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A new look at Apollonian circle packings

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We define an abstract Apollonian supergasket using the solution set of a certain Diophantine equation, showing that the solutions are in bijective correspondence with the circles of any concrete supergasket. Properties of the solution set translate directly to geometric and algebraic properties of Apollonian gaskets, facilitating their study. In particular, curvatures of individual circles are explored and geometric relationships among multiple circles are given simple algebraic expressions. All results can be applied to a concrete gasket using the curvature-center coordinates of its four defining circles. These techniques can also be applied to other types of circle packings and higher-dimensional analogs.

An Apollonian gasket is a type of circle packing in the plane generated recursively starting from a set of four mutually tangent circles. The curvatures of any four such circles are related by an equation discovered by Descartes, and every circle in a gasket generated by four circles with integer curvatures will have integer curvature. While these gaskets have been fascinating to mathematicians for some time — the use of group theory in their study was initiated by Keith Hirst [1967] and they even inspired a poem[1] — it was only relatively recently that Jeffrey Lagarias, Colin Mallows, and Allan Wilks [Lagarias et al. 2002] gave an algebraic characterization of Descartes configurations. One question in particular has inspired much work but resisted a complete answer: given the four original integer curvatures, which other curvatures can or will occur, and how frequently? Peter Sarnak [2011], Elena Fuchs [2013], and Hee Oh [2014] have recent surveys on this topic, which has seen significant progress in the past five years [Bourgain 2012; Bourgain and Kontorovich 2014; Bourgain and Fuchs 2011; Fuchs and Sanden 2011].

In this paper, inspired by recent work of Sam Northshield [2015], we provide a four-dimensional label to each circle that does not depend on the location of the circle but refers instead to its geometric relationship to the original four circles. Since we consider only the process of generating the gasket, the labels provide an abstract version of an Apollonian circle packing that can represent any concrete

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[1] *The Kiss Precise* by Frederick Soddy, 1936.

packing once an initial set of four circles is specified. These labels can be used to determine location and radius, find whether given circles in a gasket are tangent or not, perform operations such as inversion, and obtain curvature results. This technique is equally applicable to any packing generated in a similar fashion, such as the generalizations of Apollonian packings of Gerhard Guettler and Colin Mallows [2010] or packings in higher-dimensional Euclidean, spherical, or hyperbolic spaces [Lagarias et al. 2002].

1. Descartes configurations

Descartes configurations are the basic building blocks of Apollonian circle packings. We begin by providing a brief introduction; for more detail, see the paper by Lagarias, Mallows, and Wilks [Lagarias et al. 2002] or any of the surveys mentioned above.

An oriented circle in the plane consists of a circle and an orientation, thought of as a unit normal vector, of "inward" or "outward" that specifies its interior. The curvature of a circle is the inverse of its radius; the oriented curvature of an oriented circle is the curvature if the circle has an inward-pointing normal vector and the negative of the curvature otherwise. Two circles are tangent if they intersect in a single point. Lines are considered to be circles of curvature zero, and two lines that are not the same are considered to be tangent at infinity. In what follows, by a circle we will mean either an oriented circle or oriented line, tangent will mean externally tangent, and by the curvature of a circle, we will mean the oriented curvature.

A *Descartes configuration* (hereafter, configuration) consists of four circles in the plane that are pairwise externally tangent and such that no three share a point of tangency. There are four basic types of configurations, shown in Figure 1. Descartes discovered that the oriented curvatures κ_i of four oriented circles in a configuration satisfy

$$2(\kappa_1^2+\kappa_2^2+\kappa_3^2+\kappa_4^2) = (\kappa_1+\kappa_2+\kappa_3+\kappa_4)^2, \tag{1}$$

which we will call the *Descartes condition*.[2]

The Descartes condition is not enough to characterize configurations, but a characterization exists using additional information [Graham et al. 2005; Lagarias et al. 2002], and the geometry of inversion over a circle plays an important part. For a line, inversion over the line is simply reflection. For a circle C with center O and radius r, inversion over C is the Möbius transformation I_C that maps a point P to the point Q on the ray from O through P such that $r^2 = |OP||OQ|$. Each inversion is anticonformal in that it preserves magnitudes of angles but reverses their directions; further, inversion over a circle or line maps oriented circles and lines to oriented circles and lines.

