{\Bbbk} V^* \to V \otimes_{\Bbbk} V^*$ via

$$\bar{\theta}_k(v \otimes u) = \theta_k(v) \otimes \theta_k^*(u)$$

for all $v \otimes u \in V \otimes_{\Bbbk} V^*$.

Remark 2.8. As is well known, $V \otimes_{\Bbbk} V^*$ is a $\Bbbk$-algebra under the usual addition and with multiplication given by $(v \otimes u)(v' \otimes u') = (uv')(v \otimes u')$ for all $v, v' \in V$, $u, u' \in V^*$. In fact, $V \otimes_{\Bbbk} V^* \cong M(n, \Bbbk)$, as $\Bbbk$-algebras, via the map that sends $z_i \otimes z_j^*$ to the $n \times n$ matrix with 1 in the ij-entry and zeros elsewhere.

Lemma 2.9. *For every $k = 1, \ldots, n$, the linear map $\bar{\theta}_k$ is in* $\mathrm{Aut}(V \otimes_{\Bbbk} V^*)$.

Proof. Since $\bar{\theta}_k$ is linear and bijective, it remains to prove that $\bar{\theta}_k$ respects multiplication, and it suffices to consider products of pure tensors. Let $v, v' \in V$ and $u, u' \in V^*$, and write $v' = \sum_{i=1}^n v_i z_i$ and $u = \sum_{j=1}^n u_j z_j^*$, where $v_i, u_j \in \Bbbk$ for all i, j. In particular, $uv' = \sum_{i=1}^n u_i v_i$ and

$$\theta_k^*(u)\theta_k(v') = \left(\sum_{j=1}^n u_j \mu_{jk} z_j^*\right)\left(\sum_{i=1}^n v_i \mu_{ki} z_i\right) = \sum_{i=1}^n u_i v_i = uv'.$$

It follows that

$$\begin{aligned}\bar{\theta}_k\big((v \otimes u)(v' \otimes u')\big) &= \bar{\theta}_k\big((uv')(v \otimes u')\big) \\ &= uv'\theta_k(v) \otimes \theta_k^*(u'),\end{aligned}$$

whereas

$$\begin{aligned}\bar{\theta}_k(v \otimes u)\bar{\theta}_k(v' \otimes u') &= \big(\theta_k(v) \otimes \theta_k^*(u)\big)\big(\theta_k(v') \otimes \theta_k^*(u')\big) \\ &= \theta_k^*(u)\theta_k(v')\big(\theta_k(v) \otimes \theta_k^*(u')\big),\end{aligned}$$

so the result follows. □

In the following, $\Bbbk^\times$ denotes the nonzero elements of $\Bbbk$.

Lemma 2.10. *For all k, i, we have $\bar{\theta}_k = \bar{\theta}_i$ if and only if $\theta_k \in \Bbbk^\times \theta_i$.*

Proof. We have $\theta_k = \lambda\theta_i$ for some $\lambda \in \Bbbk^\times$ if and only if $\theta_k^* = \lambda^{-1}\theta_i^*$. The result follows from the definitions of $\bar{\theta}_k$ and $\bar{\theta}_i$. □

Proposition 2.11. *The map θ_k is in $\Bbbk^\times\theta_1$ for all k if and only if the algebra (S, μ) is a twist (in the sense of Definition 1.3) of the polynomial ring on n variables.*

Proof. As mentioned above, for each k, the twist of (S, μ) by θ_k yields an algebra in which the image of z_k is central. Hence, if $\theta_k \in \Bbbk^\times\theta_1$ for all k, then twisting by θ_k produces an algebra R in which the image of z_i is central for all i. Since the relations of R are induced by the relations of (S, μ), it follows that R is the polynomial ring on n variables.

Conversely, suppose (S, μ) is a twist of the polynomial ring R on n variables. It follows that there exists a degree-zero map $\theta \in \mathrm{Aut}(S, \mu)$ such that twisting (S, μ) by θ renders the image of z_k central in R for all k. Writing "$\cdot$" for the multiplication in R, this implies

$$z_k\theta(z_i) = z_k \cdot z_i = z_i \cdot z_k = z_i\theta(z_k)$$

for all i, k. However, since S is a quadratic algebra and since S_2 has a $\Bbbk$-basis $\{z_j z_l : 1 \le j \le l \le n\}$, it follows that $\theta(z_k) \in \Bbbk^\times z_k$ for all k. Writing $\theta(z_k) = \lambda_k z_k$, where $\lambda_k \in \Bbbk^\times$ for all k, we have $\mu_{ik} = \lambda_k/\lambda_i$ for all i, k and $\lambda_i\theta_i = \theta$ for all i. Thus, $\theta_k \in \Bbbk^\times\theta_1$ for all k. □

Corollary 2.12. *We have $\bar{\theta}_k = \bar{\theta}_1$ for all k if and only if (S, μ) is a twist (in the sense of Definition 1.3) of the polynomial ring on n variables.*

Proof. The result follows by combining Lemma 2.10 with Proposition 2.11. □

Lemma 2.13. *If $\bar{\theta}_k = \bar{\theta}_1$ for all k, then $\bar{\theta}_k((a_{ij})) = (\mu_{ji} a_{ij})$ for all k and for all $(a_{ij}) \in M(n, \Bbbk)$, where $M(n, \Bbbk)$ is identified with $V \otimes_{\Bbbk} V^*$ as in Remark 2.8.*

