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Fox coloring and the minimum number of colors

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We study Fox colorings of knots that are 13-colorable. We prove that any 13-colorable knot has a diagram that uses exactly five of the thirteen colors that are assigned to the arcs of the diagram. Due to an existing lower bound, this gives that the minimum number of colors of any 13-colorable knot is 5.

1. Introduction

Fox [1962] introduced a diagrammatic definition of colorability of a knot K by $\mathbb{Z}_m$ (the integers modulo m). This notion of colorability is clearly one of the simplest invariants of knots. For a natural number m greater than 1, a diagram D of a knot K is m-colorable if at every crossing, the sum of the colors of the under-arcs is twice the color of the over-arc (modulo m), as in Figure 1.

It is well known [Fox 1962] that for a prime p, a knot K is p-colorable if and only if p divides the determinant of K. The problem of finding the minimum number of colors for p-colorable knots with p prime and less than or equal to 11 was studied in [Satoh 2009; Oshiro 2010; Lopes and Matias 2012; Hayashi et al. 2012]. For example, Satoh [2009] proved that any 5-colorable knot admits a nontrivially 5-colored diagram where the coloring assignment uses only four of the five available colors. For a prime p, let K be a p-colorable knot and let $C_p(K)$ denote the minimum number of colors among all diagrams of the knot K. In [Nakamura et al. 2013], it was proved that $C_p(K) \geq \lfloor \log_2 p \rfloor + 2$. This implies that in our case, $p = 13$, the minimum number of colors of 13-colorable knots is greater than or equal to 5. In fact, the goal of this article is to prove equality, that is, $C_{13}(K) = 5$.

2. Fox coloring and the minimum number of colors of 13-colorable knots

Notation. We use $\{a\,|\,b\,|\,c\}$ to denote a crossing, as in Figure 1, where a and c are the colors of the under-arcs, b is the color of the over-arc and $a + c \equiv 2b \bmod 13$. When the crossing is of the type $\{c\,|\,c\,|\,c\}$ (trivial coloring), we will omit over- and under-crossings and draw the arcs crossing each other.

MSC2010: 57M25.

Keywords: knots, fox colorings, minimum number of colors.

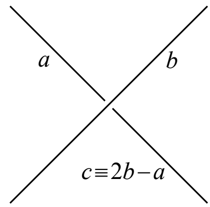


Figure 1

Theorem 2.1. *Any* 13*-colorable knot has a* 13*-colored diagram with exactly five colors. Thus*, $C_{13}(K) = 5$ *for any* 13*-colorable knot K.*

Proof. We prove this theorem using eight lemmas. In each of the following lemmas we decrease the coloring scheme of the diagram by one color c. To accomplish this we first transform any crossings of the form $\{c\,|\,c\,|\,c\}$. That is, when c is both an over-arc and an under-arc, we remove c as an over-arc by transforming any crossings of the form $\{a\,|\,c\,|\,2c{-}a\}$, where $a \in \mathbb{Z}_{13} \setminus \{c\}$. Finally, we complete each lemma by removing c as an under-arc in a case-by-case method. In these under-arc cases we must consider when c connects two crossings of the same color and when c connects two crossings of different colors. $\square$

Eliminating the color 12.

Lemma 2.2. *Any* 13*-colorable knot has a* 13*-colored diagram D with no arc colored by* 12*.*

Proof. Let $c = 12$. We first transform any crossing of the form $\{12\,|\,12\,|\,12\}$. If there is any such crossing, there is an adjacent crossing of the form $\{12\,|\,a\,|\,2a{+}1\}$ or $\{a\,|\,12\,|\,11{-}a\}$, where $a \in \mathbb{Z}_{13} \setminus \{12\}$. In either case, since $11 - a \neq 12$ and $2a + 1 \neq 12$ for any a in $\mathbb{Z}_{13} \setminus \{12\}$, we transform the diagram as in Figures 2 and 3.

Next, we remove 12 as an over-arc by transforming any crossings of the form $\{a\,|\,12\,|\,11{-}a\}$. Since $2a + 1 \neq 12$ and $3a + 2 \neq 12$ for any $a \in \mathbb{Z}_{13} \setminus \{12\}$, we transform the diagram as in Figure 4.

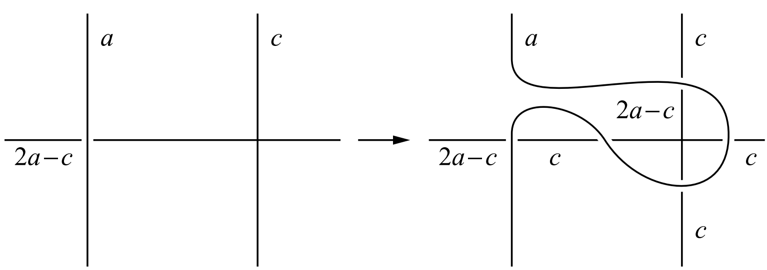


Figure 2

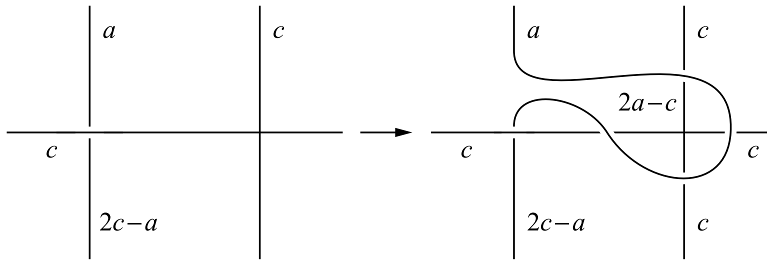


Figure 3

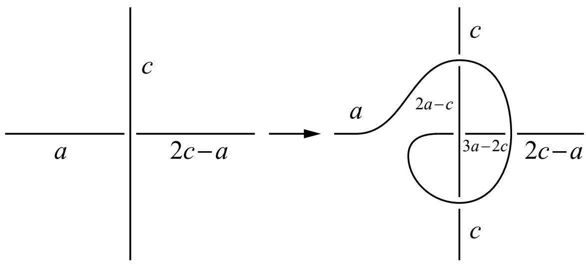


Figure 4

We complete the proof of the lemma by removing 12 as an under-arc in a case-by-case method. We first consider the case where 12 is an under-arc connecting two crossings of the form $\{12\,|\,a\,|\,2a{+}1\}$. Since $2a+1 \neq 12$, $3a+2 \neq 12$, and $4a+3 \neq 12$ for any $a \in \mathbb{Z}_{13} \setminus \{12\}$, we transform the diagram as in Figure 5.

Now we consider the case where 12 is an under-arc connecting two crossings of the forms $\{2a{+}1\,|\,a\,|\,12\}$ and $\{12\,|\,2a{+}1\,|\,4a{+}3\}$. Since $2a+1 \neq 12$ and $3a+2 \neq 12$ for any $a \in \mathbb{Z}_{13} \setminus \{12\}$, we transform the diagram as in Figure 6.

Lastly we consider the case where 12 is an under-arc connecting two crossings of the forms $\{2a{+}1\,|\,a\,|\,12\}$ and $\{12\,|\,b\,|\,2b{+}1\}$, where $a \neq b$ and $b \neq 2a+1$ for any a and b in $\mathbb{Z}_{13}\setminus\{12\}$. Since $2a-2b-1 \neq 12$ and $2a-b \neq 12$ for any a and b in $\mathbb{Z}_{13}\setminus\{12\}$ (from $a \neq b$ and $b \neq 2a+1$ respectively), we transform the diagram as in Figure 7. □

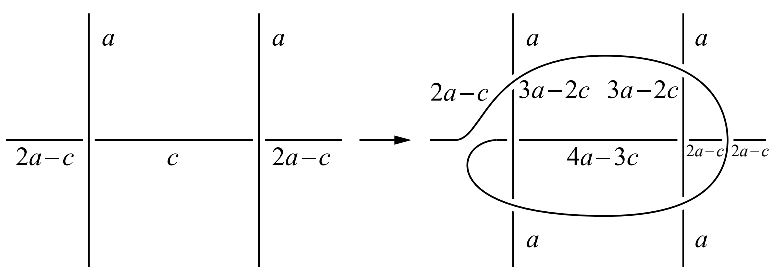


Figure 5

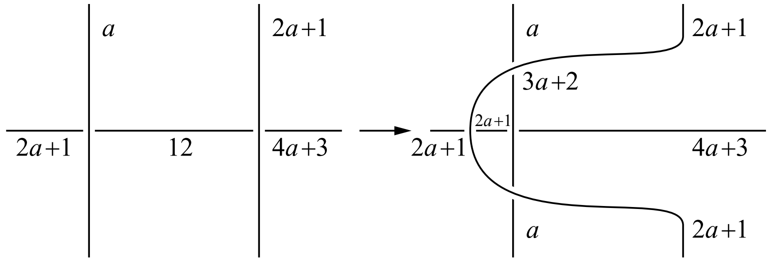


Figure 6

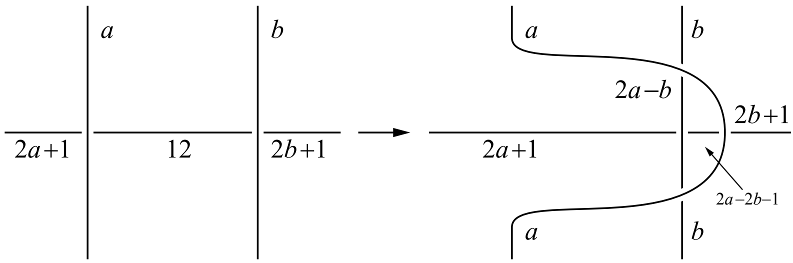


Figure 7

Eliminating the color 11.

Lemma 2.3. *Any* 13*-colorable knot has a* 13*-colored diagram D with no arc colored by* 11 *or* 12*.*

Proof. Let $c = 11$. By the previous lemma we assume that no arc in D is colored by 12. We first transform any crossing of the form $\{11\,|\,11\,|\,11\}$. If there is any such crossing, there is an adjacent crossing of the form $\{11\,|\,a\,|\,2a{+}2\}$ or $\{a\,|\,11\,|\,9{-}a\}$, where a is in $\mathbb{Z}_{13} \setminus \{11, 12\}$. If $a \neq 5, 10$, then $9 - a \neq 11, 12$ and $2a + 2 \neq 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{5, 10, 11, 12\}$, so we transform the diagram as in Figures 2 and 3.

If $a = 5$ as an under-arc, we transform the diagram as in Figure 8. Now, a cannot equal 5 as an over-arc, otherwise $2a + 2 = 12$, contradicting our assumption that no arc is colored by 12.

