involve
a journal of mathematics

Peak sets of classical Coxeter groups

Alexander Diaz-Lopez, Pamela E. Harris,
Erik Insko and Darleen Perez-Lavin

msp

2017 vol. 10, no. 2

INVOLVE 10:2 (2017)
dx.doi.org/10.2140/involve.2017.10.263

Peak sets of classical Coxeter groups

Alexander Diaz-Lopez, Pamela E. Harris,
Erik Insko and Darleen Perez-Lavin

(Communicated by Stephan Garcia)

We say a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ in the symmetric group $\mathfrak{S}_n$ has a *peak* at index i if $\pi_{i-1} < \pi_i > \pi_{i+1}$ and we let $P(\pi) = \{i \in \{1, 2, \dots, n\} \mid i \text{ is a peak of } \pi\}$. Given a set S of positive integers, we let $P(S; n)$ denote the subset of $\mathfrak{S}_n$ consisting of all permutations π where $P(\pi) = S$. In 2013, Billey, Burdzy, and Sagan proved $|P(S; n)| = p(n)2^{n-|S|-1}$, where $p(n)$ is a polynomial of degree $\max(S)-1$. In 2014, Castro-Velez et al. considered the Coxeter group of type B_n as the group of signed permutations on n letters and showed that $|P_B(S; n)| = p(n)2^{2n-|S|-1}$, where $p(n)$ is the same polynomial of degree $\max(S)-1$. In this paper we partition the sets $P(S; n) \subset \mathfrak{S}_n$ studied by Billey, Burdzy, and Sagan into subsets of permutations that end with an ascent to a fixed integer k (or a descent to a fixed integer k) and provide polynomial formulas for the cardinalities of these subsets. After embedding the Coxeter groups of Lie types C_n and D_n into $\mathfrak{S}_{2n}$, we partition these groups into bundles of permutations $\pi_1\pi_2\cdots\pi_n \mid \pi_{n+1}\cdots\pi_{2n}$ such that $\pi_1\pi_2\cdots\pi_n$ has the same relative order as some permutation $\sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$. This allows us to count the number of permutations in types C_n and D_n with a given peak set S by reducing the enumeration to calculations in the symmetric group and sums across the rows of Pascal's triangle.

1. Introduction

We say a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ in the symmetric group $\mathfrak{S}_n$ has a *peak* at index i if $\pi_{i-1} < \pi_i > \pi_{i+1}$. We let $[n] := \{1, 2, \dots, n\}$ and define the peak set of a permutation π to be the set of peaks in π:

$$P(\pi) = \{i \in [n] \mid \ i \text{ is a peak of } \pi\}.$$

Given a subset $S \subset [n]$, we denote the set of all permutations with peak set S by

$$P(S; n) = \{\pi \in \mathfrak{S}_n \mid P(\pi) = S\}.$$

MSC2010: 05A05, 05A10, 05A15.
Keywords: binomial coefficient, peak, permutation, signed permutation, permutation pattern.
This research was performed while Harris held a National Research Council Research Associateship Award at USMA/ARL.

We say a set $S \subset [n]$ is *n-admissible* (or simply *admissible* when n is understood) provided $P(S; n) \neq \varnothing$.

While the combinatorics of Coxeter groups has fascinated mathematicians for generations [Björner and Brenti 2005], the combinatorics of peaks has only recently caught the eye of the mathematical community. Stembridge [1997] was one of the first to study the combinatorics of peaks; he defined a peak analog of Stanley's theory of poset partitions. Nyman [2003] showed that taking formal sums of permutations according to their peak sets gives a nonunital subalgebra of the group algebra of the symmetric group. This motivated several papers studying peak (and descent) algebras of classical Coxeter groups [Aguiar et al. 2004; 2006b, Bergeron and Hohlweg 2006; Petersen 2007]. Peaks have also been linked to the Schubert calculus of isotropic flag manifolds [Bergeron et al. 2002; Bergeron and Sottile 2002; Billey and Haiman 1995] and the generalized Dehn–Sommerville equations [Aguiar et al. 2006a; Bergeron et al. 2000; Billera et al. 2003].

Billey, Burdzy, and Sagan [Billey et al. 2013, Theorem 1.1] counted the number of elements in the sets $P(S; n)$. For any n-admissible set S, they found these cardinalities satisfy

$$|P(S; n)| = p(n)2^{n-|S|-1}, \tag{1}$$

where $|S|$ denotes the cardinality of the set S, and where the *peak polynomial* $p(n)$ is a polynomial of degree $\max(S)-1$ that takes integral values when evaluated at integers. Their study was motivated by a problem in probability theory which explored the mass distribution on graphs as it relates to random permutations with specific peak sets; this research was presented in [Billey et al. 2015]. Billey, Burdzy, and Sagan also computed closed formulas for the peak polynomials $p(n)$ for various special cases of $P(S; n)$ using the method of finite differences, and Billey, Fahrbach, and Talmage [Billey et al. 2016] then studied the coefficients and zeros of peak polynomials.

Shortly after Billey, Burdzy, and Sagan's article appeared on the arXiv, Kasraoui [2012] proved one of their open conjectures and identified the most probable peak set for a random permutation. Then Castro-Velez et al. [2013] generalized the work of Billey, Burdzy, and Sagan to study peak sets of type-B signed permutations. They studied two sets $P_B(S; n)$ and $\hat{P}_B(S; n)$ of signed permutations with peak set S, whose formal definition we introduce in Section 3B. Their main result regarding the set $P_B(S; n)$ [Castro-Velez et al. 2013, Theorem 2.4] used induction to prove

$$|P_B(S; n)| = |P(S; n)|2^n = p(n)2^{2n-|S|-1}. \tag{2}$$

Note that $p(n)$ is the same polynomial as that of (1).

Motivated by extending the above-mentioned results to other classical Coxeter groups, our work begins by partitioning the sets $P(S; n)$ studied by Billey, Burdzy, and Sagan into subsets $P(S; n)^{\nearrow k}$ and $P(S; n)_{\searrow k}$ of permutations ending with an ascent or a descent to a fixed k, respectively. With these partitions on hand, we show in Theorems 11 and 12 that the cardinalities of these sets are governed by polynomial formulas similar to those discovered by Billey, Burdzy, and Sagan. These results are presented in Section 2.

We then embed the Coxeter groups of types C_n and D_n into $\mathfrak{S}_{2n}$ and call these embedded subgroups $\mathcal{C}_n, \mathcal{D}_n \subset \mathfrak{S}_{2n}$ the *mirrored permutations* of types C_n and D_n, respectively (Section 3). For each $\pi \in \mathfrak{S}_n$, we define the *pattern bundle* of π in types C_n and D_n in Definitions 14 and 17. Each *pattern bundle* consists of permutations $\tau_1\tau_2\cdots\tau_n \mid \tau_{n+1}\cdots\tau_{2n}$ such that $\tau_1\tau_2\cdots\tau_n$ flattens to $\pi_1\pi_2\cdots\pi_n$, meaning $\tau_1\tau_2\cdots\tau_n$ has the same relative order as $\pi_1\pi_2\cdots\pi_n$. These pattern bundles have the following properties: (1) they partition the groups $\mathcal{C}_n$ and $\mathcal{D}_n$; (2) they are indexed by the elements of $\mathfrak{S}_n$, and; (3) they have size 2^n in C_n and 2^{n-1} in D_n. This process allows us to give concise proofs of the following two identities (Theorem 24(I) and (II), respectively):

$$|P_C(S;n)| = p(n)2^{2n-|S|-1} \quad \text{and} \quad |P_D(S;n)| = p(n)2^{2n-|S|-2}.$$

We note that the polynomial appearing above is the same as that of (1). Moreover, the proof of Theorem 24(I) is much shorter than the one given by [Castro-Velez et al. 2013, Theorem 2.4], and Theorem 24(II) has not appeared before in the literature.

Finally in Section 4 we prove our main result, Theorem 26. We use the formulas for $|P(S;n)^{\nearrow k}|$ and $|P(S;n)_{\searrow k}|$ from Section 2 and sums of binomial coefficients to enumerate the set of permutations with peak set $S \subset [n]$ in C_n and D_n.

We end this introduction with a remark on the history of this collaboration. The last three authors of this article began their study of peak sets in classical Coxeter groups before Castro-Velez et al. had published their results from type B_n, and focused their study on the Coxeter (Weyl) groups of types C_n and D_n using presentations of these groups described in [Billey and Lakshmibai 2000, pp. 29, 34]. While Perez-Lavin was presenting the preliminary results of this paper at the USTARS 2014 conference held at UC Berkeley, we met Alexander Diaz-Lopez, who told us of his recently completed work with Castro-Velez et al. [2013]. Knowing that the Coxeter groups of types B and C are isomorphic, we were immediately intrigued to see what connections could be found between the two works. We were delighted to find that we used vastly different techniques to count the elements of $P_B(S;n)$ and $P_C(S;n)$, and discovered an isomorphism between the two groups which preserves peak sets (up to a reordering of the peaks). We highlight these connections and compare and contrast the two works in Section 3B.

2. Partitioning the set $P(S;n)$

To make our approach precise, we begin by setting notation and giving some definitions.

Definition 1. For a given peak set $S \subset [n-1]$, we define

$$P(S;n)^{\nearrow k} := \{\pi \in P(S;n) \mid \pi_{n-1} < \pi_n \text{ and } \pi_n = k\}, \quad \overline{P(S;n)} := \bigsqcup_{k=1}^{n} P(S;n)^{\nearrow k},$$

$$P(S;n)_{\searrow k} := \{\pi \in P(S;n) \mid \pi_{n-1} > \pi_n \text{ and } \pi_n = k\}, \quad \underline{P(S;n)} := \bigsqcup_{k=1}^{n} P(S;n)_{\searrow k}.$$

We remark that $P(S;n)^{\nearrow 1} = \varnothing$ because a permutation cannot end with an ascent to 1. Similarly $P(S;n)_{\searrow n} = \varnothing$ since a permutation cannot end with a descent to n. Therefore the sets $\overline{P(S;n)}$ and $\underline{P(S;n)}$ are the disjoint unions of sets

$$\overline{P(S;n)} = \bigsqcup_{k=2}^{n} P(S;n)^{\nearrow k} \quad \text{and} \quad \underline{P(S;n)} = \bigsqcup_{k=1}^{n-1} P(S;n)_{\searrow k}.$$

Since every $\pi \in P(S;n)$ either ends with an ascent or a descent, we see

$$P(S;n) = \overline{P(S;n)} \sqcup \underline{P(S;n)}.$$

Our next lemma counts the permutations without peaks that end with an ascent to k.

Lemma 2. *If* $2 \le k \le n$, *then* $|P(\varnothing;n)^{\nearrow k}| = 2^{k-2}$.

Proof. Let $2 \le k \le n$ and suppose $\pi = \pi_1\pi_2\cdots\pi_n \in P(\varnothing;n)^{\nearrow k}$. Hence $P(\pi) = \varnothing$ and $\pi_{n-1} < \pi_n = k$. Let us further assume that $\pi = \tau_A\, 1\, \tau_B\, k$, where τ_A and τ_B are the portions of π to the left and right of 1, respectively. Since $P(\pi) = \varnothing$, we know τ_A must decrease, while τ_B must increase. However, the values of τ_B must come from the set $\{2, 3, \dots, k-1\}$ because $\pi_{n-1} < \pi_n = k$, and there is one $\pi \in P(\varnothing;n)^{\nearrow k}$ for each subset of $\{2, 3, \dots, k-1\}$ as such a π is completely determined by which elements from that set appear in τ_B. Hence we see $|P(\varnothing;n)^{\nearrow k}| = 2^{k-2}$. □

We will next prove a recursive formula for the number of permutations with specified peak set S that end in an ascent to a fixed integer k.

Lemma 3. *Let* $S \subset [n-1]$ *be a nonempty admissible set. Let* $m = \max S$ *and fix an integer* k, *where* $1 \le k \le n$. *If* $S_1 = S \setminus \{m\}$ *and* $S_2 = S_1 \cup \{m-1\}$, *then*

$$|P(S;n)^{\nearrow k}| = \sum_{i=0}^{k-2} \binom{k-1}{i}\binom{n-k}{m-i-1} |P(S_1;m-1)| 2^{k-i-2} - |P(S_1;n)^{\nearrow k}| - |P(S_2;n)^{\nearrow k}|.$$

Proof. Observe that if $k = 1$, then the result holds trivially as all terms in the statement are identically zero. Let $2 \le k \le n$ and let $\Pi^{\nearrow k}$ denote the set of permutations ending with an ascent to k that have peak set S_1 in the first $m-1$ spots and no peaks in the last $m-n+1$, i.e.,

$$\Pi^{\nearrow k} = \{\pi \in \mathfrak{S}_n \mid P(\pi_1\pi_2\cdots\pi_{m-1}) = S_1,\ P(\pi_m\cdots\pi_n) = \varnothing, \text{ and } \pi_{n-1} < \pi_n = k\}.$$

We compute the cardinality of the set $\Pi^{\nearrow k}$ by counting the number of ways to construct a permutation in $\Pi^{\nearrow k}$.