Proof. By identifying $M(n, \Bbbk)$ with $V \otimes_{\Bbbk} V^*$, we may write $(a_{ij}) \in M(n, \Bbbk)$ as

$$(a_{ij}) = \left(z_1 \otimes \sum_{i=1}^{n} a_{1i} z_i^*\right) + \left(z_2 \otimes \sum_{i=1}^{n} a_{2i} z_i^*\right) + \cdots + \left(z_n \otimes \sum_{i=1}^{n} a_{ni} z_i^*\right).$$

If $\bar{\theta}_k = \bar{\theta}_1$ for all k, then

$$\begin{aligned}\bar{\theta}_k((a_{ij})) &= \sum_{j=1}^{n} \bar{\theta}_k\left(z_j \otimes \sum_{i=1}^{n} a_{ji} z_i^*\right) = \sum_{j=1}^{n} \bar{\theta}_j\left(z_j \otimes \sum_{i=1}^{n} a_{ji} z_i^*\right) \\ &= \sum_{j=1}^{n}\left(\theta_j(z_j) \otimes \sum_{i=1}^{n} a_{ji} \theta_j^*(z_i^*)\right) = \sum_{j=1}^{n}\left(z_j \otimes \sum_{i=1}^{n} \mu_{ij} a_{ji} z_i^*\right) = (\mu_{ji} a_{ij}). \quad \square\end{aligned}$$

Lemma 2.13 motivates the following definition.

Definition 2.14. Define $\bar{\mu} : M(n, \Bbbk) \to M(n, \Bbbk)$ by $\bar{\mu}((a_{ij})) = (\mu_{ji} a_{ij})$ for all $(a_{ij}) \in M(n, \Bbbk)$.

Moreover, $\bar{\mu} = (\ )^T \circ \hat{\mu} \circ (\ )^T$, where $\hat{\mu}$ is defined just prior to Lemma 2.2. Clearly, $\bar{\mu}$ is linear; with the assumption on μ at the start of Section 2B, $\bar{\mu}$ is also invertible.

Theorem 2.15. *The map $\bar{\mu}$ is an automorphism of $M(n, \Bbbk)$ if and only if the algebra (S, μ) is a twist of the polynomial ring on n variables.*

Proof. Identify $M(n, \Bbbk)$ with $V \otimes V^*$ as in Remark 2.8, so that we may view $\bar{\mu} : V \otimes V^* \to V \otimes V^*$. In particular, $\bar{\mu}(z_i \otimes z_j^*) = \mu_{ji}(z_i \otimes z_j^*)$ for all i, j. If (S, μ) is a twist of the polynomial ring, then $\bar{\mu} = \bar{\theta}_k$ for all k by Corollary 2.12 and Lemma 2.13. Hence $\bar{\mu}$ is an automorphism by Lemma 2.9.

Conversely, suppose $\bar{\mu}$ is an automorphism. It follows that

$$\bar{\mu}\big((z_j \otimes z_k^*)(z_k \otimes z_i^*)\big) = \bar{\mu}(z_j \otimes z_k^*)\bar{\mu}(z_k \otimes z_i^*)$$

for all i, j, k. Hence,

$$\bar{\mu}\big(z_k^* z_k (z_j \otimes z_i^*)\big) = \mu_{kj}(z_j \otimes z_k^*)\mu_{ik}(z_k \otimes z_i^*)$$

for all i, j, k, so that we have

$$\mu_{ij}(z_j \otimes z_i^*) = \mu_{ik}\mu_{kj}(z_j \otimes z_i^*)$$

for all i, j, k. It follows that $\mu_{ij} = \mu_{ik}\mu_{kj}$ for all i, j, k, so that (S, μ) is a twist of the polynomial ring by [Nafari and Vancliff 2015, Lemma 2.2]. $\square$

Corollary 2.16. *The algebra (S, μ) is a twist of the polynomial ring if and only if $[XY]^{\mu T} = Y^{\mu T} X^{\mu T}$ for all $X, Y \in M(n, \Bbbk)$.*

Proof. Identify $M(n, \Bbbk)$ with $V \otimes V^*$ as in Remark 2.8. Considering Definitions 2.1 and 2.14, $X^{\mu T} = [\bar{\mu}(X)]^T$ for all $X \in M(n, \Bbbk)$. By Theorem 2.15, (S, μ) is a twist of the polynomial ring if and only if $\bar{\mu}$ is an automorphism, that is, if and only if $\bar{\mu}(XY) = \bar{\mu}(X)\bar{\mu}(Y)$ for all $X, Y \in M(n, \Bbbk)$. However, this holds if and only if $[\bar{\mu}(XY)]^T = [\bar{\mu}(X)\bar{\mu}(Y)]^T = [\bar{\mu}(Y)]^T[\bar{\mu}(X)]^T$ for all $X, Y \in M(n, \Bbbk)$, that is, if and only if $[XY]^{\mu T} = Y^{\mu T} X^{\mu T}$ for all $X, Y \in M(n, \Bbbk)$. □

In view of this last result, it is clearer why the technical condition on μ is required in Corollary 2.7; the insight is that $\bar{\mu}$ needs to be an automorphism in order to have the μ-transpose map be an antiautomorphism, but that condition on $\bar{\mu}$ allows n automorphisms of (S, μ) to "merge" into one automorphism (denoted θ in the proof of Proposition 2.11) that twists (S, μ) to the polynomial ring.

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