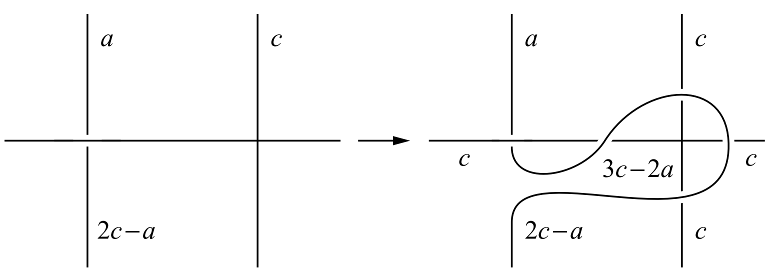


Figure 8

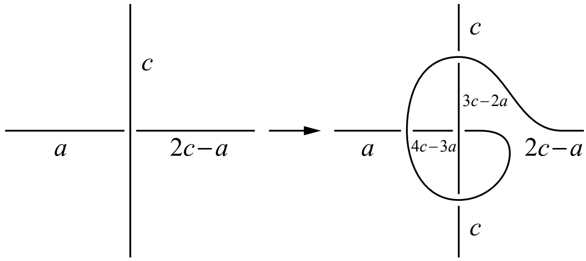


Figure 9

If $a = 10$ as an over-arc, we transform the diagram as in Figure 2. Similarly a cannot equal 10 as an under-arc, otherwise $9-a = 12$, which is a contradiction.

Next, we remove 11 as an over-arc by transforming any crossings of the form $\{a\,|\,11\,|\,9{-}a\}$. Since $9-a \neq 11, 12$, we have $a \neq 10$. Therefore if $a \neq 5, 7$ then $2a+2 \neq 11$, 12 and $3a+4 \neq 11$, 12 for any a in $\mathbb{Z}_{13} \setminus \{5, 7, 10, 11, 12\}$, and we transform the diagram as in Figure 4. If $a = 5$ or $a = 7$, we transform the diagram as in Figure 9.

We complete the proof of the lemma by removing 11 as an under-arc in a case-by-case method. We first consider the case where 11 is an under-arc connecting two crossings of the form $\{11\,|\,a\,|\,2a{+}2\}$. Since $2a+2 \neq 11$, 12, we have $a \neq 5$. If $a \neq 7, 8$, then $3a+4 \neq 11$, 12 and $4a+6 \neq 11$, 12 for any a in $\mathbb{Z}_{13} \setminus \{5, 7, 8, 11, 12\}$, and we transform the diagram as in Figure 5. If $a = 7$, we transform the diagram as in Figure 10. If $a = 8$, we transform the diagram as in Figure 11.

Now we consider the case where 11 is an under-arc connecting two crossings of the forms $\{2a{+}2\,|\,a\,|\,11\}$ and $\{11\,|\,b\,|\,2b{+}2\}$, where $a \neq b$ for any a and b in $\mathbb{Z}_{13} \setminus \{5, 11, 12\}$. (Note $a, b \neq 5$, otherwise $2a+2 = 12$ or $2b+2 = 12$.)

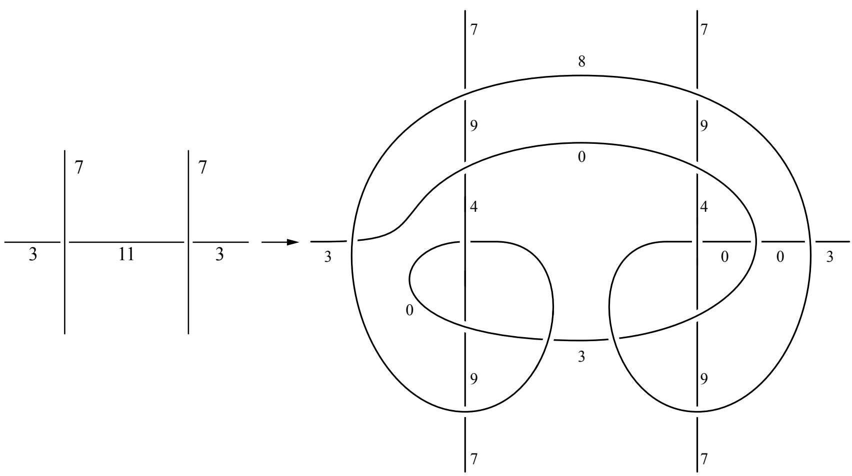


Figure 10

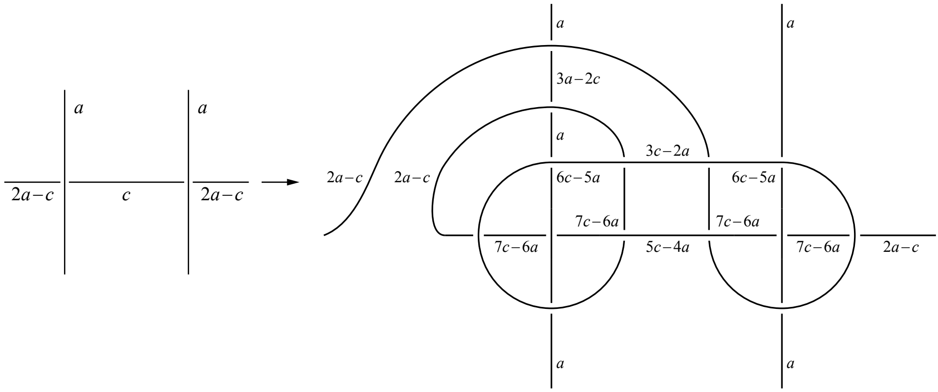


Figure 11

If $(a, b) \neq (0, 6), (6, 0), (3, 7), (7, 3)$ then either

$$2a - 2b - 2 \neq 11, 12 \quad \text{and} \quad 2a - b \neq 11, 12$$

or

$$2b - 2a - 2 \neq 11, 12 \quad \text{and} \quad 2b - a \neq 11, 12$$

for any a and b in $\mathbb{Z}_{13} \setminus \{5, 11, 12\}$, and we transform the diagram as in Figure 12.

If $(a, b) = (0, 6)$, we transform the diagram as in Figure 13. A similar transformation works for the case $(a, b) = (6, 0)$.

If $(a, b) = (3, 7)$, we transform the diagram as in Figure 14. A similar transformation works for the case $(a, b) = (7, 3)$. □

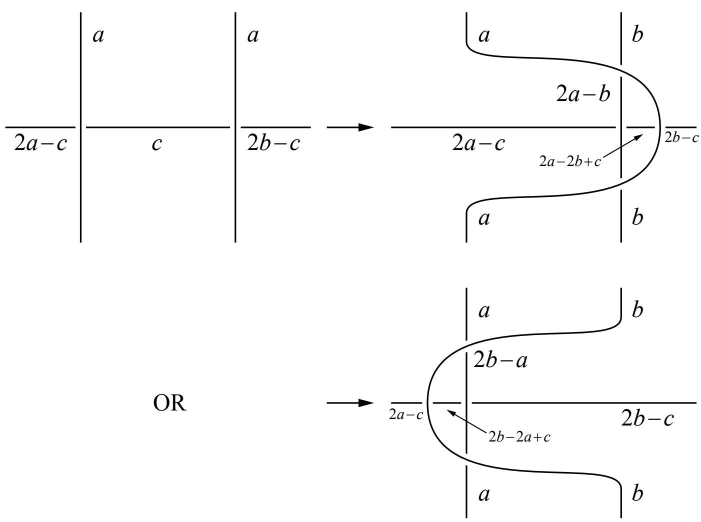


Figure 12

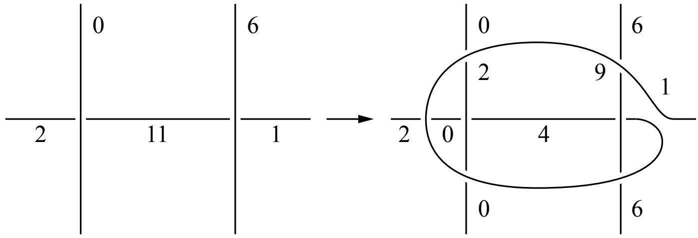


Figure 13

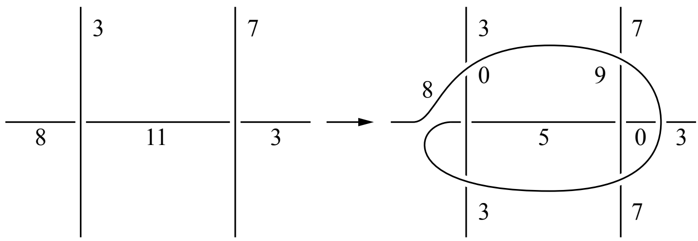


Figure 14

Eliminating the color 7.

Lemma 2.4. *Any* 13*-colorable knot has a* 13*-colored diagram D with no arc colored by* 7, 11, *or* 12*.*

Proof. Let $c = 7$. By the previous lemmas we assume that no arc in D is colored by 11 or 12. We first transform any crossing of the form $\{7|7|7\}$. If there is any such crossing, there is an adjacent crossing of the form $\{7|a|2a+6\}$ or $\{a|7|1-a\}$, where a is in $\mathbb{Z}_{13}\setminus\{7, 11, 12\}$. If $a \neq 2, 3, 9$ then $1-a \neq 7, 11, 12$ and $2a+6 \neq 7, 11, 12$ for any a in $\mathbb{Z}_{13}\setminus\{2, 3, 7, 9, 11, 12\}$, so we transform the diagram as in Figures 2 and 3.

If $a = 2$ as an over-arc, we transform the diagram as in Figure 2. Note a cannot equal 2 as an under-arc, otherwise $1 - a = 12$, contradicting our assumption that no arc is colored by 12.

Now a cannot be 3 as an over-arc or an under-arc, otherwise $1 - a = 11$ and $2a + 6 = 12$, contradicting our assumption that no arc is colored by 11 or 12. If $a = 9$ as an under-arc, we transform the diagram as in Figure 8. Note a cannot equal 9 as an over-arc, otherwise $2a + 6 = 11$, contradicting our assumption that no arc is colored by 11. Therefore any crossings of the form $\{7|7|7\}$ are removed.

Next, we remove 7 as an over-arc by transforming any crossings of the form $\{a|7|1-a\}$. Since $1-a \neq 7, 11, 12$, we have $a \neq 2, 3$. Therefore if $a \neq 0, 4, 9$ then $2a+6 \neq 7, 11, 12$ and $3a+12 \neq 7, 11, 12$ for any a in $\mathbb{Z}_{13}\setminus\{0, 2, 3, 4, 7, 9, 11, 12\}$, and we transform the diagram as in Figure 4. If $a = 0, 4, 9$, we transform the diagram as in Figure 9.