First we select a subset $P_1 = \{\pi_1, \pi_2, \ldots, \pi_{m-1}\} \subset [n] \setminus \{k\}$ (as we fix π_n to be k). When selecting P_1, we can choose i numbers from $\{1, 2, \ldots, k-1\}$ to include in P_1 for each $0 \le i \le k-1$ and then choose the remaining $m-i-1$ numbers from the set $\{k+1, k+2, \ldots, n\}$ to fill the remainder of P_1. Thus there are $\binom{k-1}{i} \cdot \binom{n-k}{m-i-1}$ ways to select the elements of P_1. By definition, there are $|P(S_1, m-1)|$ ways to arrange the $m-1$ elements of P_1 into a permutation $\pi_1\pi_2\cdots\pi_{m-1}$ satisfying $P(\pi_1\pi_2\cdots\pi_{m-1}) = S_1$.

Let $P_2 = \{\pi_m, \pi_{m+1}, \ldots, \pi_n\} = [n] \setminus P_1$, where $\pi_n = k$. There are $n-(m-1) = n-m+1$ numbers in P_2, and there are precisely $k-i-1$ elements from the set $\{1, 2, \ldots, k-1\}$ that were not chosen to be part of P_1. That means k is the $(k-i)$-th largest integer in the set P_2. By flattening the numbers in P_2, we can see there are $|P(\varnothing; n-m+1)^{\nearrow k-i}|$ ways to arrange the elements of P_2 to create a subpermutation $\pi_m\pi_{m+1}\cdots\pi_n$ that satisfies

$$P(\pi_m \cdots \pi_n) = \varnothing \quad \text{and} \quad \pi_{n-1} < \pi_n = k.$$

By Lemma 2 we know that $|P(\varnothing; n-m+1)^{\nearrow k-i}| = 2^{k-i-2}$ when $k-i \ge 2$ and it is 0 otherwise. Of course $k-i \ge 2$ when $i \le k-2$. Putting this all together, we see that the number of ways to create a permutation in $\Pi^{\nearrow k}$ is

$$|\Pi^{\nearrow k}| = \sum_{i=0}^{k-2} \binom{k-1}{i} \binom{n-k}{m-i-1} |P(S_1; m-1)| 2^{k-i-2}. \tag{3}$$

Next we consider a different way to count the elements of $\Pi^{\nearrow k}$. Note that we have not specified whether $\pi_{m-1} > \pi_m$ or $\pi_{m-1} < \pi_m$. So, in particular, based on the definition of $\Pi^{\nearrow k}$ and its restrictions on $P(\pi_1\pi_2\cdots\pi_{m-1})$ and $P(\pi_m\pi_{m+1}\cdots\pi_n)$, all of the following are possible:

$$P(\pi) = S, \quad P(\pi) = S_1, \quad \text{or} \quad P(\pi) = S_2 \quad \text{for } \pi \in \Pi^{\nearrow k}.$$

Hence

$$\Pi^{\nearrow k} = P(S; n)^{\nearrow k} \sqcup P(S_1; n)^{\nearrow k} \sqcup P(S_2; n)^{\nearrow k}.$$

Thus

$$|\Pi^{\nearrow k}| = |P(S; n)^{\nearrow k}| + |P(S_1; n)^{\nearrow k}| + |P(S_2; n)^{\nearrow k}|. \tag{4}$$

The result follows from setting (3) and (4) equal to each other and solving for the quantity $|P(S; n)^{\nearrow k}|$. □

The following lemma will be used in the proofs of Lemmas 5 and 9.

Lemma 4. *If $n \ge 2$ then*

- $|\underline{P(\varnothing; n)}| = 1$, *and*
- $|\overline{P(\varnothing; n)}| = 2^{n-1} - 1$.

Proof. The only permutation $\pi \in P(\varnothing; n)$ that ends in a descent is $n = \pi_1 > \pi_2 > \cdots > \pi_n = 1$; therefore $|\underline{P(\varnothing; n)}| = 1$. On the other hand, it is easy to see that $P(\varnothing; n) = 2^{n-1}$, as in [Billey et al. 2013, Proposition 2.1]. Since $P(\varnothing; n) = \overline{P(\varnothing; n)} \sqcup \underline{P(\varnothing; n)}$, we compute

$$|\overline{P(\varnothing; n)}| = |P(\varnothing; n)| - |\underline{P(\varnothing; n)}| = 2^{n-1} - 1. \qquad \square$$

The following result allows us to recursively enumerate the set of permutations with specified peak set S that end with an ascent.

Lemma 5. *Let $S \subset [n-1]$ be a nonempty n-admissible set, and let $m = \max S$. If we let $S_1 = S \setminus \{m\}$ and $S_2 = S_1 \cup \{m-1\}$, then*

$$|\overline{P(S; n)}| = \binom{n}{m-1}(2^{n-m} - 1)|P(S_1; m-1)| - |\overline{P(S_1; n)}| - |\overline{P(S_2; n)}|.$$

Proof. Let $S \subset [n-1]$ be an admissible set with $m = \max S$. Define the sets $S_1 = S \setminus \{m\}$, $S_2 = S_1 \cup \{m-1\}$ and

$$\Pi^{\nearrow} = \{\pi \in \mathfrak{S}_n \mid P(\pi_1\pi_2\cdots\pi_{m-1}) = S_1,\ P(\pi_m\cdots\pi_n) = \varnothing, \text{ and } \pi_{n-1} < \pi_n\}.$$

Next we compute the cardinality of $\Pi^{\nearrow}$. We observe that there are $\binom{n}{m-1}$ choices for the values of $\pi_1, \dots, \pi_{m-1}$, and by definition, there are $|P(S_1; m-1)|$ ways to arrange the values of $\pi_1, \dots, \pi_{m-1}$ so that $P(\pi_1\pi_2\cdots\pi_{m-1}) = S_1$. Once we have chosen the values of $\pi_1, \pi_2, \dots, \pi_{m-1}$, the values of

$$\pi_m, \pi_{m+1}, \pi_{m+2}, \dots, \pi_n$$

are determined. We note that there are $|\overline{P(\varnothing; n-m+1)}|$ ways to arrange the values of $\pi_m, \dots, \pi_n$, so that $P(\pi_m\cdots\pi_n) = \varnothing$ and $\pi_{n-1} < \pi_n$.

Yet Lemma 4 proved that $|\overline{P(\varnothing; n-m+1)}| = 2^{n-m} - 1$. Hence we see that

$$|\Pi^{\nearrow}| = \binom{n}{m-1}(2^{n-m} - 1)|P(S_1; m-1)|. \tag{5}$$

On the other hand $\Pi^{\nearrow} = \overline{P(S; n)} \sqcup \overline{P(S_1; n)} \sqcup \overline{P(S_2; n)}$ by the defining conditions of $\Pi^{\nearrow}$. Hence

$$|\Pi^{\nearrow}| = |\overline{P(S; n)}| + |\overline{P(S_1; n)}| + |\overline{P(S_2; n)}|. \tag{6}$$

When we set the right-hand sides of (5) and (6) equal to each other and solve for $|\overline{P(S; n)}|$, we see that

$$|\overline{P(S; n)}| = \binom{n}{m-1}(2^{n-m} - 1)|P(S_1; m-1)| - |\overline{P(S_1; n)}| - |\overline{P(S_2; n)}|. \qquad \square$$

The following examples illustrate the recursion used to prove Lemmas 3 and 5.

Example 6. We make use of Lemma 3 to compute $|P(\{3\};5)^{\nearrow 3}|$. Let S be the set $S=\{3\}\subset[5]$. Note that $m=\max S=3$. Then we compute

$$|P(\{3\};5)^{\nearrow 3}|=\left[\binom{2}{0}\binom{2}{2}2^1+\binom{2}{1}\binom{2}{1}2^0\right]|P(\varnothing;2)| \\ -|P(\varnothing;5)^{\nearrow 3}|-|P(\{2\};5)^{\nearrow 3}|. \quad (7)$$

Some small computations show that

$$P(\varnothing;2)=\{12,21\},\quad P(\varnothing;5)^{\nearrow 3}=\{54213,54123\},$$
$$P(\{2\};5)^{\nearrow 3}=\{45213,25413,45123,15423\}.$$

Accordingly, we can see that (7) gives

$$|P(\{3\};5)^{\nearrow 3}|=(2+4)(2)-2-4=6.$$

Example 7. In this example we make use of Lemma 5 to compute $|\overline{P(\{3\};5)}|$. If we let $S=\{3\}\subset[5]$ then $m=\max S=3$. We then have

$$|\overline{P(\{3\};5)}|=\binom{5}{2}(2^{5-3}-1)|P(\varnothing;2)|-|\overline{P(\varnothing;5)}|-|\overline{P(\{2\};5)}|. \quad (8)$$

Some small computations show

$$P(\varnothing;2)=\{12,21\},$$
$$\overline{P(\varnothing;5)}=\left\{\begin{matrix}54321,54213,54123,53214,53124,52134,51234,43215,\\ 43125,42135,32145,41235,31245,21345,12345\end{matrix}\right\}.$$

Direct computations yield

$$\overline{P(\{2\};5)} \\ =\left\{\begin{matrix}45312,35412,45213,25413,45123,15423,35214,25314,35124,25134,\\ 15324,15234,34215,24315,34125,24135,23145,14325,14235,13245\end{matrix}\right\}.$$

Equation (8) gives

$$|\overline{P(\{3\};5)}|=\binom{5}{2}(2^{5-3}-1)(2)-15-20=25.$$

Next we consider permutations that end in a descent to a specific value k.

Lemma 8. *Let $S\subset[n-1]$ be a nonempty admissible set, let $m=\max S$, and fix an integer k, where $1\le k\le n$. If $S_1=S\setminus\{m\}$ and $S_2=S_1\cup\{m-1\}$, then*

$$|P(S;n)_{\searrow k}|=\binom{n-k}{n-m}|P(S_1;m-1)|-|P(S_1;n)_{\searrow k}|-|P(S_2;n)_{\searrow k}|.$$

The proof of Lemma 8 follows similarly to that of Lemma 3; hence we omit the argument, but point the interested reader to the preprint version of this paper for a detailed proof [Diaz-Lopez et al. 2015].

The following result allows us to recursively enumerate the set of permutations with specified peak set S that end with a descent.

Lemma 9. *Let* $S \subset [n-1]$ *be a nonempty admissible set, and let* $m = \max S$. *If* $S_1 = S \setminus \{m\}$ *and* $S_2 = S_1 \cup \{m-1\}$, *then*

$$|\underline{P(S;n)}| = \binom{n}{m-1}|P(S_1;m-1)| - |\underline{P(S_1;n)}| - |\underline{P(S_2;n)}|.$$

Proof. By Definition 1,

$$|\underline{P(S;n)}| = \sum_{k=1}^{n-1} |P(S;n)_{\searrow k}|.$$

Using this equation and Lemma 8 we get

$$\begin{aligned}|\underline{P(S;n)}| &= \sum_{k=1}^{n-1}\Big[\binom{n-k}{n-m}|P(S_1;m-1)| - |P(S_1;n)_{\searrow k}| - |P(S_2;n)_{\searrow k}|\Big]\\ &= \binom{n}{m-1}|P(S_1;m-1)| - |\underline{P(S_1;n)}| - |\underline{P(S_2;n)}|,\end{aligned}$$

where the last equality comes from the identity $\sum_{k=0}^{n}\binom{k}{c} = \binom{n+1}{c+1}$. □

As before, we provide an example that illustrates the use of the previous results.

Example 10. Consider the set $S = \{3\} \subset [5]$; hence $m = \max S = 3$. We want to compute $|P(\{3\};5)_{\searrow 2}|$. By Lemma 8 we have

$$|P(\{3\};5)_{\searrow 2}| = \binom{3}{2}|P(\varnothing;2)| - |P(\varnothing;5)_{\searrow 2}| - |P(\{2\};5)_{\searrow 2}|.$$

Some simple computations show that

$$P(\varnothing;2) = \{12, 21\}, \quad P(\varnothing;5)_{\searrow 2} = \varnothing, \quad \text{and} \quad P(\{2\};5)_{\searrow 2} = \{15432\}.$$

Therefore

$$|P(\{3\};5)_{\searrow 2}| = 3(2) - 0 - 1 = 5.$$

In fact, $P(\{3\};5)_{\searrow 2} = \{51432, 41532, 31542, 14532, 13542\}$.

We want to compute $|\underline{P(\{3\};5)}|$. By Lemma 9 we have

$$|\underline{P(\{3\};5)}| = \binom{5}{2}|P(\varnothing;2)| - |\underline{P(\varnothing;5)}| - |\underline{P(\{2\};5)}|.$$

Again we can compute that

$$\begin{gathered}P(\varnothing;2) = \{12, 21\}, \quad \underline{P(\varnothing;5)} = \{54321\},\\ \underline{P(\{2\};5)} = \{45321, 35421, 25431, 15432\}.\end{gathered}$$

Thus

$$|\underline{P(\{3\};5)}| = 10(2) - 1 - 4 = 15.$$

In fact,

$$\underline{P(\{3\};5)} = \left\{ \begin{matrix} 53421, 43521, 34521, 52431, 42531, 32541, 24531, \\ 23541, 51432, 41532, 31542, 21543, 14532, 13542, 12543 \end{matrix} \right\}.$$

The following two theorems allow us to easily calculate closed formulas for $|\underline{P(S;n)}|$ and $|P(S;n)^{\nearrow k}|$ using the method of finite differences [Stanley 2012, Proposition 1.9.2]. We start by applying Lemma 9 in an induction argument to show $|\underline{P(S;n)}|$ is given by a polynomial $p_\delta(n)$.