[2]Descartes considered configurations without lines, but with our definitions, (1) is true for any type of configuration [Lagarias et al. 2002].

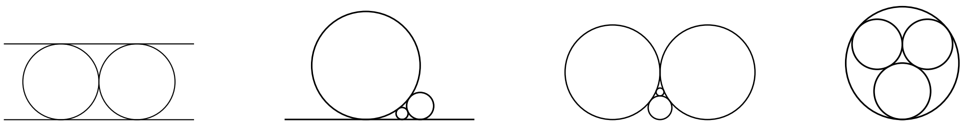

Figure 1. Descartes configurations.

Each circle that is not a line is uniquely identified by its center and curvature, since the curvature provides both radius and orientation. To uniquely identify all circles, Lagarias, Mallows, and Wilks devised *curvature-center coordinates*, which for any circle are of the form k', k, x, y, where k is the curvature and k' is the curvature of the inversion of the circle over the unit circle; if the curvature k is nonzero, then $x = kc_x$ and $y = kc_y$, where (c_x, c_y) is the center of the circle; if the curvature k is zero, then x and y are the corresponding components of the unit normal vector. For example, the curvature-center coordinates of the unit circle with the origin in its interior are $-1, 1, 0, 0$ and the curvature-center coordinates of the line $y = 1$ with the origin in its interior are $2, 0, 0, -1$.

Here is the characterization of configurations: let $C_1, \dots, C_4$ be circles, let $M = M(C_1, \dots, C_4)$ be the *curvature-center matrix* of the circles $C_1, \dots, C_4$, where each row consists of the curvature-center coordinates of the corresponding circle, and let

$$\mathrm{Q} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

(Our Q matrix is twice the Q of Lagarias et al. [2002] for notational convenience.)

Theorem 1 (augmented Euclidean Descartes theorem [Lagarias et al. 2002; Graham et al. 2005]). *Circles $C_1, \dots, C_4$ form a configuration if and only if*

$$M^{\mathrm{T}}\mathrm{Q}M = \begin{bmatrix} 0 & -8 & 0 & 0 \\ -8 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} =: \mathrm{W}. \tag{2}$$

Note that the matrix Q is related to the Descartes condition in that if $\vec{x} = (x_1\ \ x_2\ \ x_3\ \ x_4)^{\mathrm{T}}$ is a column vector then

$$\langle \vec{x}, \vec{x} \rangle_{\mathrm{Q}} := \vec{x}^{\mathrm{T}}\mathrm{Q}\vec{x} = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_1 + x_2 + x_3 + x_4)^2.$$

Indeed, the first two diagonal entries of W correspond to the Descartes condition.

Given any three mutually tangent circles C_1, C_2, and C_3 that do not share a point of tangency, there are exactly two other circles that each form a configuration with the original three [Sarnak 2011]. The operation that takes a configuration

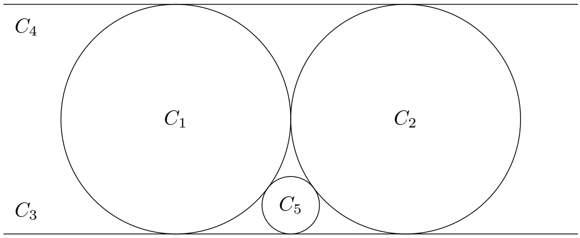

Figure 2. An example of reflection.

C_1, C_2, C_3, C_4 to the configuration C_1, C_2, C_3, C_5 is defined to be the *reflection* (of C_4 over C_1, C_2, and C_3) [Graham et al. 2005] (and when the context allows we will speak of replacing C_4 with C_5 in this fashion). In Figure 2, for example, C_5 is the reflection of C_4 over C_1, C_2, and C_3 (and C_4 is the reflection of C_5 over C_1, C_2, and C_3), and hence we can speak of replacing C_4 in the configuration C_1, C_2, C_3, C_4 with C_5 to obtain the configuration C_1, C_2, C_3, C_5.

Since inversion over a circle preserves tangency, inverting three circles of a configuration over the fourth will also result in another Descartes configuration. For example, in Figure 3, the three smallest circles invert over circle C_1 to the three largest circles.