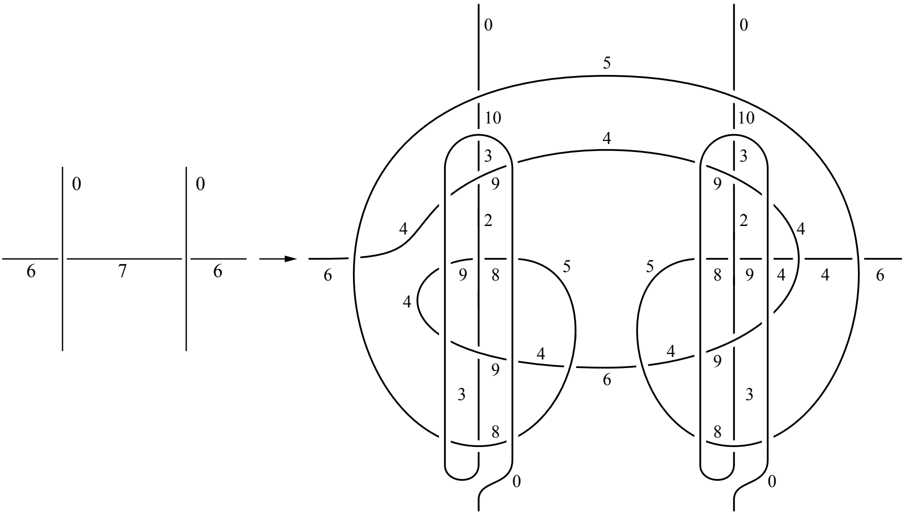


Figure 15

We complete the proof of the lemma by removing 7 as an under-arc in a case-by-case method. We first consider the case where 7 is an under-arc connecting two crossings of the form $\{7|a|2a+6\}$. Since $2a+6 \neq 7, 11, 12$, we have $a \neq 3, 9$. If $a \neq 0, 4, 5, 8$ then $3a+12 \neq 7, 11, 12$ and $4a+5 \neq 7, 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{0, 3, 4, 5, 7, 8, 9, 11, 12\}$, and we transform the diagram as in Figure 5. If $a = 0$, we transform the diagram as in Figure 15. If $a = 4$, we transform the diagram as in Figure 16. If $a = 5$, we transform the diagram as in Figure 17. If $a = 8$, we transform the diagram as in Figure 11.

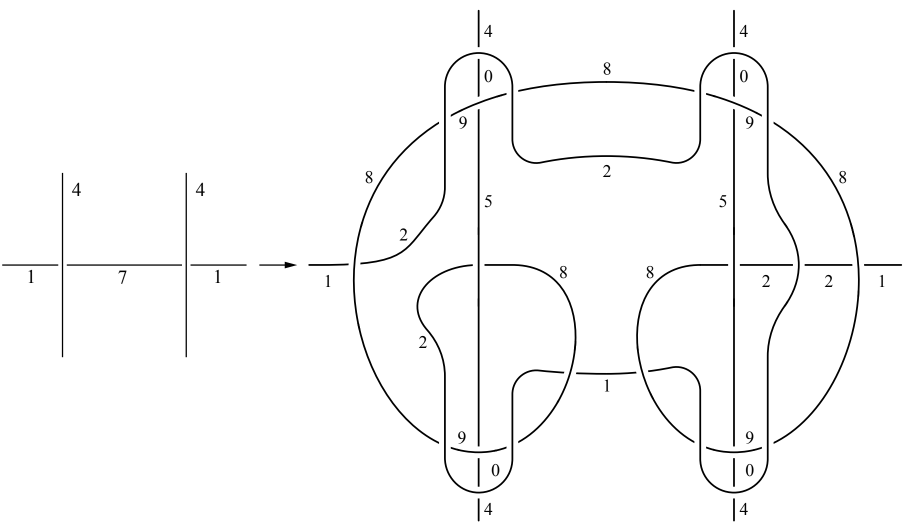


Figure 16

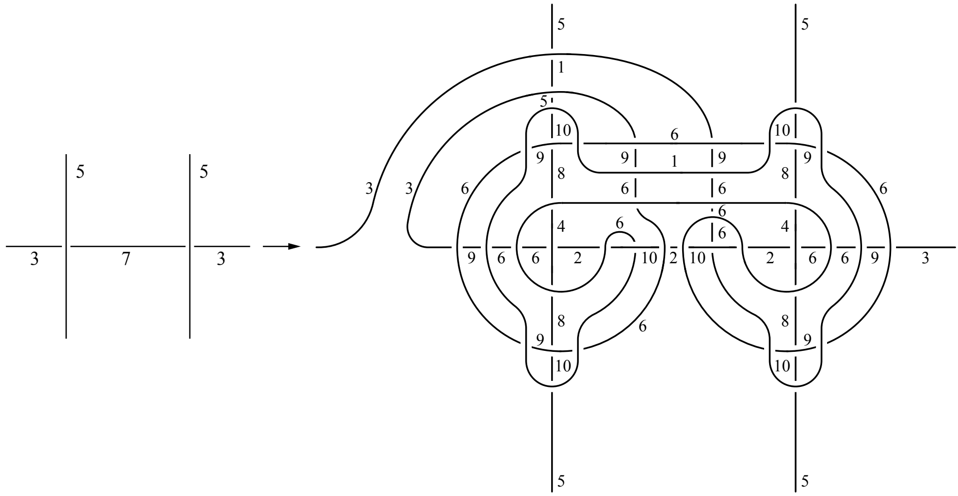


Figure 17

Now we consider the case where 7 is an under-arc connecting two crossings of the forms $\{2a{+}6|a|7\}$ and $\{7|b|2b{+}6\}$, where $a \neq b$ for any a and b in $\mathbb{Z}_{13} \setminus \{3, 7, 9, 11, 12\}$. (Note $a, b \neq 3, 9$, otherwise $2a + 6 = 11, 12$ or $2b + 6 = 11, 12$.) If

$$(a, b) \neq (0, 2), (2, 0), (0, 6), (6, 0), (1, 4), (4, 1), (4, 8), (8, 4)$$

then either

$$2a - 2b - 6 \neq 7, 11, 12 \quad \text{and} \quad 2a - b \neq 7, 11, 12$$

or

$$2b - 2a - 6 \neq 7, 11, 12 \quad \text{and} \quad 2b - a \neq 7, 11, 12$$

for any a and b in $\mathbb{Z}_{13} \setminus \{3, 7, 9, 11, 12\}$, and we transform the diagram as in Figure 12.

If $(a, b) = (0, 2)$, we transform the diagram as in Figure 18. The case $(a, b) = (2, 0)$ is similar.

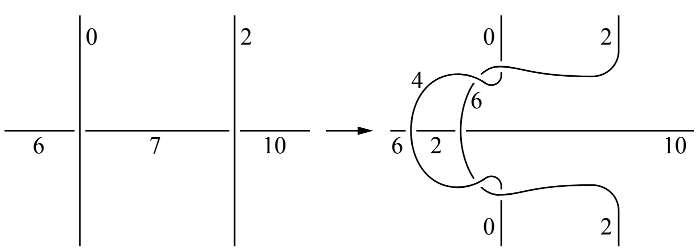


Figure 18

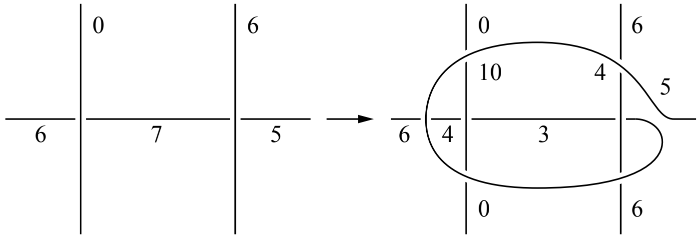


Figure 19

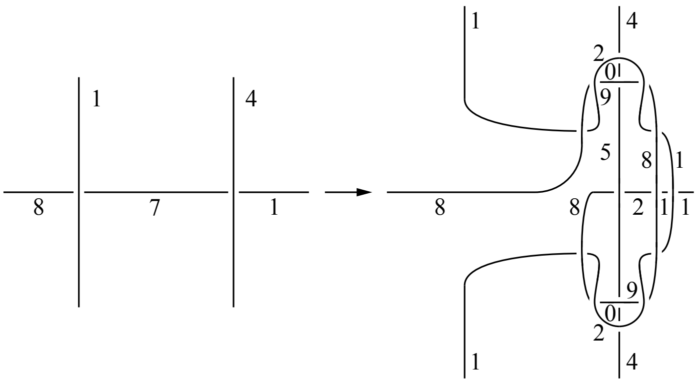


Figure 20

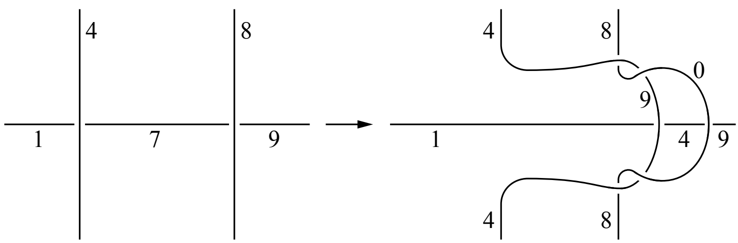


Figure 21

If $(a, b) = (0, 6)$, we transform the diagram as in Figure 19. The case $(a, b) = (6, 0)$ is similar.

If $(a, b) = (1, 4)$, we transform the diagram as in Figure 20. The case $(a, b) = (4, 1)$ is similar.

If $(a, b) = (4, 8)$, we transform the diagram as in following Figure 21. The case $(a, b) = (8, 4)$ is similar. □

Eliminating the color 8.

Lemma 2.5. *Any* 13*-colorable knot has a* 13*-colored diagram* D *with no arc colored by* 7, 8, 11, *or* 12*.*

Proof. Let $c = 8$. By the previous lemmas we assume that no arc in D is colored by 7, 11, or 12. We first transform any crossing of the form $\{8\,|\,8\,|\,8\}$. If there is any such crossing, there is an adjacent crossing of the form $\{8\,|\,a\,|\,2a{+}5\}$ or $\{a\,|\,8\,|\,3{-}a\}$, where a is in $\mathbb{Z}_{13} \setminus \{7, 8, 11, 12\}$. If $a \neq 1, 3, 4, 5, 9, 10$ then $3 - a \neq 7, 8, 11, 12$ and $2a + 5 \neq 7, 8, 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{1, 3, 4, 5, 7, 8, 9, 10, 11, 12\}$, so we transform the diagram as in Figures 2 and 3.

If $a = 4, 5, 9$ as an over-arc, we transform the diagram as in Figure 2. Note a cannot be 4, 5, or 9 as an under-arc, otherwise $3 - a = 7, 11, 12$, contradicting our assumption that no arc is colored by 7, 11, or 12. If $a = 1$ as an under-arc we transform the diagram as in Figure 8. Note a cannot be 1 as an over-arc, otherwise $2a + 5 = 7$, contradicting our assumption that no arc is colored by 7. If $a = 3$ as an under-arc, we transform the diagram as in Figure 8. Note a cannot be 3 as an over-arc, otherwise $2a + 5 = 11$, contradicting our assumption that no arc is colored by 11. If $a = 10$ as an under-arc, we transform the diagram as in Figure 8. Note a cannot be 10 as an over-arc, otherwise $2a + 5 = 12$, contradicting our assumption that no arc is colored by 12. Therefore any crossings of the form $\{8\,|\,8\,|\,8\}$ are removed.