Theorem 11. *Let $S \subset [n-1]$ be an admissible set. If $S = \varnothing$, take $m = 1$; otherwise let $m = \max S$. Then the cardinality of the set $\underline{P(S;n)}$ is given by*

$$|\underline{P(S;n)}| = p_\delta(n),$$

where $p_\delta(n)$ is a polynomial in the variable n of degree $m-1$ that returns integer values for all integers n.

Proof. We induct on the sum $\mathfrak{i} = \sum_{i\in S} i$. When $\mathfrak{i} = 0$, the set S is empty. By Lemma 4 we get $|\underline{P(\varnothing;n)}| = 1$, and so $p_\delta(n) = 1$ is a polynomial of degree 0.

Let $S \subset [n-1]$ be nonempty, with $m = \max S$ and $\sum_{i\in S} i = \mathfrak{i}$. Let $S_1 = S \setminus \{m\}$ and $S_2 = S_1 \cup \{m-1\}$, and note, in particular, that the sums $\sum_{i\in S_1} i$ and $\sum_{i\in S_2} i$ are both strictly less than $\mathfrak{i}$. By induction, we know $|\underline{P(S_1;n)}| = p_{\delta_1}(n)$ and $|\underline{P(S_2;n)}| = p_{\delta_2}(n)$, where p_{δ_1} and p_{δ_2} are polynomials of degrees less than $m-1$ that have integral values when evaluated at integers.

By (1) we have $|P(S_1;m-1)| = p(m-1)2^{(m-1)-|S_1|-1}$ and this expression returns an integer value when evaluated at any integer $m-1$ [Billey et al. 2013, Theorem 2.2]. Since the expression $p(m-1)2^{(m-1)-|S_1|-1}$ is an integer-valued constant with respect to n, we see that $p(m-1)2^{(m-1)-|S_1|-1}\binom{n}{m-1}$ is a polynomial in the variable n of degree $m-1$. These facts, together with Lemma 9, imply

$$\begin{aligned} |\underline{P(S;n)}| &= \binom{n}{m-1}|P(S_1;m-1)| - |\underline{P(S_1;n)}| - |\underline{P(S_2;n)}| \\ &= \binom{n}{m-1} p(m-1)2^{(m-1)-|S_1|-1} - p_{\delta_1} - p_{\delta_2} \\ &= p_\delta, \end{aligned}$$

where p_δ is a polynomial in the variable n of degree $m-1$ that has integer values when evaluated at integers. □

Using Lemma 3, we show $|P(S;n)^{\nearrow k}|$ is given by a polynomial.

Theorem 12. *Let $S \subset [n-1]$ be an admissible set. If $S = \varnothing$ take $m = 1$; otherwise let $m = \max S$. Fix an integer k satisfying $2 \le k \le n$; then the cardinality of the set $P(S;n)^{\nearrow k}$ is given by*

$$|P(S;n)^{\nearrow k}| = p_{\alpha(k)}(n),$$

where $p_{\alpha(k)}(n)$ is a polynomial of degree $m-1$ that returns integer values for all integers n.

Proof. We proceed by induction on the sum $\mathfrak{i} = \sum_{i\in S} i$. When $\mathfrak{i} = 0$ the set S is empty. By Lemma 2 we get $|P(\varnothing; n)^{\nearrow k}| = 2^{k-2}$, which is a polynomial of degree 0 in the indeterminate n.

Consider a nonempty subset $S \subset [n-1]$ with $m = \max S$ and $\sum_{i\in S} i = \mathfrak{i}$. Let $S_1 = S \setminus \{m\}$ and $S_2 = S_1 \cup \{m-1\}$, and note, in particular, that the sums $\sum_{i\in S_1} i$ and $\sum_{i\in S_2} i$ are both strictly less than $\mathfrak{i}$. By induction, we know

$$|P(S_1; n)^{\nearrow k}| = p_{\alpha_1(k)}(n) \quad \text{and} \quad |P(S_2; n)^{\nearrow k}| = p_{\alpha_2(k)}(n),$$

where $p_{\alpha_1(k)}(n)$ and $p_{\alpha_2(k)}(n)$ are each polynomials of degrees less than $m-1$ that have integer values when evaluated at integers.

By (1) we know $|P(S_1; m-1)| = p(m-1)2^{(m-1)-|S_1|-1}$ is an integer-valued function when evaluated at any integer $m-1$, and it is a constant function with respect to n. Hence the expression $\binom{k-1}{i}|P(S_1; m-1)|2^{k-i-2}$ is a polynomial expression in n that has degree $m-1$ when $i = 0$ and degree less than or equal to $m-1$ for $1 \le i \le k-2$. These facts, together with Lemma 3, imply

$$\begin{aligned}|P(S;n)^{\nearrow k}| &= \sum_{i=0}^{k-2}\binom{k-1}{i}\binom{n-k}{m-i-1}|P(S_1;m-1)|2^{k-i-2}\\ &\qquad\qquad -|P(S_1;n)^{\nearrow k}|-|P(S_2;n)^{\nearrow k}|\\ &= \sum_{i=0}^{k-2}\binom{k-1}{i}\binom{n-k}{m-i-1}|P(S_1;m-1)|2^{k-i-2}-p_{\alpha_1(k)}(n)-p_{\alpha_2(k)}(n)\\ &= p_{\alpha(k)}(n)\end{aligned}$$

is a polynomial in the variable n of degree $m-1$ that returns integer values when evaluated at integers. □

Below we show an example of how to find the polynomial $p_{\alpha(k)}(n)$.

Example 13. It is well known that any sequence given by a polynomial of degree d can be completely determined by any consecutive $d+1$ values by the method of finite differences [Stanley 2012, Proposition 1.9.2]. Theorems 11 and 12 give us a way of finding explicit formulas $p_{\alpha(k)}(n)$ and $p_\delta(n)$ for an admissible set S.

For instance if $S = \{2, 4\}$ and $k = 6$, Theorem 12 tells us $p_{\alpha(k)}(n)$ is a polynomial of degree 3. Hence we require four consecutive terms to compute $p_{\alpha(k)}(n)$. One can compute that the first few values of $p_{\alpha(6)}(n) = |P(S; n)^{\nearrow 6}|$ are

$$\begin{aligned}&p_{\alpha(6)}(6) = 16, \quad &&p_{\alpha(6)}(7) = 80, \quad &&p_{\alpha(6)}(8) = 224,\\ &p_{\alpha(6)}(9) = 480, \quad &&p_{\alpha(6)}(10) = 880, \quad &&p_{\alpha(6)}(11) = 1456, \quad \ldots.\end{aligned}$$

We then take four successive differences until we get a row of zeros in the following array:

$$\begin{array}{cccccc} 16 & 80 & 224 & 480 & 880 & 1456 \ \dots \\ 64 & 144 & 256 & 400 & 576 & \dots \\ 80 & 112 & 144 & 176 & \dots & \\ 32 & 32 & 32 & \dots & & \\ 0 & 0 & \dots & & & \end{array}$$

Since the first value we considered in the first row of the array above is the value of $p_{\alpha(6)}(n)$ at $n=6$, we can write the polynomial $p_{\alpha(6)}(n)$ in the basis $\binom{n-6}{j}$ as

$$p_{\alpha(6)}(n) = 16\binom{n-6}{6} + 64\binom{n-6}{7} + 80\binom{n-6}{8} + 32\binom{n-6}{9}.$$

The sequence given by $\frac{1}{16}p_{\alpha(6)}(n)$ in this example is sequence A000330 in Sloane's *On-line encyclopedia of integer sequences* [OEIS 1996] with the index n shifted by 6.

3. Pattern bundles of Coxeter groups of types *C* and *D*

In this section, we describe embeddings of the Coxeter groups of types C_n and D_n into the symmetric group $\mathfrak{S}_{2n}$. We then partition these groups into subsets, which we call *pattern bundles* and denote by $\mathcal{C}_n(\pi)$ and $\mathcal{D}_n(\pi)$, that correspond to permutations π of $\mathfrak{S}_n$. Each of the *type-C_n pattern bundles* $\mathcal{C}_n(\pi)$ contains 2^n elements, and the *type-D_n pattern bundles* $\mathcal{D}_n(\pi)$ contain 2^{n-1} elements. These sets allow us to give a concise proof of Theorem 24, and they play an instrumental role in our proof of Theorem 26.

3A. ***Pattern bundle algorithms for $\mathcal{C}_n$ and $\mathcal{D}_n$.*** We define the group of type-C_n *mirrored permutations* to be the subgroup $\mathcal{C}_n \subset \mathfrak{S}_{2n}$ consisting of all permutations $\pi_1\pi_2\cdots\pi_n \,|\, \pi_{n+1}\pi_{n+2}\cdots\pi_{2n} \in \mathfrak{S}_{2n}$, where $\pi_i = k$ if and only if $\pi_{2n-i+1} = 2n-k+1$. In other words, if we place a "mirror" between π_n and π_{n+1}, then the numbers i and $2n-i+1$ must be the same distance from the mirror for each $1 \le i \le n$. A simple transposition s_i with $1 \le i \le n-1$ acts on a mirrored permutation $\pi \in \mathcal{C}_n \subset \mathfrak{S}_{2n}$ (on the right) by simultaneously transposing π_i with π_{i+1} and π_{2n-i} with π_{2n-i+1}. The simple transposition s_n acts on a mirrored permutation $\pi \in \mathcal{C}_n \subset \mathfrak{S}_{2n}$ by transposing π_n with π_{n+1}.

Similarly, we define the group of type-D_n *mirrored permutations* as the subgroup $\mathcal{D}_n \subset \mathfrak{S}_{2n}$ consisting of all permutations $\pi_1\pi_2\cdots\pi_n \,|\, \pi_{n+1}\pi_{n+2}\cdots\pi_{2n} \in \mathfrak{S}_{2n}$, where $\pi_i = k$ if and only if $\pi_{2n-i+1} = 2n-k+1$ and the set $\{\pi_1, \pi_2, \dots, \pi_n\}$ always contains an even number of elements from the set $\{n+1, n+2, \dots, 2n\}$. A simple transposition s_i with $1 \le i \le n-1$ acts on a mirrored permutation $\pi \in \mathcal{D}_n \subset \mathfrak{S}_{2n}$ (on the right) by simultaneously transposing π_i with π_{i+1} and π_{2n-i} with π_{2n-i+1}. The simple transposition s_n acts on a mirrored permutation $\pi \in \mathcal{D}_n \subset S_{2n}$ by transposing $\pi_{n-1}\pi_n$ with $\pi_{n+1}\pi_{n+2}$.

Definition 14. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. We define the *pattern bundle of* π in type C_n (denoted $\mathcal{C}_n(\pi)$) to be the set of all mirrored permutations

$$\tau = \tau_1\tau_2\cdots\tau_n \mid \tau_{n+1}\tau_{n+2}\cdots\tau_{2n} \in \mathcal{C}_n$$

such that $\tau_1\tau_2\cdots\tau_n$ has the same relative order as $\pi_1\pi_2\cdots\pi_n$.

We could equivalently describe $\mathcal{C}_n(\pi)$ as the set of mirrored permutations which contain the *permutation pattern* π in the first n entries. We will show that these sets partition $\mathcal{C}_n$ into subsets of size 2^n. For every $\pi \in \mathfrak{S}_n$, we will describe how to construct the *pattern bundle* $\mathcal{C}_n(\pi) \subset \mathcal{C}_n$ of π using the following process:

Algorithm 15 (pattern bundle algorithm for $\mathcal{C}_n(\pi)$).

(1) Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ and write it as a mirrored permutation

$$\pi_1\pi_2\cdots\pi_n \mid \pi_{n+1}\pi_{n+2}\cdots\pi_{2n} \in \mathfrak{S}_{2n}.$$

(2) Let $I_n = \{\pi_1, \pi_2, \ldots, \pi_n\}$. Fix $0 \le j \le n$ and select j elements from the set I_n. Then let Π be the set consisting of the j selected elements.

(3) The set $I_n \setminus \Pi$ consists of $n - j$ elements. Denote this subset of I_n by Π^c.

(4) Let $\overline{\Pi^c}$ denote the set containing $\pi_{2n-i_k+1} = 2n - \pi_{i_k} + 1$ for each $\pi_{i_k} \in \Pi^c$. Note that $|\overline{\Pi^c}| = n - j$.

(5) List the n elements of the set

$$\overline{\Pi^c} \sqcup \Pi$$

so that they are in the same relative order as π and call them $\tau_1\tau_2\cdots\tau_n$. (Note that the set $\overline{\Pi^c}$ consists of the integers that were switched in Step (4), and the set Π consists of the ones that were fixed in Step (2).)

(6) The order of the remaining entries $\tau_{n+1}\tau_{n+2}\cdots\tau_{2n}$ is determined by that of $\tau_1\tau_2\cdots\tau_n$ since we must have $\tau_{2n-i+1} = 2n - \tau_i + 1$ for $1 \le i \le n$.