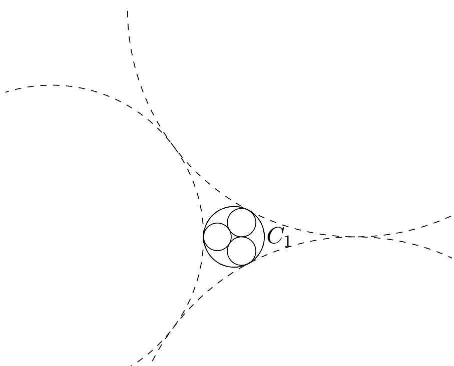

Figure 3. An example of inversion.

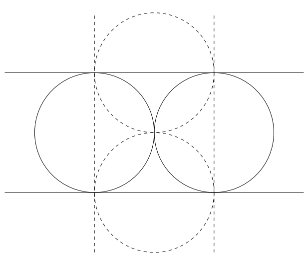

Figure 4. A configuration (solid lines) and its dual (dashed lines).

Finally, each configuration $C_1, \dots, C_4$ also has a dual configuration $C'_1, \dots, C'_4$ such that each C'_i does not intersect C_i and goes through the three points of tangency of the other three C_j with $j \neq i$. For example, in Figure 4, a configuration (solid lines) is superimposed with its dual (dashed lines).

2. Apollonian gaskets

Apollonian Gaskets can be defined geometrically and algebraically. In this section, we will review the geometric construction.

Given three mutually tangent circles, there are exactly two other circles that form a configuration with the original three. Thus, starting with a configuration of four circles, any three of the four define a new configuration not including the other circle. Repeatedly creating new configurations in this fashion, a circle packing (a collection of circles with mutually disjoint interiors) is created, called an Apollonian circle packing or Apollonian gasket; see Figure 5.

If $\kappa_1, \dots, \kappa_5$ are the curvatures of five circles $C_1, \dots, C_5$ such that C_1, C_2, C_3, C_4 and C_1, C_2, C_3, C_5 are configurations, the Descartes condition implies

$$\kappa_5 = 2\kappa_1 + 2\kappa_2 + 2\kappa_3 - \kappa_4. \quad (3)$$

Thus, in an Apollonian gasket, because each circle belongs to a configuration that can be obtained from the original one by repeated replacement operations, if the original curvatures are integers then the curvatures of all the circles in the gasket will also be integers.

The gasket with starting curvatures 0, 0, 2, and 2 contains another set of well-known circles called the Ford circles, shown in Figure 6, which can be defined as follows. For $r > 0$ and arbitrary real a, let $C(a, r)$ be the circle with radius r above and tangent to the x-axis at $x = a$. For relatively prime integers c and d with $d \neq 0$, let $C_{c,d} = C(c/d, 1/(2d^2))$; the set of all such $C_{c,d}$ are the Ford circles. These circles have a number of interesting properties. To see they are part of the $(2, 2, 0, 0)$-gasket (which we will call the *Ford gasket*) invokes one of these properties: if $C_{a,b}$ and $C_{c,d}$ are mutually tangent, then $C_{a+c,b+d}$ forms a

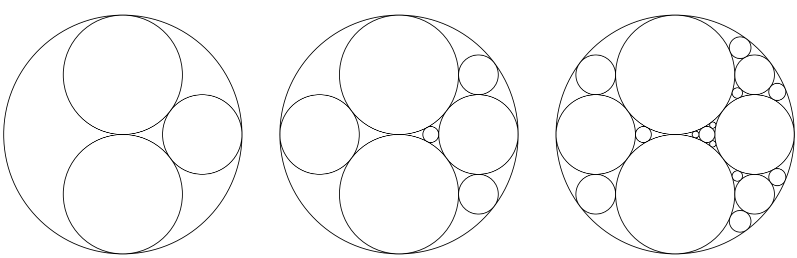

Figure 5. An Apollonian gasket.

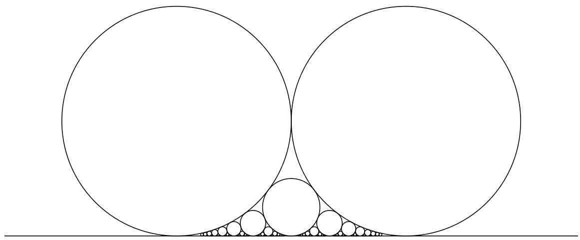

Figure 6. Ford circles.