Next, we remove 8 as an over-arc by transforming any crossings of the form $\{a\,|\,8\,|\,3{-}a\}$. Since $3 - a \neq 7, 8, 11, 12$, we have $a \neq 4, 5, 9$. Therefore if $a \neq 1, 3, 10$ then $2a + 5 \neq 7, 8, 11, 12$ and $3a + 10 \neq 7, 8, 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{1, 3, 4, 5, 7, 8, 9, 10, 11, 12\}$, and we transform the diagram as in Figure 4. If $a = 1, 3$ or 10, we transform the diagram as in Figure 9.

We complete the proof of the lemma by removing 8 as an under-arc in a case-by-case method. We first consider the case where 8 is an under-arc connecting two crossings of the form $\{8\,|\,a\,|\,2a{+}5\}$. Since $2a + 5 \neq 7, 8, 11, 12$, we have $a \neq 1, 3, 10$. If $a \neq 5, 9$ then $3a + 10 \neq 7, 8, 11, 12$ and $4a + 2 \neq 7, 8, 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{1, 3, 5, 7, 8, 9, 10, 11, 12\}$, and we transform the diagram as in Figure 5. If $a = 5$, we transform the diagram as in Figure 22. If $a = 9$, we transform the diagram as in Figure 23.

Now we consider the case where 8 is an under-arc connecting two crossings of the forms $\{2a{+}5\,|\,a\,|\,8\}$ and $\{8\,|\,b\,|\,2b{+}5\}$, where $a \neq b$ for any a and b in $\mathbb{Z}_{13} \setminus \{1, 3, 7, 8, 10, 11, 12\}$. (Note $a, b \neq 1, 3, 10$, otherwise $2a + 5 = 7, 11, 12$ or $2b + 5 = 7, 11, 12$.) If $(a, b) \neq (0, 2), (2, 0), (0, 6), (6, 0), (2, 5), (5, 2)$ then either

$$2a - 2b - 5 \neq 7, 8, 11, 12 \quad \text{and} \quad 2a - b \neq 7, 8, 11, 12$$

or

$$2b - 2a - 5 \neq 7, 8, 11, 12 \quad \text{and} \quad 2b - a \neq 7, 8, 11, 12$$

for any a and b in $\mathbb{Z}_{13} \setminus \{1, 3, 7, 8, 10, 11, 12\}$, and we transform the diagram as in Figure 12.

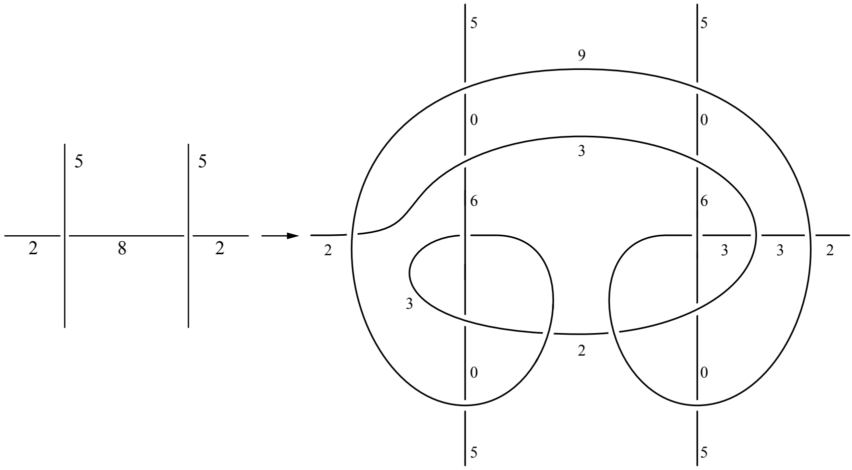


Figure 22

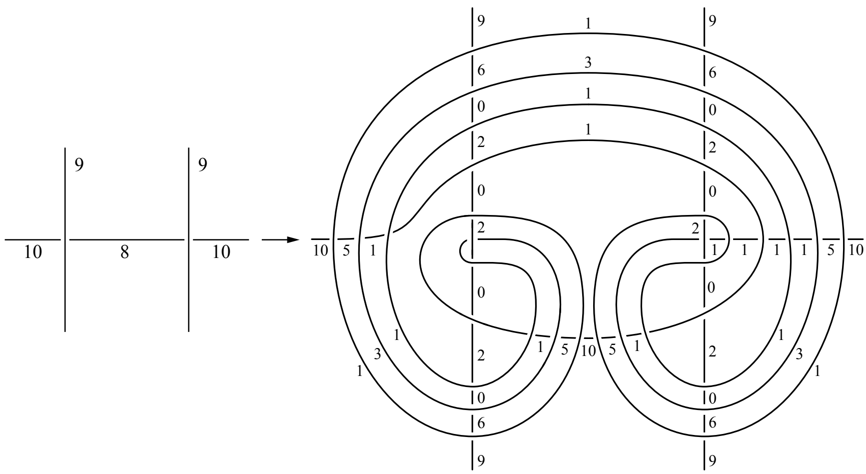


Figure 23

If $(a, b) = (0, 2)$, we transform the diagram as in Figure 24. The case $(a, b) = (2, 0)$ is similar.

If $(a, b) = (0, 6)$, we transform the diagram as in Figure 25. The case $(a, b) = (6, 0)$ is similar.

If $(a, b) = (2, 5)$, we transform the diagram as in Figure 26. The case $(a, b) = (5, 2)$ is similar. □

Eliminating the color 6.

Lemma 2.6. *Any* 13*-colorable knot has a* 13*-colored diagram D with no arc colored by* 6, 7, 8, 11, *or* 12.

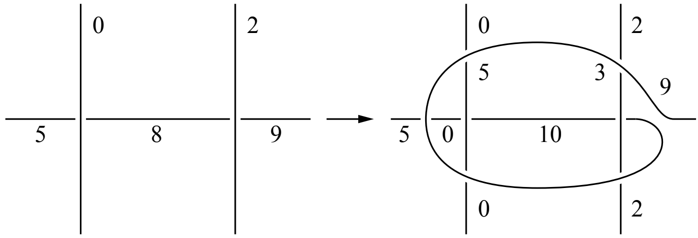


Figure 24

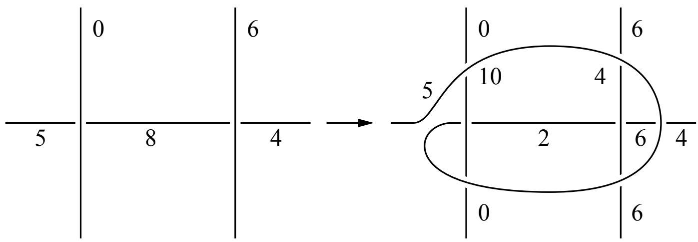


Figure 25

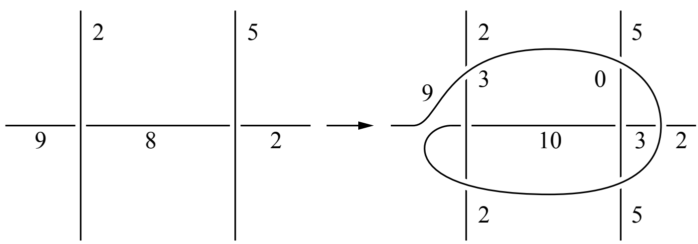


Figure 26

Proof. Let $c = 6$. By the previous lemmas we assume that no arc in D is colored by 7, 8, 11, or 12. We first transform any crossing of the form $\{6\,|\,6\,|\,6\}$. If there is any such crossing, there is an adjacent crossing of the form $\{6\,|\,a\,|\,2a{+}7\}$ or $\{a\,|\,6\,|\,12{-}a\}$, where a is in $\mathbb{Z}_{13} \setminus \{6, 7, 8, 11, 12\}$. With the exceptions of $a = 0, 2, 9$ as an over-arc (when $2a + 7 = 7, 8, 11, 12$) and $a = 0, 1, 4, 5$ as an under-arc (when $12 - a = 7, 8, 11, 12$), we transform the diagram as in Figures 2 and 3.

Now we must check when $a = 0, 2, 9$ as an under-arc. First and foremost a cannot equal 0 as an under-arc, otherwise $12 - a = 12$, contradicting our assumption that no arc is colored by 12. If $a = 2, 9$ as an under-arc, we transform the diagram as in Figure 8. Therefore any crossings of the form $\{6\,|\,6\,|\,6\}$ are removed.

Next, we remove 6 as an over-arc by transforming any crossings of the form $\{a\,|\,6\,|\,12{-}a\}$. Since $12 - a \neq 6, 7, 8, 11, 12$, we have $a \neq 0, 1, 4, 5$. With the exceptions of $a = 2, 9$ (when $2a + 7 = 6, 7, 8, 11, 12$ and $3a + 1 = 6, 7, 8, 11, 12$),

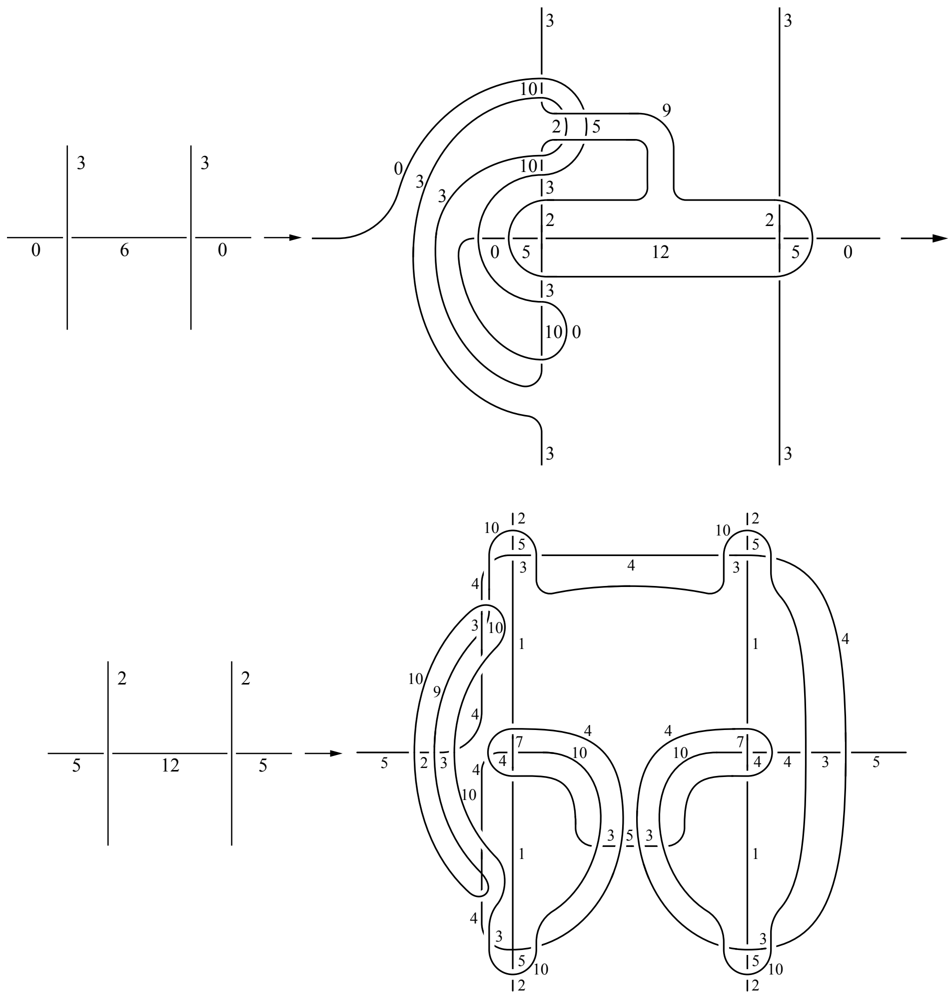


Figure 27

we transform the diagram as in Figure 4. If $a = 2$ or $a = 9$, we transform the diagram as in Figure 9.