(7) Output the mirrored permutation $\tau_1\tau_2\cdots\tau_n \mid \tau_{n+1}\tau_{n+2}\cdots\tau_{2n} \in \mathcal{C}_n \subset \mathfrak{S}_{2n}$ and stop.

Step (5) ensures all of the constructed elements will have the same relative order as π. It follows that the set $\mathcal{C}_n(\pi)$ described in Definition 14 denotes all elements of $\mathcal{C}_n$ created from π by Algorithm 15. Notice in Step (2), we must choose j values to fix. When we let j range from 0 to n, we see that the total number of elements in $\mathcal{C}_n(\pi)$ is given by

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

We conclude that $|\mathcal{C}_n(\pi)| = 2^n$ for all $\pi \in \mathfrak{S}_n$.

Note that if $\tau = \tau_1\tau_2\cdots\tau_n \mid \tau_{n+1}\tau_{n+2}\cdots\tau_{2n} \in \mathcal{C}_n$, then $\tau_1\cdots\tau_n$ has the same relative order as exactly one element $\pi \in \mathfrak{S}_n$. It follows that if σ and π are distinct permutations of $\mathfrak{S}_n$, then $\mathcal{C}_n(\sigma) \cap \mathcal{C}_n(\pi) = \varnothing$. Therefore, this process creates all $2^n n!$ elements of $\mathcal{C}_n$.

Example 16. Using Algorithm 15, we have partitioned the elements of $\mathcal{C}_3$ into the pattern bundles $\mathcal{C}_n(\pi)$:

$$\mathcal{C}_3 = \left\{\begin{matrix} \mathbf{123|456}, & \mathbf{132|546}, & \mathbf{213|465}, & \mathbf{231|645}, & \mathbf{312|564}, & \mathbf{321|654}, \\ 124|356, & 142|536, & 214|365, & 241|635, & 412|563, & 421|653, \\ 135|246, & 153|426, & 315|264, & 351|624, & 513|462, & 531|642, \\ 145|236, & 154|326, & 326|154, & 362|514, & 514|362, & 541|632, \\ 236|145, & 263|415, & 415|326, & 451|623, & 623|451, & 632|541, \\ 246|135, & 264|315, & 426|153, & 462|513, & 624|351, & 642|531, \\ 356|124, & 365|214, & 536|142, & 563|412, & 635|241, & 653|421, \\ 456|123, & 465|213, & 546|132, & 564|312, & 645|231, & 654|321 \end{matrix}\right\}.$$

One can see that the elements of $\pi \in \mathfrak{S}_3$ correspond to the elements in the top row (in bold font). Each column consists of the pattern bundle $\mathcal{C}(\pi)$ corresponding to each $\pi \in \mathfrak{S}_3$.

Definition 17. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. We define the *pattern bundle* $\mathcal{D}_n(\pi)$ to be the set of all mirrored permutations $\tau = \tau_1\tau_2\cdots\tau_n \mid \tau_{n+1}\cdots\tau_{2n} \in \mathcal{D}_n$ such that $\tau_1\tau_2\cdots\tau_n$ has the same relative order as $\pi_1\pi_2\cdots\pi_n$.

For every $\pi \in \mathfrak{S}_n$, we construct the subsets $\mathcal{D}_n(\pi) \subset \mathcal{D}_n$ using the following process:

Algorithm 18 (pattern bundle algorithm for $\mathcal{D}_n(\pi)$).

(1) Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ and write it as a mirrored permutation

$$\pi_1\pi_2\cdots\pi_n \mid \pi_{n+1}\pi_{n+2}\cdots\pi_{2n} \in \mathfrak{S}_{2n}.$$

(2) If n is even, then pick an even number $2j$, with $0 \le j \le \frac{1}{2}n$. Select a subset of $2j$ elements from the set $\{\pi_1, \pi_2, \ldots, \pi_n\}$ to keep fixed. Then let Π be the set consisting of the $2j$ selected elements.
If n is odd, then pick an odd number $2j+1$ with $1 \le j \le \frac{1}{2}(n-1)$. Select a subset of $2j+1$ elements from the set $\{\pi_1, \pi_2, \ldots, \pi_n\}$ to keep fixed. Then let Π be the set consisting of the $2j+1$ selected elements.

(3) If n is even, let the set of remaining $n-2j$ elements be denoted as Π^c. (Note that $n-2j$ is an even integer.)
If n is odd, let the set of remaining $n-2j-1$ elements be denoted as Π^c. (Note that $n-2j-1$ is an even integer.)

(4) Let the set $\overline{\Pi^c}$ denote the set of mirror images from the elements of Π^c. In other words, for each $\pi_{i_k} \in \Pi^c$, the mirror image π_{2n-i_k+1} is in $\overline{\Pi^c}$.

(5) List elements of the set $\Pi \sqcup \overline{\Pi^c}$ so they are in the same relative order as π and call the resulting permutation $\tau_1\tau_2\cdots\tau_n$.

(6) Then the entries of $\tau_{n+1}\tau_{n+2}\cdots\tau_{2n}$ are determined by the relation $\tau_{2n-i_k+1} = 2n - \tau_{i_k} + 1$.

(7) Output the mirrored permutation $\tau_1\tau_2\cdots\tau_n \mid \tau_{n+1}\tau_{n+2}\cdots\tau_{2n} \in \mathcal{D}_n \subset \mathfrak{S}_{2n}$ and stop.

By Definition 17 the set $\mathcal{D}_n(\pi)$ is the subset of all elements of $\mathcal{D}_n$ which are created from π by Algorithm 18. This is because Step (5) ensures that all of the constructed elements will have the same relative order as π.

In Step (2) we choose an even/odd number of entries to fix, so that we always exchange an even number of entries with their mirror image. This ensures each τ constructed via Algorithm 18 is a type-$\mathcal{D}_n$ mirrored permutation. When n is even, we can see from Step (2) that the total number of permutations created by Algorithm 18 is given by $\sum_{j=0}^{n/2}\binom{n}{2j}$. When n is odd, we can use the identity

$$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j+1} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}, \quad \text{where } 2k = n-(2j+1),$$

to see that the total number of elements created by Algorithm 18 is also given by the formula

$$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}.$$

Pascal's identity for computing binomial coefficients states that for all integers n and k with $1 \le k \le n-1$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Using this identity we can see that

$$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j} = \sum_{j=0}^{n-1}\binom{n-1}{j} = 2^{n-1}.$$

So for every element $\pi \in \mathfrak{S}_n$, we create 2^{n-1} elements of $\mathcal{D}_n$. Hence $|\mathcal{D}_n(\pi)| = 2^{n-1}$. Also notice that for each choice of π, the 2^{n-1} elements of $\mathcal{D}_n(\pi)$ will be distinct due to the choice of which elements get sent to their mirror image. Namely, if σ and π are distinct permutations of $\mathfrak{S}_n$, then $\mathcal{D}_n(\sigma) \cap \mathcal{D}_n(\pi) = \varnothing$. Therefore, this process creates all $2^{n-1}n!$ distinct elements of $\mathcal{D}_n$.

Example 19. Using Algorithm 18, we have partitioned the set $\mathcal{D}_3$ into the pattern bundles $\mathcal{D}_n(\pi)$:

$$\mathcal{D}_3 = \begin{Bmatrix} \mathbf{123|456}, & \mathbf{132|546}, & \mathbf{213|465}, & \mathbf{231|645}, & \mathbf{312|564}, & \mathbf{321|654}, \\ 145|236, & 154|326, & 214|365, & 241|635, & 412|563, & 421|653, \\ 246|135, & 264|315, & 426|153, & 462|513, & 624|351, & 642|531, \\ 356|124, & 365|214, & 536|142, & 563|412, & 635|241, & 653|421 \end{Bmatrix}.$$

Note that the elements in the top row (in bold font) are the elements of $\mathfrak{S}_3$, while the elements in each column are the elements of the pattern bundle $\mathcal{D}_3(\pi)$ for each respective $\pi \in \mathfrak{S}_3$.

3B. ***Peak sets in types B and C.*** Castro-Velez et al. [2013] studied the sets of type-$\mathcal{B}_n$ *signed permutations* (defined below) with a given peak set $R \subset [n-1]$. It is well known that the group of signed permutations of type $\mathcal{B}_n$ is isomorphic to the Coxeter groups of types B_n and C_n. In this section, we describe one such isomorphism between the group of signed permutations $\mathcal{B}_n$ and the mirrored permutations $\mathcal{C}_n$ and show how the peak sets in mirrored permutations studied in this paper correspond with the ones studied by Castro-Velez et al. [2013]. It is important to know that even though we compute the cardinalities of similar sets, our methods are completely different and yield different equations. In particular, Castro-Velez et al. use induction arguments similar to those used by Billey, Burdzy, and Sagan in the realm of signed permutations to derive their formulas, whereas we use the pattern bundles of type $\mathcal{C}_n$ to reduce the problem to calculations in the symmetric group.

Let $\mathcal{B}_n$ denote the group of *signed permutations* on n letters

$$\mathcal{B}_n := \big\{\beta_1\beta_2\cdots\beta_n \mid \beta_i \in \{-n, -n+1, \dots, -1, 1, \dots, n\} \\ \text{and } \{|\beta_1|, |\beta_2|, \dots, |\beta_n|\} = [n]\big\}.$$

We say that a signed permutation $\beta \in \mathcal{B}_n$ has a peak at i if $\beta_{i-1} < \beta_i > \beta_{i+1}$.

Definition 20. Let $R \subseteq [n-1]$. Then the sets $P_B(R; n)$ and $\hat{P}_B(R; n)$ are defined as

$$P_B(R; n) := \{\beta \in \mathcal{B}_n \mid P(\beta_1\cdots\beta_n) = R\},$$
$$\hat{P}_B(R; n) := \{\beta \in \mathcal{B}_n \mid P(\beta_0\beta_1\cdots\beta_n) = R, \text{ where } \beta_0 = 0\}.$$

In this paper we study the sets of mirrored permutations of types C_n and D_n that have a given peak set S.

Definition 21. Let $\mathcal{C}_n$ and $\mathcal{D}_n$ be the mirrored permutations of types C and D, respectively. For $S \subseteq [n-1]$, we define the sets $P_C(S; n)$ and $P_D(S; n)$ as

$$P_C(S; n) := \{\pi \in \mathcal{C}_n \mid P(\pi_1\pi_2\cdots\pi_n) = S\}, \tag{9}$$
$$P_D(S; n) := \{\pi \in \mathcal{D}_n \mid P(\pi_1\pi_2\cdots\pi_n) = S\}. \tag{10}$$

Let $S \subseteq [n]$ we define the sets $\hat{P}_C(S; n)$ and $\hat{P}_D(S; n)$ as

$$\hat{P}_C(S; n) := \{\pi \in \mathcal{C}_n \mid P(\pi_1\pi_2 \cdots \pi_n \,|\, \pi_{n+1}) = S\}, \tag{11}$$

$$\hat{P}_D(S; n) := \{\pi \in \mathcal{D}_n \mid P(\pi_1\pi_2 \cdots \pi_n \,|\, \pi_{n+1}) = S\}. \tag{12}$$

Note that $\hat{P}_C(S; n)$ and $\hat{P}_D(S; n)$ differ from $P_C(S; n)$ and $P_D(S; n)$ in that they allow for a peak in the n-th position when $\pi_{n-1} < \pi_n > \pi_{n+1}$. The following proposition provides a bijection between the peak sets $\hat{P}_B(R; n)$ considered by Castro-Velez et al. [2013] and $\hat{P}_C(S; n)$ considered in this paper.

Proposition 22. *Let* $S = \{i_1, i_2, \ldots, i_k\} \subset \{2, 3, \ldots, n\}$ *and*

$$R = \{n - i_1 + 1,\ n - i_2 + 1, \ldots, n - i_k + 1\} \subset [n - 1].$$

Then there is a bijection between $\mathcal{C}_n$ *and* $\mathcal{B}_n$ *that maps* $\hat{P}_C(S; n)$ *to* $\hat{P}_B(R; n)$.

The above result states that the peaks of $\pi_1\pi_2 \cdots \pi_n \,|\, \pi_{n+1}$ correspond bijectively with the peaks of a signed permutation $\beta_0\beta_1\beta_2 \cdots \beta_n$, where $\beta_0 = 0$, and the peaks of $\pi_1\pi_2 \cdots \pi_n$ correspond with those of $\beta_1\beta_2 \cdots \beta_n$. Before proceeding to the proof of Proposition 22, we set some preliminaries.