Descartes configuration with $C_{a,b}$, $C_{c,d}$, and the x-axis. The claim then follows from $C_{0,1}$, $C_{1,1}$, and the x-axis being part of the original four gasket circles.

Sam Northshield [2015] recently discovered a new characterization and labeling for the Ford circles. For integers s and t with $s+t>0$, define

$$\langle s,t\rangle = C\left(\frac{s}{s+t}, \frac{1}{(s+t)^2}\right).$$

Then the set of Ford circles is exactly the set of those $\langle s,t\rangle$ with integer s and t that satisfy two conditions: $s+t>0$ and there exists an integer u such that $\gcd(s,t,u)=1$ and $s^2+t^2+u^2=(s+t+u)^2$. This characterization also allowed Northshield to study natural generalizations of the Ford circles in higher dimensions.

3. The Apollonian group

Each geometric operation described above has a matrix counterpart. For example, consider two configurations C_1, C_2, C_3, C_4 and C_1, C_2, C_3, C_5, let

$$\mathrm{S}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{bmatrix}$$

and let $M = M(C_1, C_2, C_3, C_4)$. We claim that $\mathrm{S}_4 M = M(C_1, C_2, C_3, C_5)$. Since $\mathrm{S}_4^{\mathrm{T}}\mathrm{Q}\mathrm{S}_4 = \mathrm{Q}$, we have $(\mathrm{S}_4 M)^{\mathrm{T}}\mathrm{Q}\mathrm{S}_4 M = M^{\mathrm{T}}\mathrm{Q}M = \mathrm{W}$, so that $\mathrm{S}_4 M$ is also a configuration. Since S_4 does not change C_1, C_2, or C_3, it follows that $\mathrm{S}_4 M$ must be the unique configuration obtained by reflection of C_4. This provides an alternate way of defining an Apollonian gasket.

The Apollonian group $\mathcal{A}$ is generated by S_4 along with

$$\mathrm{S}_1 = \begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{S}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{S}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

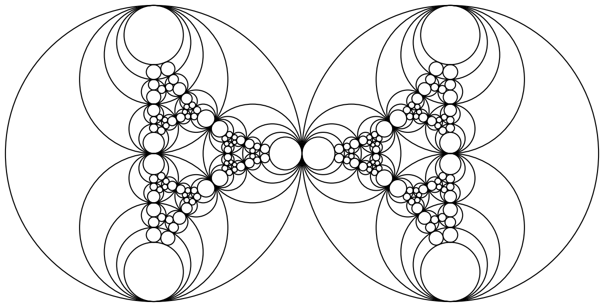

Figure 7. A dual Apollonian packing.

These matrices satisfy $\mathrm{S}_i^2 = \mathrm{I}$ and $\mathrm{S}_i^{\mathrm{T}}\mathrm{Q}\mathrm{S}_i = \mathrm{Q}$ for each i. With this notation, the Apollonian gasket generated by an initial Descartes configuration whose circles have curvature-center matrix M consists of the circles in the configurations of the orbit of M under the left action of $\mathcal{A}$.

Given a column vector of initial curvatures $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)^{\mathrm{T}}$ that satisfy the Descartes condition, in light of (3) and the above, multiplication by S_i can be viewed as removing curvature κ_i and substituting the curvature of its replacement. Thus the curvatures that occur in an Apollonian gasket with initial curvature vector v are those that occur in the vectors of the orbit of v under the action of the Apollonian group.

One can verify that the matrix $\mathrm{T}_i := \mathrm{S}_i^{\mathrm{T}}$ corresponds to inversion over the i-th circle of a configuration, that the matrix $\mathrm{D} := -\frac{1}{2}\mathrm{Q}$ gives $\mathrm{D}M(C_1, \dots, C_4) = M(C_1', \dots, C_4')$, and that $\mathrm{D} = \mathrm{D}^{-1} = \mathrm{D}^{\mathrm{T}}$. These matrices are related by $\mathrm{S}_i\mathrm{D} = \mathrm{D}\mathrm{T}_i$ for each i. As a result, the dual Apollonian group $\mathcal{A}^{\perp}$ generated by $T_1, \dots, T_4$ is conjugate to the Apollonian group. The orbit of a configuration under $\mathcal{A}^{\perp}$ is called a dual Apollonian packing; see Figure 7.