We complete the proof of the lemma by removing 6 as an under-arc in a case-by-case method. We first consider the case where 6 is an under-arc connecting two crossings of the form $\{6|a|2a{+}7\}$. Since $2a + 7 \neq 6, 7, 8, 11, 12$, we have $a \neq 0, 2, 9$. If $a \neq 1, 3, 4$ then $3a + 1 \neq 6, 7, 8, 11, 12$ and $4a + 8 \neq 6, 7, 8, 11, 12$, so we transform the diagram as in Figure 5. If $a = 1$, we transform the diagram as in Figure 11. If $a = 3$, we transform the diagram as in Figures 27 and 28. If $a = 4$, we transform the diagram as in Figure 11.

Now we consider the case where 6 is an under-arc connecting two crossings of the forms $\{2a{+}7|a|6\}$ and $\{6|b|2b{+}7\}$, where $a \neq b$ for any a and b in

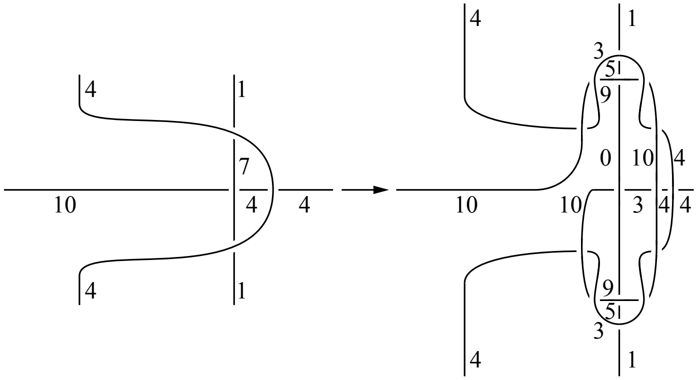


Figure 28

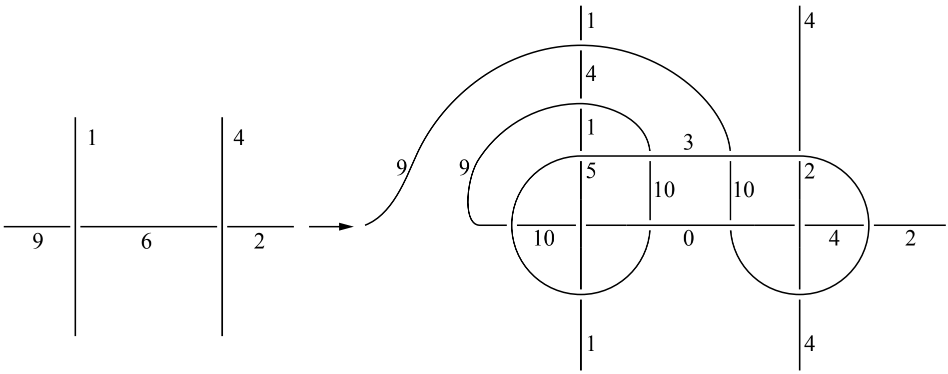


Figure 29

$\mathbb{Z}_{13} \setminus \{0, 2, 6, 7, 8, 9, 11, 12\}$. (Note $a, b \neq 0, 2, 9$, otherwise $2a + 7 = 7, 8, 11, 12$ or $2b + 7 = 7, 8, 11, 12$.)

If $(a, b) \neq (1, 4), (4, 1)$ then either

$$2a - 2b - 7 \neq 6, 7, 8, 11, 12 \quad \text{and} \quad 2a - b \neq 6, 7, 8, 11, 12$$

or

$$2b - 2a - 7 \neq 6, 7, 8, 11, 12 \quad \text{and} \quad 2b - a \neq 6, 7, 8, 11, 12$$

for any a and b in $\mathbb{Z}_{13} \setminus \{0, 2, 6, 7, 8, 9, 11, 12\}$, and we transform the diagram as in Figure 12.

If $(a, b) = (1, 4)$, we transform the diagram as in Figure 29. The case $(a, b) = (4, 1)$ is similar. $\square$

Eliminating the color 1.

Lemma 2.7. *Any* 13*-colorable knot has a* 13*-colored diagram D with no arc colored by* 1, 6, 7, 8, 11, *or* 12.

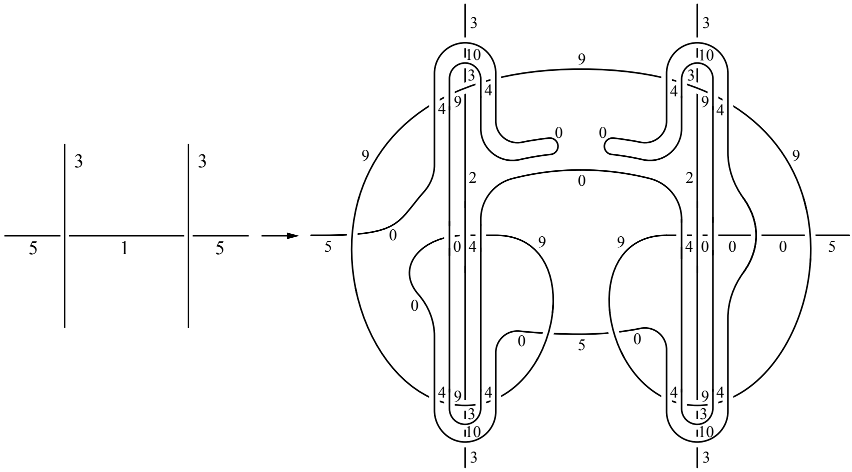


Figure 30

Proof. Let $c = 1$. By the previous lemmas we assume that no arc in D is colored by 6, 7, 8, 11, or 12. We first transform any crossing of the form $\{1\,|\,1\,|\,1\}$. If there is any such crossing, there is an adjacent crossing of the form $\{1\,|\,a\,|\,2a{+}12\}$ or $\{a\,|\,1\,|\,2{-}a\}$, where a is in $\mathbb{Z}_{13} \setminus \{1, 6, 7, 8, 11, 12\}$. With the exceptions of $a = 0, 4, 10$ as an over-arc (when $2a + 12 = 6, 7, 8, 11, 12$) and $a = 3, 4, 9$ as an under-arc (when $2 - a = 6, 7, 8, 11, 12$), we transform the diagram as in Figures 2 and 3.

Now we must check when $a = 0, 4, 10$ as an under-arc. We know a cannot be 4 as an under-arc, otherwise $2 - a = 11$, contradicting our assumption that no arc is colored by 11. If $a = 0$ or $a = 10$ as an under-arc, we transform the diagram as in Figure 8. Therefore any crossings of the form $\{1\,|\,1\,|\,1\}$ are removed.

Next, we remove 1 as an over-arc by transforming any crossings of the form $\{a\,|\,1\,|\,2{-}a\}$. Since $2{-}a \neq 1, 6, 7, 8, 11, 12$, we have $a \neq 3, 4, 9$. With the exceptions of $a = 0, 10$ (when $2a + 12 = 1, 6, 7, 8, 11, 12$ and $3a + 11 = 1, 6, 7, 8, 11, 12$), we transform the diagram as in Figure 4. If $a = 0$ or $a = 10$, we transform the diagram as in Figure 9.

We complete the proof by removing 1 as an under-arc in a case-by-case method. We first consider the case where 1 is an under-arc connecting two crossings of the form $\{1\,|\,a\,|\,2a{+}12\}$. Since $2a + 12 \neq 1, 6, 7, 8, 11, 12$, we have $a \neq 0, 4, 10$. If $a \neq 3, 9$ then $3a + 11 \neq 1, 6, 7, 8, 11, 12$ and $4a + 10 \neq 1, 6, 7, 8, 11, 12$, so we transform the diagram as in Figure 5. If $a = 3$, we transform the diagram as in Figure 30. If $a = 9$, we transform the diagram as in Figure 31.

Now we consider the case where 1 is an under-arc connecting two crossings of the forms $\{2a{+}12\,|\,a\,|\,1\}$ and $\{1\,|\,b\,|\,2b{+}12\}$, where $a \neq b$ for any a and b in $\mathbb{Z}_{13} \setminus \{0, 1, 4, 6, 7, 8, 10, 11, 12\}$. (Note $a, b \neq 0, 4, 10$, otherwise $2a + 12 = 1, 6, 7, 8, 11, 12$ or $2b + 12 = 1, 6, 7, 8, 11, 12$.) If $(a, b) \neq (2, 5), (5, 2), (3, 5), (5, 3)$ then

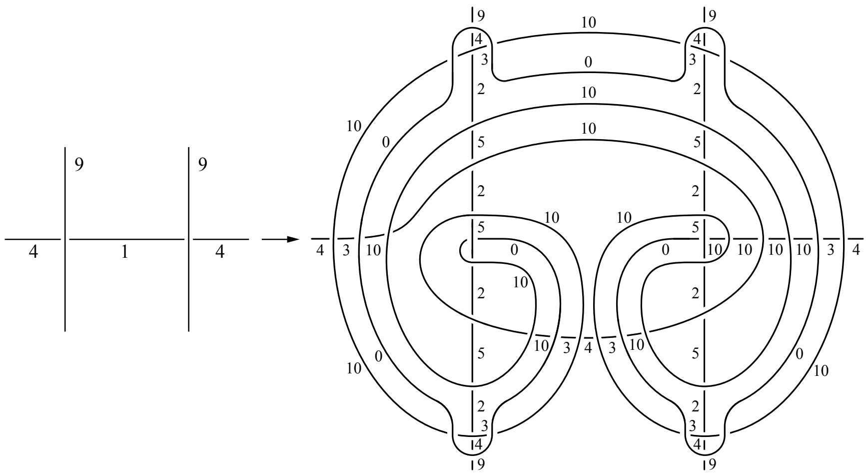


Figure 31

either

$$2a - 2b - 12 \neq 1, 6, 7, 8, 11, 12 \quad \text{and} \quad 2a - b \neq 1, 6, 7, 8, 11, 12$$

or

$$2b - 2a - 12 \neq 1, 6, 7, 8, 11, 12 \quad \text{and} \quad 2b - a \neq 1, 6, 7, 8, 11, 12$$

for any a and b in $\mathbb{Z}_{13} \setminus \{0, 1, 4, 6, 7, 8, 10, 11, 12\}$, and we transform the diagram as in Figure 12.