Billey and Lakshmibai [2000, Definition 8.3.2] note that a mirrored permutation

$$\pi_1\pi_2 \cdots \pi_n \,|\, \pi_{n+1}\pi_{n+2} \cdots \pi_{2n} \in \mathcal{C}_n$$

can be represented by either side of the mirror, $\pi_1\pi_2 \cdots \pi_n$ or $\pi_{n+1}\pi_{n+2} \cdots \pi_{2n}$, and we use the latter $\pi_{n+1}\pi_{n+2} \cdots \pi_{2n}$ to define a map F from $\mathcal{C}_n$ to $\mathcal{B}_n$ as

$$F : \mathcal{C}_n \to \mathcal{B}_n,$$
$$\pi_1\pi_2 \cdots \pi_n \,|\, \pi_{n+1}\pi_{n+2} \cdots \pi_{2n} \mapsto \beta_1\beta_2 \cdots \beta_n,$$

where

$$\beta_i = \begin{cases} \pi_{n+i} - n & \text{if } \pi_{n+i} > n, \\ \pi_{n+i} - n - 1 & \text{if } \pi_{n+i} \le n. \end{cases}$$

We consider a signed permutation $\beta = \beta_1\beta_2 \cdots \beta_n$ in $\mathcal{B}_n$ as $\beta_0\beta_1 \cdots \beta_n$, where $\beta_0 = 0$, thus allowing for a peak at position 1. We note that the map F respects the relative order of the sequence $\pi_n\pi_{n+1}\pi_{n+2} \cdots \pi_{2n}$; i.e., for $0 \le i \le n - 1$, if $\pi_{n+i} < \pi_{n+i+1}$ then $\beta_i < \beta_{i+1}$, and similarly if $\pi_{n+i} > \pi_{n+i+1}$ then $\beta_i > \beta_{i+1}$.

We also define an automorphism $G : \mathcal{B}_n \to \mathcal{B}_n$ which switches the sign of each β_i in $\beta_0\beta_1\beta_2 \cdots \beta_n$ (keeping $\beta_0 = 0$ fixed). To avoid cumbersome notation, for each β_i, we set $\overline{\beta}_i = -\beta_i$. The following table illustrates how the maps F and G map the group of mirrored permutations $\mathcal{C}_2$ bijectively to the group of signed permutations $\mathcal{B}_2$:

$\pi \in \mathcal{C}_2$	$F(\pi) \in \mathcal{B}_2$	$G(F(\pi)) \in \mathcal{B}_2$
$12\|34$	012	$0\bar{1}\bar{2}$
$21\|43$	021	$0\bar{2}\bar{1}$
$13\|24$	$0\bar{1}2$	$01\bar{2}$
$24\|13$	$0\bar{2}1$	$02\bar{1}$
$31\|42$	$02\bar{1}$	$0\bar{2}1$
$42\|31$	$01\bar{2}$	$0\bar{1}2$
$34\|12$	$0\bar{2}\bar{1}$	021
$43\|21$	$0\bar{1}\bar{2}$	012

With the above notation at hand we now proceed to the proof.

Proof of Proposition 22. Let $\pi = \pi_1\pi_2 \cdots \pi_n \mid \pi_{n+1} \cdots \pi_{2n}$ be a mirrored permutation and $F(\pi) = \beta_0\beta_1\beta_2 \cdots \beta_n$. Then we see $G(F(\pi)) = \beta_0\overline{\beta_1}\,\overline{\beta_2} \cdots \overline{\beta_n}$. Suppose $\pi_i < \pi_{i+1}$ for some $i \in \{1, 2, \ldots, n\}$. Looking at the mirror images of π_i and π_{i+1}, we get $2n - \pi_i + 1 > 2n - \pi_{i+1} + 1$; thus $\pi_{2n-i+1} > \pi_{2n-(i+1)+1}$. Since the map F respects the relative order of $\pi_n\pi_{n+1} \cdots \pi_{2n}$, we have $\beta_{n-i+1} > \beta_{n-(i+1)+1}$, and thus $\overline{\beta_{n-i+1}} < \overline{\beta_{n-(i+1)+1}}$. Using the same argument but replacing "<" with ">" and vice versa, we get that if $\pi_i > \pi_{i+1}$ then $\overline{\beta_{n-i+1}} > \overline{\beta_{n-(i+1)+1}}$. Therefore if $\pi \in \hat{P}_C(S; n)$ then $G(F(\pi)) \in \hat{P}_B(R; n)$, and if $\pi \notin \hat{P}_C(S; n)$ then $G(F(\pi)) \notin \hat{P}_B(R; n)$. Since both G and F are bijections, we conclude that $G(F(\hat{P}_C(S; n))) = \hat{P}_B(R; n)$. □

We can also consider signed permutations $\beta \in \mathcal{B}_n$ without the convention that $\beta_0 = 0$. In that case we obtain the following result.

Corollary 23. *Let* $S = \{i_1, i_2, \ldots, i_k\} \subset \{2, 3, \ldots, n-1\}$ *and*

$$R = \{n - i_1 + 1,\ n - i_2 + 1, \ldots, n - i_k + 1\} \subset \{2, \ldots, n-1\}.$$

Then the bijection $G \circ F : \mathcal{C}_n \to \mathcal{B}_n$ *maps* $P_C(S; n)$ *to* $P_B(R; n)$.

Proof. The proof of this corollary proceeds exactly as the proof of Proposition 22. □

3C. *The sets $P_C(S; n)$ and $P_D(S; n)$.* In this subsection, we use the fact that $\mathcal{C}_n(\pi)$ and $\mathcal{D}_n(\pi)$ partition $\mathcal{C}_n$ and $\mathcal{D}_n$ to give concise proofs that $|P_C(S; n)| = p(n)2^{2n-|S|-1}$ and $|P_D(S; n)| = p(n)2^{2n-|S|-2}$, where $p(n)$ is the polynomial given in [Billey et al. 2013, Theorem 2.2].

Theorem 24. *Let* $S \subseteq [n-1]$. *Then*

(I) $|P_C(S; n)| = p(n)2^{2n-|S|-1}$,

(II) $|P_D(S; n)| = p(n)2^{2n-|S|-2}$.

Proof. To prove part (I), note that Billey et al. [2013, Theorem 2.2] showed that $|P(S; n)| = p(n)2^{n-|S|-1}$, where $p(n)$ is a polynomial with degree $\max(S) - 1$. Algorithm 15 showed that each $\pi \in P(S; n)$ corresponds to a subset $\mathcal{C}_n(\pi) \subset \mathcal{C}_n$

which contains 2^n elements. By construction these elements have the exact same peak set as π. In other words, for every $\tau \in \mathcal{C}_n(\pi)$, the peak sets $P(\tau) = P(\pi) = S$ agree. We compute that $|P_C(S; n)| = p(n)2^{n-|S|-1}2^n = p(n)2^{2n-|S|-1}$.

Part (II) follows similarly, replacing $\mathcal{C}_n$ with $\mathcal{D}_n$, Algorithm 15 with Algorithm 18, and 2^n with 2^{n-1}. □

4. Peak sets of the Coxeter groups of types *C* and *D*

In this section we use specific sums of binomial coefficients and the partitions

$$P(S; n) = \overline{P(S; n)} \sqcup \underline{P(S; n)} \quad \text{and} \quad \overline{P(S; n)} = \bigsqcup_{k=2}^{n} P(S; n)^{\nearrow k}$$

to describe the cardinality of the sets $\hat{P}_C(S; n)$, $\hat{P}_C(S \cup \{n\}; n)$, $\hat{P}_D(S; n)$, and $\hat{P}_D(S \cup \{n\}; n)$. We begin by setting the following notation:

Definition 25. Let $\Phi(n, k)$ denote the sum of the last $n - j + 1$ terms of the n-th row in Pascal's triangle,

$$\Phi(n, k) = \sum_{i=k}^{n} \binom{n}{i} = \binom{n}{k} + \binom{n}{k+1} + \cdots + \binom{n}{n},$$

and let

$$\Psi(n, k) = 2^n - \Phi(n, k).$$

We can now state our main result.

Theorem 26. *Type C: Let $\hat{P}_C(S; n)$ denote the set of elements of $\mathcal{C}_n$ with peak set $S \subset [n-1]$. Then*

$$|\hat{P}_C(S; n)| = \sum_{k=1}^{n} |P(S; n)^{\nearrow k}| \cdot \Phi(n, k) + |\underline{P(S; n)}| \cdot 2^n$$

and

$$|\hat{P}_C(S \cup \{n\}; n)| = \sum_{k=1}^{n} |P(S; n)^{\nearrow k}| \cdot \Psi(n, k).$$

Type D: Let $\hat{P}_D(S; n)$ denote the set of elements of $\mathcal{D}_n$ with peak set $S \subset [n-1]$. If n is even, then

$$|\hat{P}_D(S; n)| = \sum_{k=1}^{n/2} \left(|P(S; n)^{\nearrow 2k-1}| + |P(S; n)^{\nearrow 2k}|\right) \Phi(n-1, 2k-1) + |\underline{P(S; n)}| 2^{n-1}$$

and

$$|\hat{P}_D(S \cup \{n\}; n)| = \sum_{k=1}^{n/2} \left(|P(S; n)^{\nearrow 2k-1}| + |P(S; n)^{\nearrow 2k}|\right) \Psi(n-1, 2k-1).$$

If n is odd, then

$$|\hat{P}_D(S;n)| = \sum_{k=1}^{(n-1)/2} \big(|P(S;n)^{\nearrow 2k+1}| + |P(S;n)^{\nearrow 2k}|\big)\Phi(n-1,2k) + |\underline{P(S;n)}|2^{n-1}$$

and

$$|\hat{P}_D(S\cup\{n\};n)| = \sum_{k=1}^{(n-1)/2} \big(|P(S;n)^{\nearrow 2k+1}| + |P(S;n)^{\nearrow 2k}|\big)\Psi(n-1,2k).$$

Since the proofs of the type-C and type-D results in Theorem 26 require some specific identities involving the functions Φ and Ψ, we present these results and proofs in Sections 4A and 4B, respectively.

Note that Proposition 22 shows that $|\hat{P}_B(R;n)| = |\hat{P}_C(S;n)|$. Castro-Velez et al. [2013, Theorem 3.2] gave a recursive formula for computing the cardinality of the set $\hat{P}_B(R;n)$. Theorem 26 provides an alternate formula for $|\hat{P}_C(S;n)| = |\hat{P}_B(R;n)|$ using the sums of binomial coefficients $\Phi(n,k)$ and $\Psi(n,k)$, and the cardinalities of sets $\underline{P(S;n)}$ and $P(S;n)^{\nearrow k}$.

4A. ***Peak sets of the Coxeter groups of type C.*** The following lemma uses the functions $\Phi(n,k)$ and $\Psi(n,k)$ to count the number of elements in $\mathcal{C}_n(\pi)$ having an ascent in the n-th position. This lemma is the key step in the type-C proof of Theorem 26.

Lemma 27. *If $\pi \in P(S;n)^{\nearrow k}$ then there are $\Phi(n,k)$ elements $\tau \in \mathcal{C}_n(\pi)$ with $\tau_n \le n$ and $\Psi(n,k)$ elements $\tau \in \mathcal{C}_n(\pi)$ with $\tau_n > n$.*

Proof. Suppose that $\pi = \pi_1\pi_2\cdots\pi_n \in P(S;n)^{\nearrow k}$, so $\pi_{n-1} < \pi_n = k$. If $\tau = \tau_1\tau_2\cdots\tau_n \mid \tau_{n+1}\tau_{n+2}\cdots\tau_{2n} \in \mathcal{C}_n(\pi)$, then τ_n is the k-th largest integer in the set $\{\tau_1,\tau_2,\dots,\tau_n\}$ because τ has the same relative order as π and $\pi_n = k$. Therefore if at least k elements of the set $\{\tau_1,\tau_2,\dots,\tau_n\}$ have $\tau_i \le n$ then we conclude $\tau_n \le n$.

We will show there are $\binom{n}{j}$ elements $\tau \in \mathcal{C}_n(\pi)$, where exactly j elements of the set $\{\tau_1,\tau_2,\dots,\tau_n\}$ satisfy $\tau_n \le n$. To construct such a τ, we start with $\pi = \pi_1\pi_2\cdots\pi_n$, and then we choose j elements of the set $\{\pi_1,\pi_2,\dots,\pi_n\}$ to remain fixed. We replace the remaining $n-j$ elements of $\{\pi_1,\pi_2,\dots,\pi_n\}$ with their mirror images, which are all greater than n. Finally, we list the elements of the resulting set so that they have the same relative order as π and call them $\tau_1\tau_2\cdots\tau_n$. The subpermutation $\tau_{n+1}\tau_{n+2}\cdots\tau_{2n}$ is then completely determined by the subpermutation $\tau_1\tau_2\cdots\tau_n$. Thus there are $\binom{n}{j}$ mirrored permutations τ of the form $\tau = \tau_1\tau_2\cdots\tau_n \mid \tau_{n+1}\tau_{n+2}\cdots\tau_{2n} \in \mathcal{C}_n(\pi)$, where j of the elements in $\{\tau_1,\tau_2,\dots,\tau_n\}$ satisfy $\tau_i \le n$.

Considering all integers j with $k \le j \le n$, we see that the number of elements

$$\tau = \tau_1\tau_2\cdots\tau_n \mid \tau_{n+1}\tau_{n+2}\cdots\tau_{2n} \in \mathcal{C}_n(\pi)$$

with at least k of the elements in $\{\tau_1, \tau_2, \dots, \tau_n\}$ satisfying $\tau_i \le n$ is exactly

$$\Phi(n, k) = \sum_{j=k}^{n} \binom{n}{j}.$$

Thus there are $\Phi(n, k)$ elements $\tau \in \mathcal{C}_n(\pi)$ with $\tau_n \le n$. The other $2^n - \Phi(n, k) = \Psi(n, k)$ elements of $\mathcal{C}_n(\pi)$ must have $\tau_n > n$. □

With the above result at hand, we now give the following proof.

Proof of Theorem 26, type C. Let $\pi = \pi_1\pi_2 \cdots \pi_n \in P(S; n)$, and recall that $\mathcal{C}_n(\pi)$ is the set of elements of $\mathcal{C}_n$ whose first n entries have the same relative order as π, and $|\mathcal{C}_n(\pi)| = 2^n$ for any $\pi \in \mathfrak{S}_n$. Let $\tau = \tau_1\tau_2 \cdots \tau_n \mid \tau_{n+1} \cdots \tau_{2n} \in \mathcal{C}_n$ denote a mirrored permutation of type C_n. Then there are two possibilities:

- Either τ has the same peak set as π so that $\tau \in \hat{P}_C(S; n)$, or
- τ has an additional peak at n, in which case $\tau \in \hat{P}_C(S \cup \{n\}; n)$.

There are two cases in which $\tau \in \hat{P}_C(S; n)$:

Case 1: If π ends with a descent, i.e., $\pi_{n-1} > \pi_n$, then every $\tau \in \mathcal{C}_n(\pi)$ also has $\tau_{n-1} > \tau_n$, and thus τ is in $\hat{P}_C(S; n)$ because it cannot possibly have a peak at n if it ends with a descent. We conclude that if $\pi \in \underline{P(S; n)}$ then all 2^n elements $\tau \in \mathcal{C}_n(\pi)$ are in $\hat{P}_C(S; n)$.

Case 2: If π ends with an ascent, i.e., $\pi_{n-1} < \pi_n$, then $\tau_{n-1} < \tau_n$ for all $\tau \in \mathcal{C}_n(\pi)$ as well. (Recall that for any $\sigma \in \mathcal{C}_n$, our map into $\mathfrak{S}_{2n}$ identifies σ_i with its *mirror* σ_{n-i+1} by $\sigma_{n-i+1} = 2n - \sigma_i + 1$.) Hence, if $\tau_n \le n$, then $\tau_{n+1} = 2n - \tau_n + 1 > \tau_n$. In this case $\tau_{n-1} < \tau_n < \tau_{n+1}$, and τ does not have a peak at n. So $\tau \in \hat{P}_C(S; n)$. Therefore we conclude that if $\pi \in \overline{P(S; n)}$ and if $\tau \in \mathcal{C}_n(\pi)$ satisfies $\tau_n \le n$ then τ is an element of $\hat{P}_C(S; n)$. By Lemma 27 we conclude that if $\pi \in P(S; n)^{\nearrow k}$ then $\Phi(n, k)$ of the elements in $\mathcal{C}_n(\pi)$ are in $\hat{P}_C(S; n)$.

Case 3: There is only one case in which $\tau \in \hat{P}_C(S \cup \{n\}; n)$. If $\pi \in \overline{P(S; n)}$ and $\tau \in \mathcal{C}_n(\pi)$ is such that $\tau_n > n$, then τ must satisfy $\tau_{n-1} < \tau_n > \tau_{n+1}$ because $\tau_{n+1} = 2n - \tau_n + 1 < n$. Therefore τ is an element of $\hat{P}_C(S \cup \{n\}; n)$. Applying Lemma 27 we conclude that if $\pi \in P(S; n)^{\nearrow k}$ then $\Psi(n, k)$ of the elements in $\mathcal{C}_n(\pi)$ are in $\hat{P}_C(S \cup \{n\}; n)$.

From Cases 1 and 2, we conclude that the cardinality of $\hat{P}_C(S; n)$ is given by

$$|\hat{P}_C(S; n)| = \sum_{k=1}^{n} |P(S; n)^{\nearrow k}| \cdot \Phi(n, k) + |\underline{P(S; n)}| \cdot 2^n.$$

From Case 3, we get

$$|\hat{P}_C(S \cup \{n\}; n)| = \sum_{k=1}^{n} |P(S; n)^{\nearrow k}| \cdot \Psi(n, k).$$ □

The following example illustrates the type-C formulas proven in Theorem 26.

Example 28. Using the results of this section, we compute the sets $\hat{P}_C(S; 3)$, where $S \subset [3]$. First, the group $\mathfrak{S}_3$ can be partitioned as $\mathfrak{S}_3 = P(\varnothing; 3) \sqcup P(\{2\}; 3)$, where

$$P(\varnothing; 3) = \{123, 321, 213, 312\} \quad \text{and} \quad P(\{2\}; 3) = \{132, 231\}.$$

To calculate the peak sets in type C_3, we will further partition the sets $P(\varnothing; 3)$ and $P(\{2\}; 3)$ using Definition 1. Hence we compute

$$P(\varnothing; 3) = \underline{P(\varnothing; 3)} \sqcup P(\varnothing; 3)^{\nearrow 2} \sqcup P(\varnothing; 3)^{\nearrow 3},$$

where $\underline{P(\varnothing; 3)} = \{321\}$, $P(\varnothing; 3)^{\nearrow 2} = \{312\}$, and $P(\varnothing; 3)^{\nearrow 3} = \{123, 213\}$.

We also compute the set

$$P(\{2\}; 3) = \underline{P(\{2\}; 3)} = \{231, 132\}.$$

Of the 48 elements of the Coxeter group $\mathcal{C}_3$, only $2^3|P(\varnothing; 3)| = 2^3 \cdot 4 = 32$ elements are in $\hat{P}_C(\varnothing; 3) \sqcup \hat{P}_C(\{3\}; 3)$. Of these 32 permutations, we observe that 18 lie in $\hat{P}_C(\{3\}; 3)$ and 14 lie in $\hat{P}_C(\varnothing; 3)$. We calculate $|\hat{P}_C(\varnothing; 3)|$ using Theorem 26:

$$\begin{aligned} |\hat{P}_C(\varnothing; 3)| &= (|\underline{P(\varnothing; 3)}| \cdot 2^3) + (|P(\varnothing; 3)^{\nearrow 2}| \cdot \Phi(3, 2)) + (|P(\varnothing; 3)^{\nearrow 3}| \cdot \Phi(3, 3)) \\ &= (1 \cdot 8) + (1 \cdot 4) + (2 \cdot 1) = 14. \end{aligned}$$

Hence $|\hat{P}_C(\{3\}; 3)| = 2^3 \cdot 4 - 14 = 18$. Since $P(\{2\}; 3) = \underline{P(\{2\}; 3)}$, we have

$$|\hat{P}_C(\{2\}; 3)| = |\underline{P(\{2\}; 3)}| \cdot 2^3 = |P(\{2\}; 3)| \cdot 2^3 = 16.$$

Indeed one may confirm that $\hat{P}_C(\{2\}; 3)$ is the union of the sets

$$\mathcal{C}_3(231) = \begin{Bmatrix} 231|645, & 241|635, \\ 351|624, & 362|514, \\ 451|623, & 462|513, \\ 356|124, & 564|312 \end{Bmatrix} \quad \text{and} \quad \mathcal{C}_3(132) = \begin{Bmatrix} 132|645, & 142|536, \\ 153|426, & 263|415, \\ 154|326, & 264|315, \\ 365|214, & 465|213 \end{Bmatrix}.$$

4B. ***Peak sets of the Coxeter group of type D.*** In this section, we use the functions $\Phi(n, k)$ and $\Psi(n, k)$ to describe the cardinalities of $\hat{P}_D(S; n)$ and $\hat{P}_D(S \cup \{n\}; n)$. The results depend on the parity of n. We begin by providing the following lemmas (similar to Lemma 27), which are used in the type-D proof of Theorem 26.

Lemma 29. *Let n be even, and let $1 \le k \le \frac{1}{2}n$. If $\pi \in P(S; n)^{\nearrow 2k} \sqcup P(S; n)^{\nearrow 2k-1}$, then there are $\Phi(n-1, 2k-1)$ elements $\tau \in \mathcal{D}_n(\pi)$ with $\tau_n \le n$, and $\Psi(n-1, 2k-1)$ elements $\tau \in \mathcal{D}_n(\pi)$ with $\tau_n > n$.*

Proof. Suppose $\pi = \pi_1\pi_2 \cdots \pi_n \in P(S; n)^{\nearrow 2k}$, so $\pi_n = 2k$ and $\pi_{n-1} = i$ for some integer $i < 2k$. If $\tau = \tau_1\tau_2 \cdots \tau_n \mid \tau_{n+1}\tau_{n+2} \cdots \tau_{2n} \in \mathcal{D}_n(\pi)$, then τ_n is the $2k$-th largest integer in the set $\{\tau_1, \tau_2, \ldots, \tau_n\}$ because τ has the same relative order as π and $\pi_n = 2k$. Therefore if at least $2k$ elements of the set $\{\tau_1, \tau_2, \dots, \tau_n\}$ satisfy $\tau_i \le n$ then we can conclude that $\tau_n \le n$. Moreover, $\tau_n \le n$ if and only if $\tau_n < \tau_{n+1} = 2n - \tau_n + 1$. Thus we wish to count the number of $\tau \in \mathcal{D}_n(\pi)$ with $\tau_n \le n$.

In the construction of $\mathcal{D}_n(\pi)$, the total number of τ with at least $2k$ of the elements from $\{\tau_1, \tau_2, \dots, \tau_n\}$ fixed (and less than or equal to n) is given by the sum

$$\binom{n}{2k} + \binom{n}{2k+2} + \cdots + \binom{n}{n-2} + \binom{n}{n} \tag{13}$$

when n is even. Using the identity $\binom{n}{2k} = \binom{n-1}{2k-1} + \binom{n-1}{2k}$, we can see that the quantity in (13) equals

$$\left[\binom{n-1}{2k-1} + \binom{n-1}{2k}\right] + \left[\binom{n-1}{2k+1} + \binom{n-1}{2k+2}\right] + \cdots + \binom{n-1}{n-1} = \Phi(n-1, 2k-1)$$

when n is even.

Suppose $\pi = \pi_1\pi_2 \cdots \pi_n \in P(S; n)^{\nearrow 2k-1}$, so $\pi_n = 2k - 1$ and $\pi_{n-1} = i$ for some integer $i < 2k - 1$. If $\tau = \tau_1\tau_2 \cdots \tau_n \mid \tau_{n+1}\tau_{n+2} \cdots \tau_{2n} \in \mathcal{D}_n(\pi)$, then τ_n is the $(2k-1)$-th largest integer in the set $\{\tau_1, \tau_2, \dots, \tau_n\}$ because τ has the same relative order as π and $\pi_n = 2k - 1$. Therefore if at least $2k - 1$ elements of the set $\{\tau_1, \tau_2, \dots, \tau_n\}$ satisfy $\tau_i \le n$ then we can conclude that $\tau_n \le n$. Moreover, $\tau_n \le n$ if and only if $\tau_n < \tau_{n+1} = 2n - \tau_n + 1$. So again, the number of elements with $\tau_n \le n$ is $\Phi(n - 1, 2k - 1)$.

We conclude that when $\pi \in P(S; n)^{\nearrow 2k} \sqcup P(S; n)^{\nearrow 2k-1}$, there are $\Phi(n-1, 2k-1)$ mirrored permutations $\tau \in \mathcal{D}_n(\pi)$ with $\tau_n < \tau_{n+1}$. Since there are 2^{n-1} elements in $\mathcal{D}_n(\pi)$, we see that there are $\Psi(n-1, 2k-1)$ elements $\tau \in \mathcal{D}_n(\pi)$ with $\tau_n > \tau_{n+1}$. □

Lemma 30. *Let n be odd and let $1 \le k \le \frac{1}{2}(n - 1)$. If $\pi \in P(S; n)^{2k}$ or $\pi \in P(S; n)^{2k+1}$ then there are $\Phi(n - 1, 2k)$ elements $\tau \in \mathcal{D}_n(\pi)$ with $\tau_n \le n$ and $\Psi(n - 1, 2k)$ elements $\tau \in \mathcal{D}_n(\pi)$ with $\tau_n > n$.*

The proof of Lemma 30 follows similarly to that of Lemma 29; hence we omit the argument, but point the interested reader to the arXiv preprint of this paper for a detailed proof [Diaz-Lopez et al. 2015]. We are now ready to enumerate the sets $\hat{P}_D(S; n)$ and $\hat{P}_D(S \cup \{n\}; n)$.

Proof of Theorem 26, type D. Let $\pi \in P(S; n)$, $\tau = \tau_1\tau_2 \cdots \tau_n \mid \tau_{n+1}\tau_{n+2} \cdots \tau_{2n} \in \mathcal{D}_n$, and recall that $\mathcal{D}_n(\pi)$ consists of the elements of $\mathcal{D}_n$ which have the same relative order as π. There are 2^{n-1} such elements. Since $\tau \in \mathcal{D}_n(\pi)$, its first n entries $\tau_1\tau_2 \cdots \tau_n$ have the same relative order as $\pi_1\pi_2 \cdots \pi_n$, and just as in the type-$\mathcal{C}_n$ case, there are two possibilities:

- Either τ has the same peak set as π so that $\tau \in \hat{P}_D(S; n)$, or
- τ has an additional peak at n, in which case $\tau \in \hat{P}_D(S \cup \{n\}; n)$.