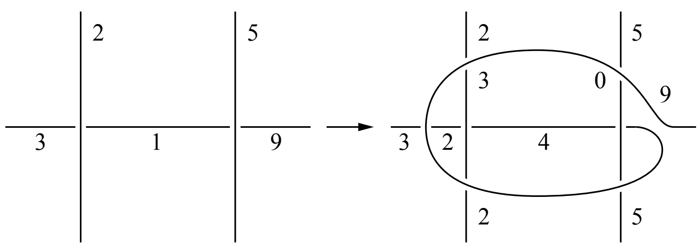


Figure 32

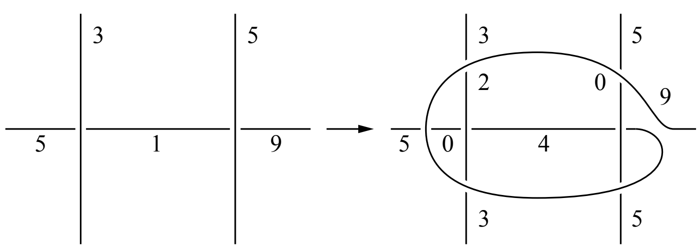


Figure 33

If $(a, b) = (2, 5)$, we transform the diagram as in Figure 32. The case $(a, b) = (5, 2)$ is similar.

If $(a, b) = (3, 5)$, we transform the diagram as in Figure 33. The case $(a, b) = (5, 3)$ is similar. □

Eliminating the color 10.

Lemma 2.8. *Any* 13*-colorable knot has a* 13*-colored diagram D with no arc colored by* 1, 6, 7, 8, 10, 11, *or* 12*.*

Proof. Let $c = 10$. By the previous lemmas we assume that no arc in D is colored by 1, 6, 7, 8, 11, or 12. We first transform any crossing of the form $\{10|10|10\}$. If there is any such crossing, there is an adjacent crossing of the form $\{10|a|2a+3\}$ or $\{a|10|7-a\}$, where a is in $\mathbb{Z}_{13} \setminus \{1, 6, 7, 8, 10, 11, 12\}$. With the exceptions of $a = 2, 4, 9$ as an over-arc (when $2a+3 = 1, 6, 7, 8, 11, 12$) and $a = 0, 9$ as an under-arc (when $7-a = 1, 6, 7, 8, 11, 12$), we transform the diagram as in Figures 2 and 3.

Now we must check when $a = 2, 4, 9$ as an under-arc. We know a cannot be 9 as an under-arc, otherwise $7 - a = 11$, contradicting our assumption that no arc is colored by 11. If $a = 2$ or $a = 4$ as an under-arc, we transform the diagram as in Figure 8. Therefore any crossings of the form $\{10|10|10\}$ are removed.

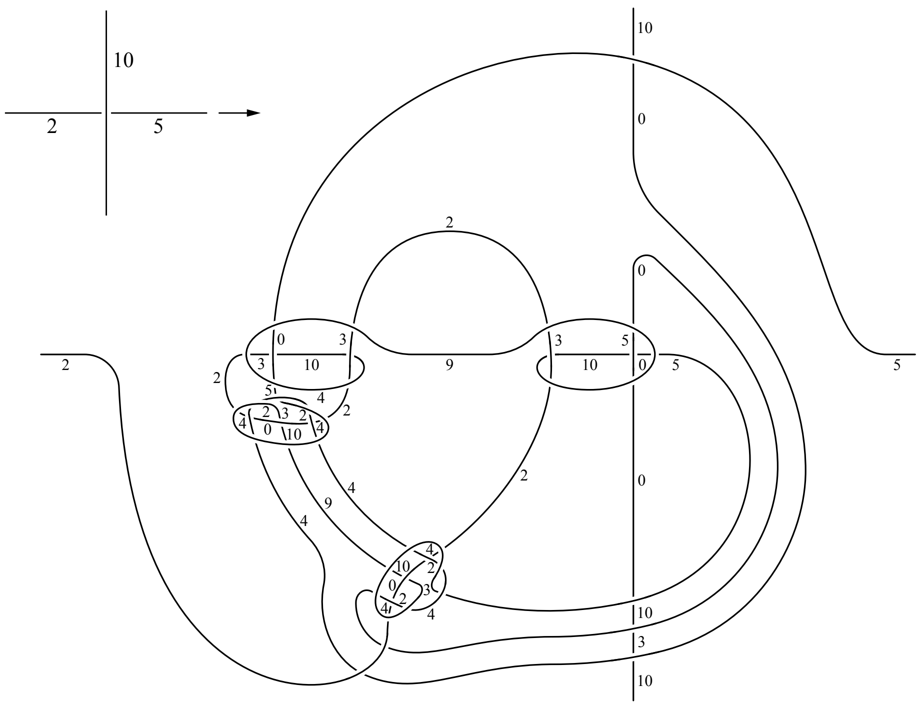


Figure 34

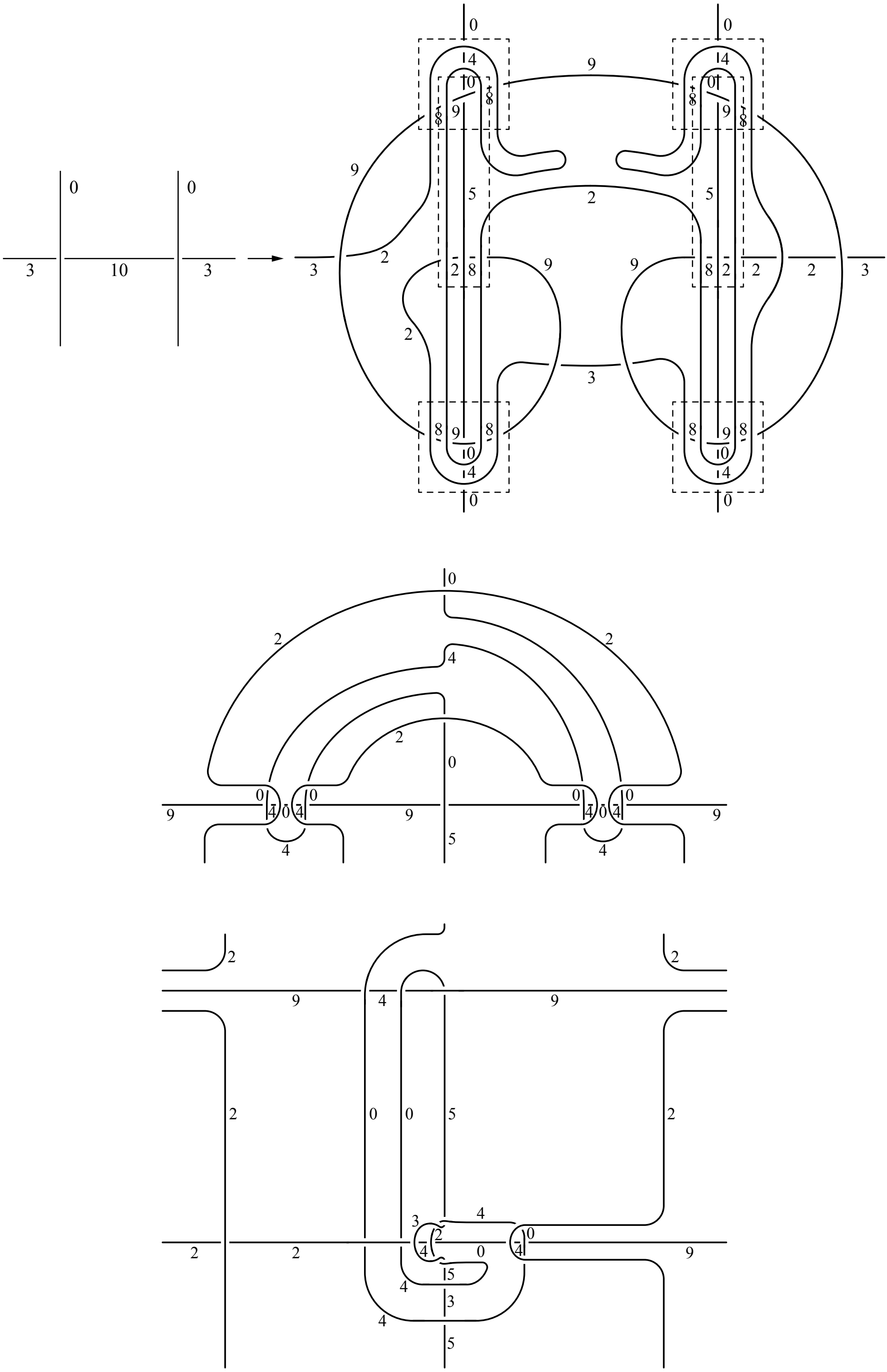


Figure 35

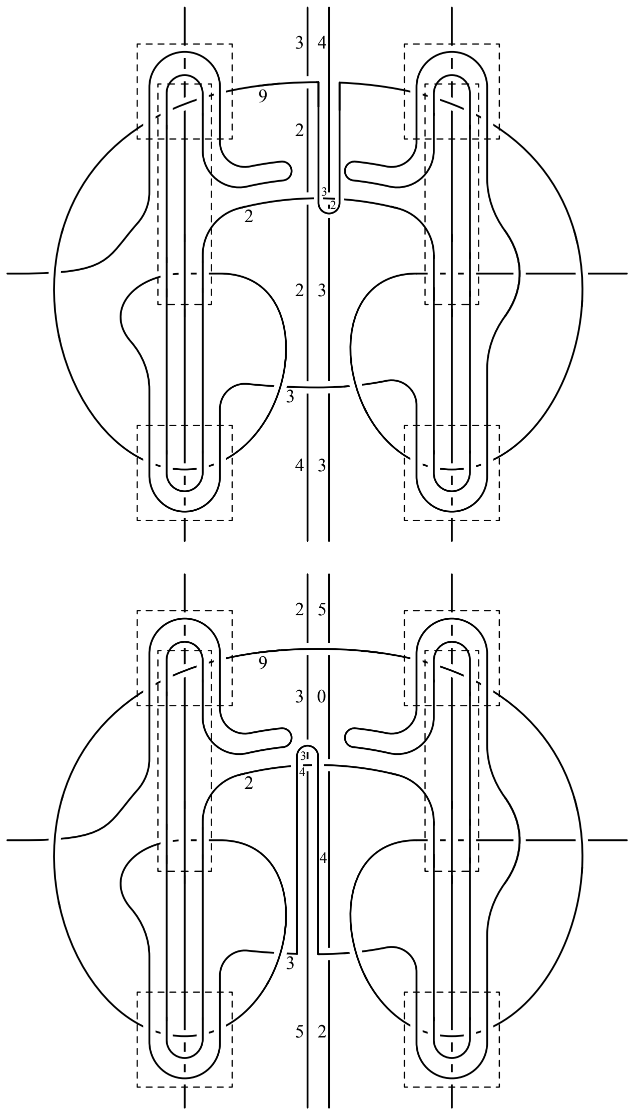


Figure 36

Next, we remove 10 as an over-arc by transforming any crossings of the form $\{a|10|7-a\}$. Since $7-a \neq 1, 6, 7, 8, 10, 11, 12$, we have $a \neq 0, 9$. With the exceptions of $a = 2, 4, 5$ (when $2a+3 = 1, 6, 7, 8, 10, 11, 12$ and $3a+6 = 1, 6, 7, 8, 10, 11, 12$), we transform the diagram as in Figure 4. If $a = 2$, we transform the diagram as in Figure 34. If $a = 4$, we transform the diagram as in Figure 9. If $a = 5$, since $7-a = 2$, we transform the diagram similarly to Figure 34, i.e., $a = 2$.