There are two cases in which $\tau \in \hat{P}_D(S; n)$:

Case 1: If π ends with a descent, i.e., $\pi_{n-1} > \pi_n$, then every $\tau \in \mathcal{D}_n(\pi)$ also has $\tau_{n-1} > \tau_n$, and thus τ is in $\hat{P}_D(S;n)$ because it cannot possibly have a peak at n if it has a descent at $n-1$. We conclude that if $\pi \in \underline{P(S;n)}$ then all 2^{n-1} elements of $\mathcal{D}_n(\pi)$ are in $\hat{P}_D(S;n)$.

Case 2: If π ends with an ascent, $\pi_{n-1} < \pi_n$, then $\tau_{n-1} < \tau_n$ for all $\tau \in \mathcal{D}_n(\pi)$ as well. (Recall that for any $\sigma \in \mathcal{D}_n$, our map into $\mathfrak{S}_{2n}$ identifies σ_i with σ_{n-i+1} by $\sigma_{n-i+1} = 2n - \sigma_i + 1$.) Hence, if $\tau_n \leq n$, then $\tau_{n+1} = 2n - \tau_n + 1 > \tau_n$. In this case $\tau_{n-1} < \tau_n < \tau_{n+1}$, and τ does not have a peak at n. So $\tau \in \hat{P}_D(S;n)$. Therefore we conclude that if $\pi \in \overline{P(S;n)}$ and if $\tau \in \mathcal{D}_n(\pi)$ satisfies $\tau_n \leq n$ then τ is an element of $\hat{P}_D(S;n)$. By Lemma 29 we conclude that if $\pi \in P(S;n)^{\nearrow k}$ then $\Phi(n-1, 2k-1)$ of the elements in $\mathcal{D}_n(\pi)$ are in $\hat{P}_D(S;n)$.

Case 3: There is only one case in which $\tau \in \hat{P}_D(S \cup \{n\};n)$. If $\pi \in \overline{P(S;n)}$ and $\tau \in \mathcal{D}_n(\pi)$ is such that $\tau_n > n$, then τ must satisfy $\tau_{n-1} < \tau_n > \tau_{n+1}$ because $\tau_{n+1} = 2n - \tau_n + 1 < n$. Therefore τ is an element of $\hat{P}_D(S \cup \{n\};n)$.

We have shown if π is in $\underline{P(S;n)}$, then all 2^{n-1} elements $\mathcal{D}_n(\pi)$ are in $\hat{P}_C(S;n)$. Lemma 29 showed when n is even and $\pi \in P(S;n)^{\nearrow 2k}$ or $\pi \in P(S;n)^{\nearrow 2k-1}$, then $\Phi(n, 2k-1)$ of the elements of $\mathcal{D}_n(\pi)$ are in $\hat{P}_D(S;n)$. Thus we conclude when n is even, the cardinality of $\hat{P}_D(S;n)$ is given by the formula

$$|\hat{P}_D(S;n)| = \sum_{k=1}^{n} \big(|P(S;n)^{k-1}| + |P(S;n)^{2k}|\big) \cdot \Phi(n, 2k-1) + |\underline{P(S;n)}| \cdot 2^{n-1}.$$

Lemma 29 also showed if $\pi \in P(S;n)^{\nearrow 2k}$ or $\pi \in P(S;n)^{\nearrow 2k-1}$ then $\Psi(n-1, 2k-1)$ elements from $\mathcal{D}_n(\pi)$ are in the set $\hat{P}_D(S \cup \{n\};n)$, and thus

$$|\hat{P}_D(S \cup \{n\};n)| = \sum_{k=1}^{n/2} \big(|P(S;n)^{\nearrow 2k-1}| + |P(S;n)^{\nearrow 2k}|\big) \cdot \Psi(n-1, 2k-1)$$

when n is even.

Lemma 30 showed when n is odd and $\pi \in P(S;n)^{\nearrow 2k}$ or $\pi \in P(S;n)^{\nearrow 2k-1}$, then $\Phi(n-1, 2k)$ of the elements of $\mathcal{D}_n(\pi)$ are in $\hat{P}_D(S;n)$. Thus we conclude that when n is odd, the cardinality of $\hat{P}_D(S;n)$ is given by the formula

$$|\hat{P}_D(S;n)| = \sum_{k=1}^{(n-1)/2} \big(|P(S;n)^{\nearrow 2k+1}| + |P(S;n)^{\nearrow 2k}|\big) \cdot \Phi(n-1, 2k) + |\underline{P(S;n)}| \cdot 2^{n-1}.$$

Lemma 30 also showed if $\pi \in P(S;n)^{\nearrow 2k}$ or $\pi \in P(S;n)^{\nearrow 2k+1}$ then $\Psi(n-1, 2k)$ elements from $\mathcal{D}_n(\pi)$ are in the set $\hat{P}_D(S \cup \{n\};n)$, and thus

$$|\hat{P}_D(S \cup \{n\};n)| = \sum_{k=1}^{(n-1)/2} \big(|P(S;n)^{\nearrow 2k+1}| + |P(S;n)^{\nearrow 2k}|\big) \cdot \Psi(n-1, 2k)$$

when n is odd. This proves the formula for the cardinality of $\hat{P}_D(S \cup \{n\};n)$. $\square$

4C. ***Special case: empty peak set in types C and D.*** In this section we consider the special case of $S = \varnothing$ in types C_n and D_n.

Proposition 31. *Let* $n \geq 2$ *and* $m \geq 4$, *then*

(I) $|\hat{P}_C(\varnothing; n)| = \frac{1}{2}(3^n + 1)$,

(II) $|\hat{P}_D(\varnothing; m)| = \frac{1}{4}3^m + \frac{1}{4}(-1)^m + \frac{1}{2}$.

Proposition 31(I) was originally proved by Castro-Velez et al. [2013, Theorem 2.4] in type B_n. However, the proof given here is a combinatorial argument involving ternary sequences (in the letters A, B and C) with an even number of B's that restricts naturally to a proof of a similar result involving the mirrored permutations with no peaks in type D_n as well.

The integer sequence given by Proposition 31(I) is sequence A007051 in [OEIS 1996] after the first three iterations. Let T_n denote the set of ternary sequences (in the letters A, B and C) of length n with an even number of B's. It is noted on Sloane's OEIS that $\frac{1}{2}(3^n + 1)$ counts all such sequences.

Proof of Proposition 31(I). To prove that $|\hat{P}_C(\varnothing; n)| = |T_n| = \frac{1}{2}(3^n + 1)$, we prove there is a bijection between the sets T_n and $\hat{P}_C(\varnothing; n)$.

Every permutation $\pi \in \hat{P}_C(\varnothing; n)$ has the form $\pi = \pi_A\pi_B\pi_C \mid \overline{\pi_C\pi_B\pi_A}$, where π_A is a sequence of numbers in descending order and each $\pi_i \in \pi_A$ is greater than n, π_B is a sequence of numbers in descending order and each $\pi_i \in \pi_B$ is less than or equal to n, and π_C is a sequence of numbers in ascending order and each $\pi_i \in \pi_C$ is less than or equal to n. Note that the mirror image $\overline{\pi_C\pi_B\pi_A}$ is determined uniquely by $\pi_A\pi_B\pi_C$, so to condense notation in this proof we will refrain from writing it. It is possible for at most two of the parts π_A, π_B, or π_C to be empty. Moreover, there is always a choice of whether to include the minimum element of the subpermutation $\pi_B\pi_C$ as the last element in π_B or the first element in π_C. We always choose to make the length of π_B even by including/excluding this minimum element depending on the parity of π_B.

More precisely, let $\pi = \pi_A\pi_B\pi_C \in \hat{P}_C(\varnothing; n)$, where

$$\pi_A = [\pi_1 > \cdots > \pi_k], \quad \pi_B = [\pi_{k+1} > \cdots > \pi_{k+j}], \quad \text{and} \quad \pi_C = [\pi_{k+j+1} < \cdots < \pi_n].$$

Define a set map $\Delta : \hat{P}_C(\varnothing; n) \to T_n$ by assigning a ternary sequence $\Delta(\pi) = x$ in T_n to each element $\pi \in \hat{P}_C(\varnothing; n)$ by setting

$$\Delta(\pi)_i = x_i = \begin{cases} A & \text{if } i \in \{2n - \pi_1 + 1, \ldots, 2n - \pi_k + 1\}, \\ B & \text{if } i \in \{\pi_{k+1}, \ldots, \pi_{k+j}\}, \\ C & \text{if } i \in \{\pi_{k+j+1}, \ldots, \pi_n\}. \end{cases}$$

Note that there is an even number of B's by the way we defined π_B. Hence $\Delta(\pi) = x \in T_n$.

We can also define a set map $\Theta : T_n \to \hat{P}_C(\varnothing; n)$ by reversing this process. That is to say, given a ternary sequence $x = x_1 x_2 \cdots x_n$ in T_n, define $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ as

$$\mathcal{A} = \{1 \le i \le n : x_i = A\}, \quad \mathcal{B} = \{1 \le i \le n : x_i = B\}, \quad \text{and} \quad \mathcal{C} = \{1 \le i \le n : x_i = C\}.$$

List the elements of $\mathcal{A}$ and $\mathcal{C}$ in ascending order and $\mathcal{B}$ in descending order:

$$\mathcal{A} = [a_1 < a_2 < \cdots < a_k], \quad \mathcal{B} = [b_{k+1} > b_{k+2} > \cdots > b_{k+j}],$$
$$\mathcal{C} = [c_{k+j+1} < c_{k+j+2} < \cdots < c_n].$$

Then define $\Theta(x) = \pi$, where

$$\pi_i = \begin{cases} 2n - a_i + 1 & \text{if } 1 \le i \le k, \\ b_i & \text{if } k+1 \le i \le k+j, \\ c_i & \text{if } k+j+1 \le i \le n. \end{cases}$$

Notice that after π_i is determined for $1 \le i \le n$, the rest of π is determined.

To show $\Theta \circ \Delta = \mathrm{Id}$, let $\pi = \pi_A \pi_B \pi_C \in \hat{P}_C(\varnothing; n)$, where

$$\pi_A = [\pi_1 > \cdots > \pi_k], \quad \pi_B = [\pi_{k+1} > \cdots > \pi_{k+j}], \quad \text{and} \quad \pi_C = [\pi_{k+j+1} < \cdots < \pi_n],$$

and set $\sigma = \Theta(\Delta(\pi)) = \sigma_1 \cdots \sigma_n$. Then

$$\Delta(\pi)_i = x_i = A \quad \text{for } i \in \{2n - \pi_1 + 1, \ldots, 2n - \pi_k + 1\},$$

so $\mathcal{A} = [2n - \pi_1 + 1 < \cdots < 2n - \pi_k + 1]$. By the definition of Θ, we get $\sigma_i = 2n - (2n - \pi_i + 1) + 1$ for $1 \le i \le k$; thus $\sigma_i = \pi_i$ for $1 \le i \le k$.

Similarly, $\Delta(\pi)_i = x_i = B$ for $i \in \{\pi_{k+1}, \ldots, \pi_{k+j}\}$; thus $\mathcal{B} = [\pi_{k+1} > \cdots > \pi_{k+j}]$. By the definition of Θ, we get $\sigma_i = \pi_i$ for $k+1 \le i \le k+j$. Finally, $\Delta(\pi)_i = x_i = C$ for $i \in \{\pi_{k+j+1}, \ldots, \pi_n\}$; thus $\mathcal{C} = [\pi_{k+j+1} < \cdots < \pi_n]$. By the definition of Θ, we see that $\sigma_i = \pi_i$ for $k+j+1 \le i \le n$. Therefore $\sigma_i = \pi_i$ for $1 \le i \le n$, which implies $\Theta(\Delta(\pi)) = \sigma = \pi$ for all $\pi \in \hat{P}_C(\varnothing; n)$. A similar argument shows $\Delta(\Theta(x)) = x$ for all $x \in T_n$. □

The integer sequence given by Proposition 31(II) is sequence A122983 in [OEIS 1996] after the first three iterations. To prove this result, we let T_n denote the set of ternary sequences (in the letters A, B and C) of length n with an even number of A's and B's. It is noted on Sloane's OEIS that $\frac{1}{4}3^n + \frac{1}{4}(-1)^n + \frac{1}{2}$ counts all such sequences. In the following proof we construct a bijection from T_n to $\hat{P}_D(\varnothing; n)$ by using the maps Δ and Θ, similar to the proof of Proposition 31(I).

Proof of Proposition 31(II). The proof follows as the proof of Proposition 31(I), with the additional condition that the length of π_A is even since every element π in $\hat{P}_D(\varnothing; n)$ has an even number of entries in $\pi_1 \pi_2 \cdots \pi_n$ that are greater than n. We point the interested reader to the arXiv preprint version of this paper for a detailed proof [Diaz-Lopez et al. 2015]. □

We will illustrate the bijection between Δ and Θ, described in the proof of Proposition 31, with the following example.

Example 32. Type C: Consider the permutation $\pi \in \mathcal{C}_{10}$, where

$$\pi = 20\ 18\ 13\ 10\ 9\ 7\ 4\ 2\ 5\ 6 \,|\, 15\ 16\ 19\ 17\ 14\ 12\ 11\ 8\ 3\ 1.$$

Let $\Delta(\pi) = x \in T_n$. Since

$$\pi_A = 20\ 18\ 13, \quad \pi_B = 10\ 9\ 7\ 4, \quad \text{and} \quad \pi_C = 2\ 5\ 6,$$

we have $x_i = A$ for $i \in \{1, 3, 8\}$, $x_i = B$ for $i \in \{4, 7, 9, 10\}$, and $x_i = C$ for $i \in \{2, 5, 6\}$. Thus $\Delta(\pi) = x = ACABCCBABB$.

Consider $\Theta(\Delta(\pi)) \in \hat{P}_C(\varnothing; 10)$. Since $\Delta(\pi) = x = ACABCCBABB$, the lists $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are defined as

$$\mathcal{A} = [1 < 3 < 8], \quad \mathcal{B} = [10 > 9 > 7 > 4], \quad \text{and} \quad \mathcal{C} = [2 < 5 < 6].$$

Using the definition of Θ, we get

$$\Theta(\Delta(\pi)) = \Theta(x) = 20\ 18\ 13\ 10\ 9\ 7\ 4\ 2\ 5\ 6 \,|\, 15\ 16\ 19\ 17\ 14\ 12\ 11\ 8\ 3\ 1 = \pi.$$

Type D: Consider the permutation $\pi \in \mathcal{D}_{10}$, where

$$\pi = 20\ 18\ 13\ 11\ 9\ 7\ 4\ 2\ 5\ 6 \,|\, 15\ 16\ 19\ 17\ 14\ 12\ 10\ 8\ 3\ 1.$$

Let $\Delta(\pi) = x \in T_n$. Since

$$\pi_A = 20\ 18\ 13\ 11, \quad \pi_B = 9\ 7\ 4\ 2, \quad \text{and} \quad \pi_C = 5\ 6,$$

we have $x_i = A$ for $i \in \{1, 3, 8, 10\}$, $x_i = B$ for $i \in \{2, 4, 7, 9\}$, and $x_i = C$ for $i \in \{5, 6\}$. Thus

$$\Delta(\pi) = x = ABABCCBABA.$$

Consider $\Theta(\Delta(\pi)) \in \hat{P}_D(\varnothing; 10)$. Since $\Delta(\pi) = x = ABABCCBABA$, the lists $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are defined as

$$\mathcal{A} = [1 < 3 < 8 < 10], \quad \mathcal{B} = [9 > 7 > 4 > 2], \quad \text{and} \quad \mathcal{C} = [5 < 6].$$

Using the definition of Θ, we get

$$\Theta(\Delta(\pi)) = \Theta(x) = 20\ 18\ 13\ 11\ 9\ 7\ 4\ 2\ 5\ 6 \,|\, 15\ 16\ 19\ 17\ 14\ 12\ 10\ 8\ 3\ 1 = \pi.$$

5. Questions and future work

We end this paper with a few questions of interest. We suspect that the sets we call pattern bundles have appeared elsewhere in the literature on Coxeter groups, but we do not know of such a reference. (Note that the pattern bundles are the fibers of an order-preserving flattening map from $\mathcal{C}_n$ to $\mathfrak{S}_n$ that differs from the usual 2^n to 1 projection of signed permutations to $\mathfrak{S}_n$, which forgets the negative signs.) If these sets have not been studied before, then our first question is:

Problem 1. Can the pattern bundles of types C_n and D_n be used to study other permutation statistics (such as descent sets for instance)?

We can also ask whether these techniques can be applied to study other groups having *suitably nice* embeddings into $\mathfrak{S}_N$, and whether the peak set of the image encodes any information about the embedded group.

Problem 2. Can the methods used in this paper be applied to study peak sets of groups such as the dihedral groups or Coxeter groups of exceptional type by embedding them into $\mathfrak{S}_N$ for some N?

We provide recursive formulas for the quantities $|\hat{P}_C(S; n)|$ and $|\hat{P}_D(S; n)|$ in Theorem 26 that can be used to find closed formulas for any particular choice of peak set S. Several of the special cases we consider in this paper give closed formulas for integer sequences appearing in [OEIS 1996]. Hence we believe the following would be an interesting undergraduate student research project.

Problem 3 (undergraduate student research project). Can one compute closed formulas for some families of peak sets and analyze which of these appear on the OEIS?

This leads us to our final question:

Problem 4. Can one discover closed combinatorial formulas for $|\hat{P}_C(S; n)|$ and $|\hat{P}_D(S; n)|$ in general?

Acknowledgements

The authors would like to thank the *Underrepresented Students in Topology and Algebra Symposium* (USTARS); if not for our chance encounter at USTARS this collaboration may not have materialized! We also thank Sara Billey, Christophe Hohlweg, and Bruce Sagan for helpful conversations about this paper. Pamela E. Harris gratefully acknowledges travel support from the Photonics Research Center and the Mathematical Sciences Center of Excellence at the United States Military Academy.

References

[Aguiar et al. 2004] M. Aguiar, N. Bergeron, and K. Nyman, "The peak algebra and the descent algebras of types B and D", *Trans. Amer. Math. Soc.* **356**:7 (2004), 2781–2824. MR Zbl

[Aguiar et al. 2006a] M. Aguiar, N. Bergeron, and F. Sottile, "Combinatorial Hopf algebras and generalized Dehn–Sommerville relations", *Compos. Math.* **142**:1 (2006), 1–30. MR Zbl

[Aguiar et al. 2006b] M. Aguiar, K. Nyman, and R. Orellana, "New results on the peak algebra", *J. Algebraic Combin.* **23**:2 (2006), 149–188. MR Zbl

[Bergeron and Hohlweg 2006] N. Bergeron and C. Hohlweg, "Coloured peak algebras and Hopf algebras", *J. Algebraic Combin.* **24**:3 (2006), 299–330. MR Zbl

[Bergeron and Sottile 2002] N. Bergeron and F. Sottile, "Skew Schubert functions and the Pieri formula for flag manifolds", *Trans. Amer. Math. Soc.* **354**:2 (2002), 651–673. MR Zbl

[Bergeron et al. 2000] N. Bergeron, S. Mykytiuk, F. Sottile, and S. van Willigenburg, "Noncommutative Pieri operators on posets", *J. Combin. Theory Ser. A* **91**:1–2 (2000), 84–110. MR Zbl

[Bergeron et al. 2002] N. Bergeron, S. Mykytiuk, F. Sottile, and S. van Willigenburg, "Shifted quasi-symmetric functions and the Hopf algebra of peak functions", *Discrete Math.* **246**:1–3 (2002), 57–66. MR Zbl

[Billera et al. 2003] L. J. Billera, S. K. Hsiao, and S. van Willigenburg, "Peak quasisymmetric functions and Eulerian enumeration", *Adv. Math.* **176**:2 (2003), 248–276. MR Zbl

[Billey and Haiman 1995] S. Billey and M. Haiman, "Schubert polynomials for the classical groups", *J. Amer. Math. Soc.* **8**:2 (1995), 443–482. MR Zbl

[Billey and Lakshmibai 2000] S. Billey and V. Lakshmibai, *Singular loci of Schubert varieties*, Progress in Mathematics **182**, Birkhäuser, Boston, 2000. MR Zbl

[Billey et al. 2013] S. Billey, K. Burdzy, and B. E. Sagan, "Permutations with given peak set", *J. Integer Seq.* **16**:6 (2013), Article 13.6.1, 18. MR Zbl

[Billey et al. 2015] S. Billey, K. Burdzy, S. Pal, and B. E. Sagan, "On meteors, earthworms and WIMPs", *Ann. Appl. Probab.* **25**:4 (2015), 1729–1779. MR Zbl

[Billey et al. 2016] S. Billey, M. Fahrbach, and A. Talmage, "Coefficients and roots of peak polynomials", *Exp. Math.* **25**:2 (2016), 165–175. MR Zbl

[Björner and Brenti 2005] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics **231**, Springer, New York, 2005. MR Zbl

[Castro-Velez et al. 2013] F. Castro-Velez, A. Diaz-Lopez, R. Orellana, J. Pastrana, and R. Zevallos, "Number of permutations with same peak set for signed permutations", preprint, 2013. To appear in *J. Comb.* arXiv

[Diaz-Lopez et al. 2015] A. Diaz-Lopez, P. E. Harris, E. Insko, and D. Perez-Lavin, "Peaks Sets of Classical Coxeter Groups", preprint, 2015. arXiv

[Kasraoui 2012] A. Kasraoui, "The most frequent peak set of a random permutation", preprint, 2012. arXiv

[Nyman 2003] K. L. Nyman, "The peak algebra of the symmetric group", *J. Algebraic Combin.* **17**:3 (2003), 309–322. MR Zbl

[OEIS 1996] OEIS, "The on-line encyclopedia of integer sequences", 1996, http://oeis.org.

[Petersen 2007] T. K. Petersen, "Enriched P-partitions and peak algebras", *Adv. Math.* **209**:2 (2007), 561–610. MR Zbl

[Stanley 2012] R. P. Stanley, *Enumerative combinatorics, Volume 1*, 2nd ed., Cambridge Studies in Advanced Mathematics **49**, Cambridge University Press, 2012. MR Zbl

[Stembridge 1997] J. R. Stembridge, "Enriched P-partitions", *Trans. Amer. Math. Soc.* **349**:2 (1997), 763–788. MR Zbl

Received: 2015-09-11 Revised: 2016-01-21 Accepted: 2016-02-07

adiazlo1@swarthmore.edu *Department of Mathematics and Statistics, Swarthmore College, Swarthmore, PA 19081, United States*

peh2@williams.edu *Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267, United States*

einsko@fgcu.edu *Department of Mathematics, Florida Gulf Coast University, Fort Myers, FL 33965, United States*

darleenpl@uky.edu *Department of Mathematics, University of Kentucky, Lexington, KY 40506, United States*

mathematical sciences publishers msp

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

MANAGING EDITOR

Kenneth S. Berenhaut — Wake Forest University, USA

BOARD OF EDITORS

Colin Adams	Williams College, USA
John V. Baxley	Wake Forest University, NC, USA
Arthur T. Benjamin	Harvey Mudd College, USA
Martin Bohner	Missouri U of Science and Technology, USA
Nigel Boston	University of Wisconsin, USA
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA
Pietro Cerone	La Trobe University, Australia
Scott Chapman	Sam Houston State University, USA
Joshua N. Cooper	University of South Carolina, USA
Jem N. Corcoran	University of Colorado, USA
Toka Diagana	Howard University, USA
Michael Dorff	Brigham Young University, USA
Sever S. Dragomir	Victoria University, Australia
Behrouz Emamizadeh	The Petroleum Institute, UAE
Joel Foisy	SUNY Potsdam, USA
Errin W. Fulp	Wake Forest University, USA
Joseph Gallian	University of Minnesota Duluth, USA
Stephan R. Garcia	Pomona College, USA
Anant Godbole	East Tennessee State University, USA
Ron Gould	Emory University, USA
Andrew Granville	Université Montréal, Canada
Jerrold Griggs	University of South Carolina, USA
Sat Gupta	U of North Carolina, Greensboro, USA
Jim Haglund	University of Pennsylvania, USA
Johnny Henderson	Baylor University, USA
Jim Hoste	Pitzer College, USA
Natalia Hritonenko	Prairie View A&M University, USA
Glenn H. Hurlbert	Arizona State University,USA
Charles R. Johnson	College of William and Mary, USA
K. B. Kulasekera	Clemson University, USA
Gerry Ladas	University of Rhode Island, USA
Suzanne Lenhart	University of Tennessee, USA
Chi-Kwong Li	College of William and Mary, USA
Robert B. Lund	Clemson University, USA
Gaven J. Martin	Massey University, New Zealand
Mary Meyer	Colorado State University, USA
Emil Minchev	Ruse, Bulgaria
Frank Morgan	Williams College, USA
Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Zuhair Nashed	University of Central Florida, USA
Ken Ono	Emory University, USA
Timothy E. O'Brien	Loyola University Chicago, USA
Joseph O'Rourke	Smith College, USA
Yuval Peres	Microsoft Research, USA
Y.-F. S. Pétermann	Université de Genève, Switzerland
Robert J. Plemmons	Wake Forest University, USA
Carl B. Pomerance	Dartmouth College, USA
Vadim Ponomarenko	San Diego State University, USA
Bjorn Poonen	UC Berkeley, USA
James Propp	U Mass Lowell, USA
Józeph H. Przytycki	George Washington University, USA
Richard Rebarber	University of Nebraska, USA
Robert W. Robinson	University of Georgia, USA
Filip Saidak	U of North Carolina, Greensboro, USA
James A. Sellers	Penn State University, USA
Andrew J. Sterge	Honorary Editor
Ann Trenk	Wellesley College, USA
Ravi Vakil	Stanford University, USA
Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
Ram U. Verma	University of Toledo, USA
John C. Wierman	Johns Hopkins University, USA
Michael E. Zieve	University of Michigan, USA

PRODUCTION

Silvio Levy, Scientific Editor

Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2017 is US $175/year for the electronic version, and $235/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2017 Mathematical Sciences Publishers

involve

2017 vol. 10 no. 2

1944-4176(2017)10:2;1-B