We complete the proof by removing 10 as an under-arc in a case-by-case method. We first consider the case where 10 is an under-arc connecting two crossings of the

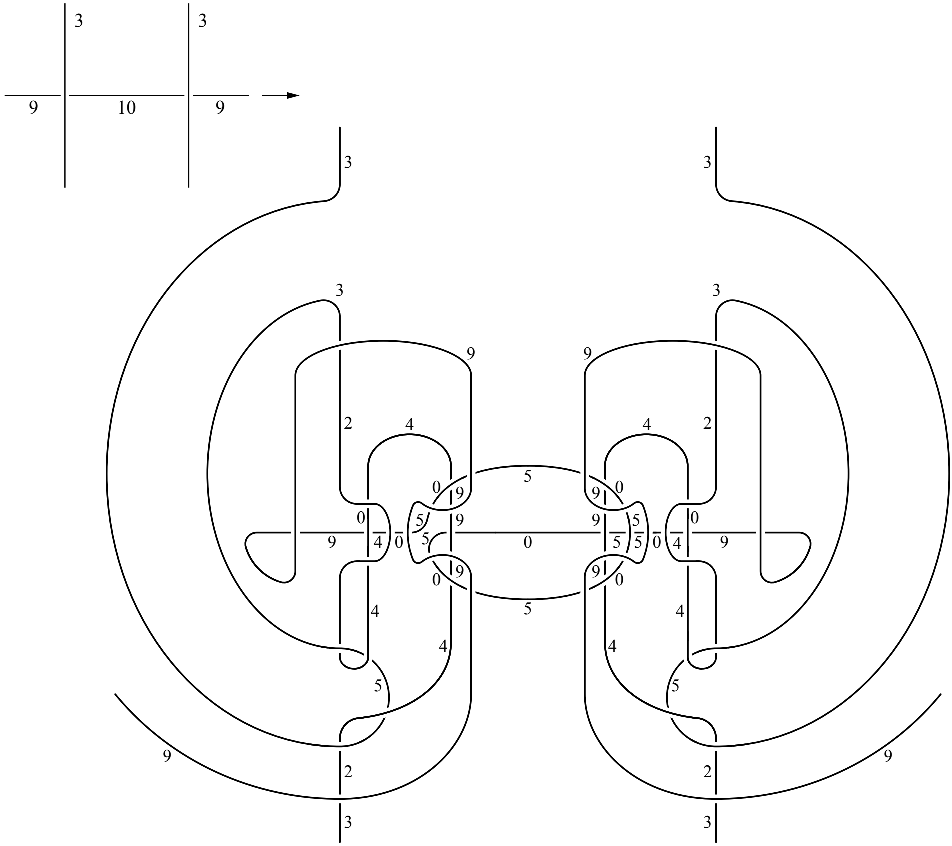


Figure 37

form $\{10|a|2a{+}3\}$. Since $2a+3 \neq 1, 6, 7, 8, 10, 11, 12$, we have $a \neq 2, 4, 9$. So, we need to check $a = 0, 3, 5$. If $a = 0$, we transform the diagram as in Figure 35, and we shall refer to this transformation throughout Lemma 2.8. As such, two variations of this transformation are given in Figure 36. If $a = 3$, we transform the diagram as in Figure 37. If $a = 5$, we transform the diagram as in Figure 38. Note the center of $a = 5$ as well as the six dashed boxes are the same transformations we used for $a = 0$ and its variations. Also, there are two arcs colored by 10, each of which are transformed by $a = 3$ as in Figure 37.

Now we consider the case where 10 is an under-arc. There are six such cases: $(a, b) = (0, 3), (3, 0), (0, 5), (5, 0), (3, 5), (5, 3)$. If $(a, b) = (0, 3)$, we transform the diagram as in Figure 39. For eliminating the 10 arc, see the variations of $a = 0$ in Figure 36. The case $(a, b) = (3, 0)$ is similar.

If $(a, b) = (0, 5)$, we transform the diagram as in Figure 40. For eliminating the 10 arc, see $a = 5$ in Figure 38; however, we will use the variations of $a = 0$ in Figure 36 for the center. The case $(a, b) = (5, 0)$ is similar.

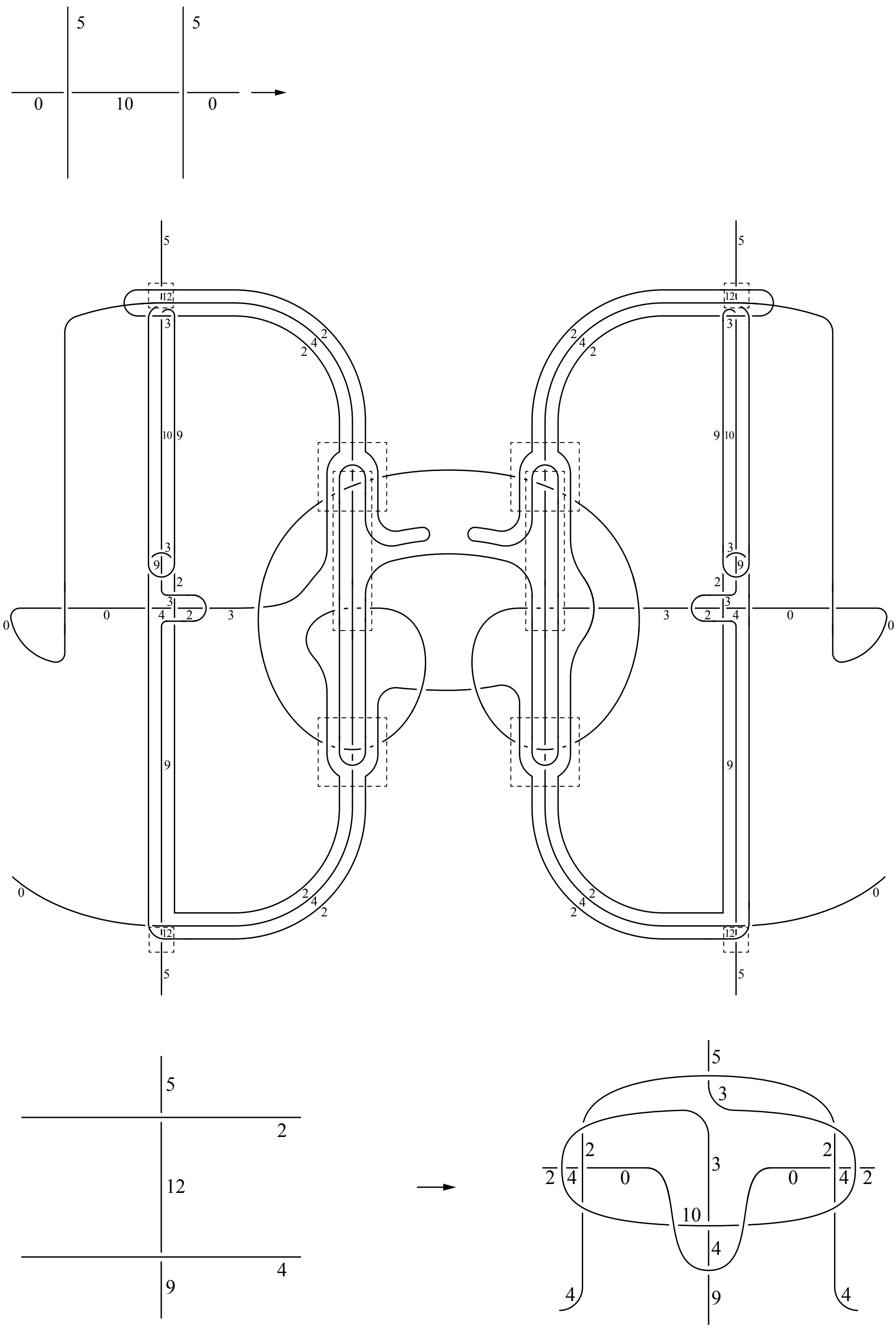


Figure 38

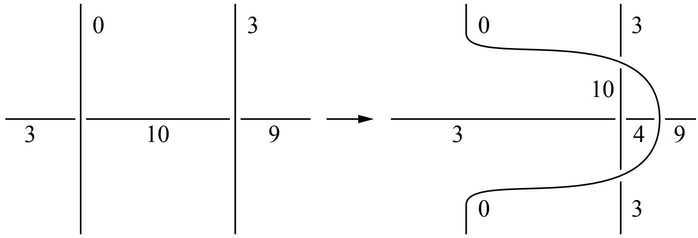


Figure 39

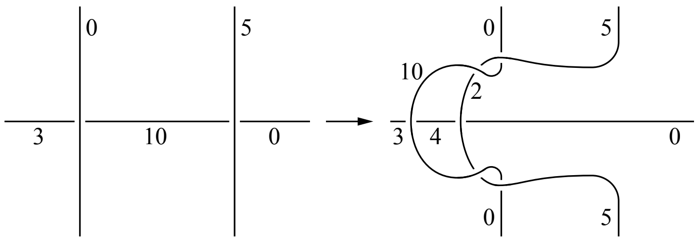


Figure 40

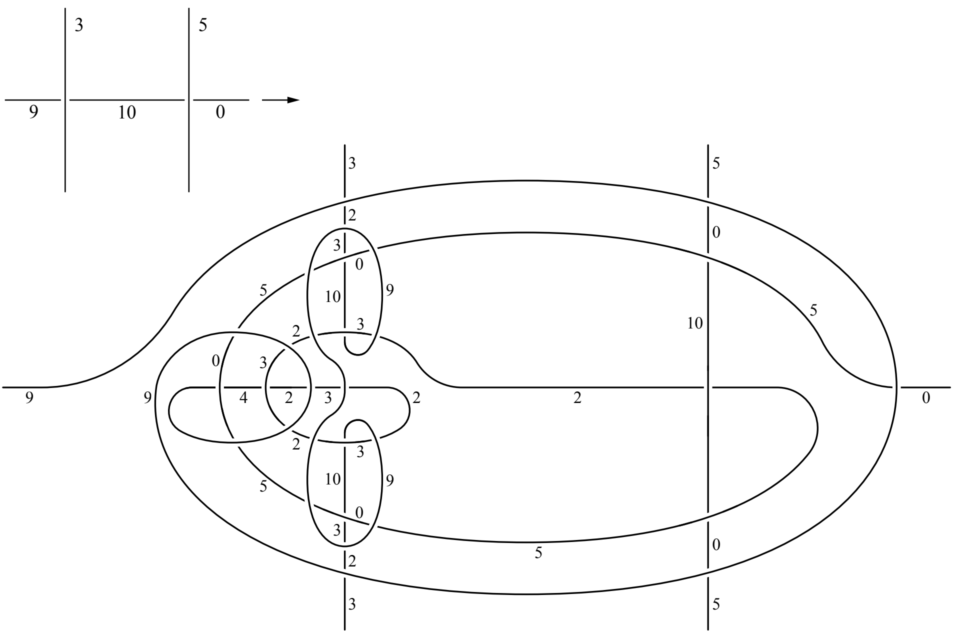


Figure 41

If $(a, b) = (3, 5)$, we transform the diagram as in Figure 41. For eliminating the 10 arcs, see the $(a, b) = (0, 3)$ case in Figure 39 and the $a = 5$ case in Figure 38 using the variations in Figure 36. The case $(a, b) = (5, 3)$ is similar. □

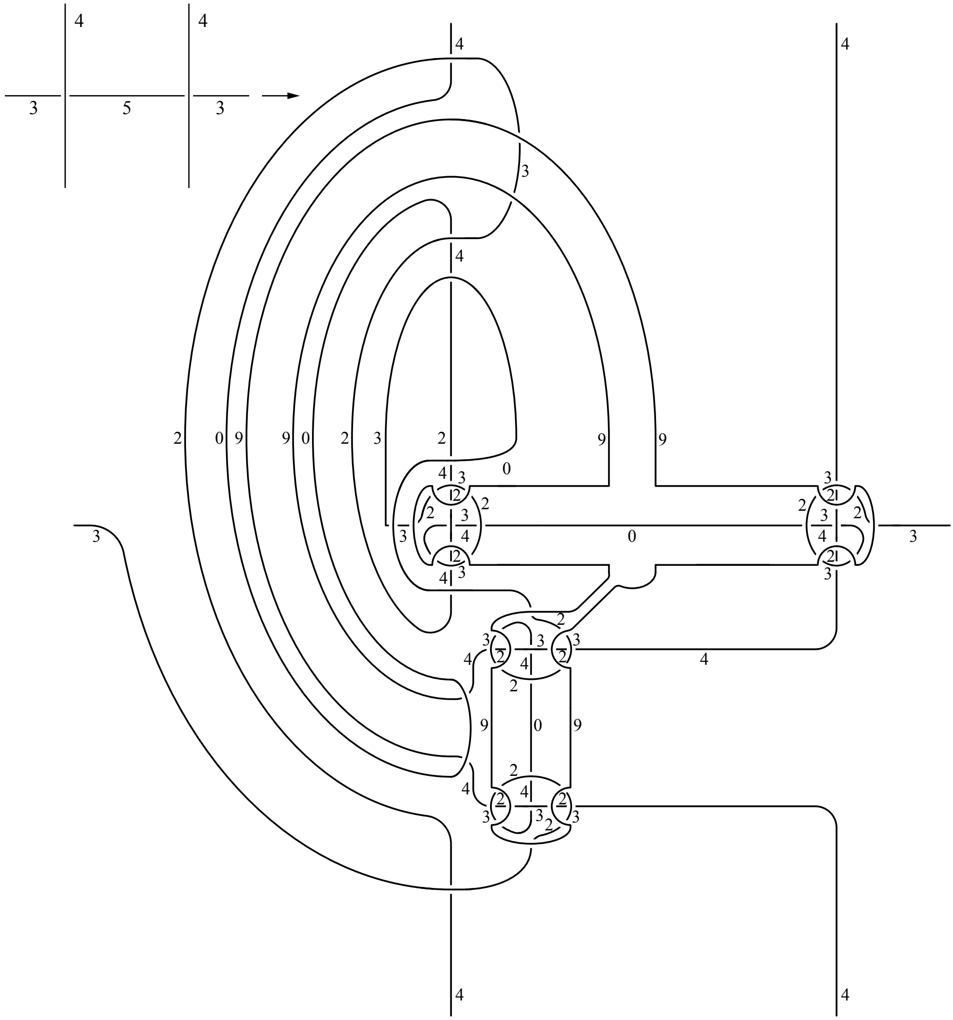
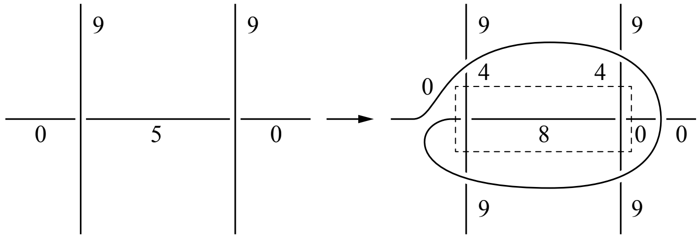


Figure 42

Eliminating the color 5.

Lemma 2.9. *Any* 13*-colorable knot has a* 13*-colored diagram D with no arc colored by* 1, 5, 6, 7, 8, 10, 11, *or* 12*.*

Proof. Let $c = 5$. By the previous lemmas we assume that no arc in D is colored by 1, 6, 7, 8, 10, 11, or 12. We first transform any crossing of the form $\{5\,|\,5\,|\,5\}$. If there is any such crossing, there is an adjacent crossing of the form $\{5\,|\,a\,|\,2a{+}8\}$, where a is in $\mathbb{Z}_{13} \setminus \{1, 5, 6, 7, 8, 10, 11, 12\}$. Since $10 - a = 1, 6, 7, 8, 10, 11, 12$ when $a = 0, 2, 3, 4, 9$, we know a cannot be an under-arc. Therefore, with the exceptions of $a = 0, 2, 3$ as an over-arc (when $2a + 8 = 1, 5, 6, 7, 8, 10, 11, 12$), we transform the diagram as in Figure 2. Therefore any crossings of the form $\{5\,|\,5\,|\,5\}$ are removed.



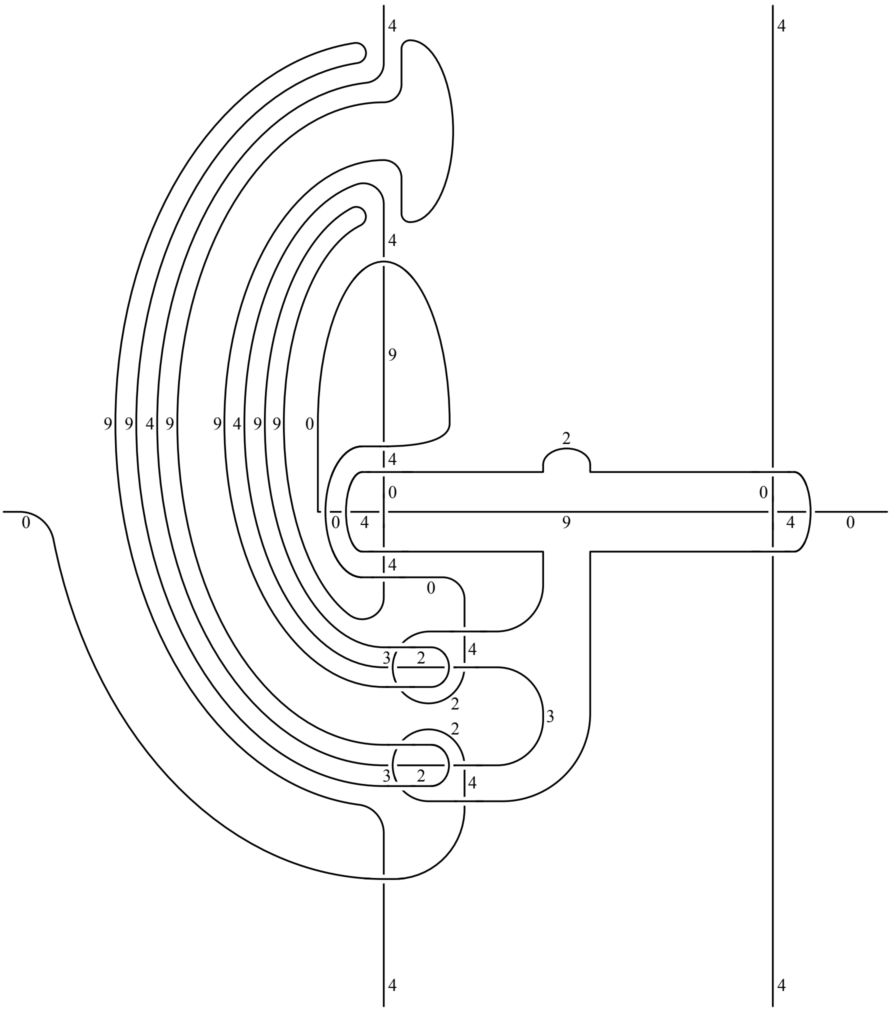


Figure 43

Next, we remove 5 as an over-arc by transforming any crossings of the form $\{a|5|10{-}a\}$. Since $10 - a \neq 1, 5, 6, 7, 8, 10, 11, 12$, we have $a \neq 0, 2, 3, 4, 9$. Therefore, 5 cannot be an over-arc.

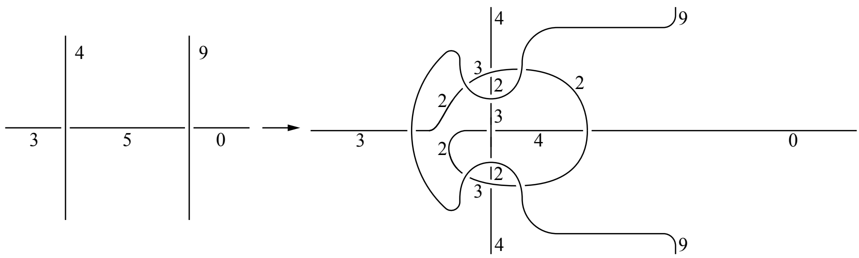


Figure 44

We complete the proof of Lemma 2.8 by removing 5 as an under-arc in a case-by-case method. We first consider the case where 5 is an under-arc connecting two crossings of the form $\{5|a|2a+8\}$. Since $2a+8 \neq 1, 5, 6, 7, 8, 10, 11, 12$, we have $a \neq 0, 2, 3$. So, we need to check $a = 4, 9$. If $a = 4$, we transform the diagram as in Figure 42. If $a = 9$, we transform the diagram as in Figure 43.

Now we consider the case where 5 is an under-arc connecting two crossings of the forms $\{5|a|2a+8\}$ and $\{5|b|2b+8\}$. Since $2a+8, 2b+8 \neq 1, 5, 6, 7, 8, 10, 11, 12$, there are two cases that we need to consider: $(a, b) = (4, 9), (9, 4)$. If $(a, b) = (4, 9)$, we transform the diagram as in Figure 44. The case $(a, b) = (9, 4)$ is similar. $\square$

At the same time we were working on this problem, Bento and Lopes [2015] proved the same result using different techniques.

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