4. An abstract supergasket

Having now reviewed the geometric and algebraic constructions of Apollonian circle packings, we proceed to transpose the algebraic viewpoint; instead of looking at configurations, we will focus on identifying individual circles. From now on, for convenience, we will view (a, b, c, d) both as a point and as a vector. We will also use it to identify a circle: given a configuration with curvature-center matrix M, let (a, b, c, d) be the circle whose curvature-center coordinates are given by the vector $(a, b, c, d)M$.

There are two motivations for this notation. One is to extend Northshield's coordinates for Ford circles. The other is to view the process of generating an Apollonian gasket in an abstract fashion: if M is the curvature-center matrix of the configuration that generates an Apollonian gasket, then by definition any configuration in the gasket has curvature-center matrix of the form AM, where A is an element of the Apollonian group $\mathcal{A}$. In particular, $M = \mathrm{I}M$, and we can view the rows of the identity matrix I as giving the four original circles, which correspond to the labels $\mathrm{e}_1 = (1, 0, 0, 0)$, $\mathrm{e}_2 = (0, 1, 0, 0)$, $\mathrm{e}_3 = (0, 0, 1, 0)$, and $\mathrm{e}_4 = (0, 0, 0, 1)$.

Information contained in these labels can be applied to any gasket by using the corresponding curvature-center matrix. For example, using the curvature-center coordinates of the first four circles in the Ford gasket, the reader can verify that each label (a, b, c, d) with $a + b \neq 0$ corresponds to the circle with

$$x = \frac{b}{a+b}, \quad y = \frac{a+b-c+d}{2(a+b)}, \quad k = 2(a+b), \tag{4}$$

where (x, y) is the center and k is the curvature, while labels of the form $(a, -a, c, d)$ correspond to lines.

While any label (a, b, c, d) corresponds to a circle, which ones give circles in the gasket? This question is equivalent to asking what rows can occur in matrices in $\mathcal{A}$. If l is a circle in the gasket, then l is a row of some matrix $A \in \mathcal{A}$ and, for any i, we have $AS_i \in \mathcal{A}$. Then $l\mathrm{S}_i$ is a row of $A\mathrm{S}_i$, and so $l\mathrm{S}_i$ is the label of a circle in the gasket. Since any $A \in \mathcal{A}$ can be written as a word in the S_i, any vector corresponding to the label of a circle in the gasket can be written as $\mathrm{e}_i A$ for some $A \in \mathcal{A}$ and some $1 \leq i \leq 4$. Thus the question becomes what are the orbits of the e_i under $\mathcal{A}$?

Let

$$f_{\mathrm{Q}}(a, b, c, d) = 2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2.$$

Then $f_{\mathrm{Q}}(\mathrm{e}_i) = \langle \mathrm{e}_i, \mathrm{e}_i \rangle_{\mathrm{Q}} = 1$ for each i. Moreover, since $f_{\mathrm{Q}}(\mathrm{e}_i) = 1$ and $\langle u\mathrm{S}_i, u\mathrm{S}_i \rangle_{\mathrm{Q}} = \langle u, u \rangle_{\mathrm{Q}}$ for each i and every vector u, each label (a, b, c, d) of a circle in the gasket satisfies $f_{\mathrm{Q}}(a, b, c, d) = 1$. Unfortunately, this condition does not characterize the gasket circles.[3] One way to discover this is to start plotting integer solutions to $f_{\mathrm{Q}}(a, b, c, d) = 1$ using (4); in doing so, an interesting picture emerges (see Figure 8).

The group $\mathcal{A}^S$ generated by the S_i and the T_i is the super Apollonian group, and an orbit of a configuration under the super Apollonian group is a superpacking or *supergasket* [Graham et al. 2006]. In fact, as we will prove, integer solutions to $f_{\mathrm{Q}}(a, b, c, d) = 1$ correspond bijectively to the circles of any Apollonian supergasket. The rest of this section is devoted to proving this characterization.

Let $\mathcal{I}$ be the set of integer solutions to $f_{\mathrm{Q}}(a, b, c, d) = 1$. Note first that $\langle u\mathrm{S}_i, u\mathrm{S}_i \rangle_{\mathrm{Q}} = \langle u\mathrm{T}_i, u\mathrm{T}_i \rangle_{\mathrm{Q}} = \langle u, u \rangle_{\mathrm{Q}}$ for each i and every vector u, so that $\langle \mathrm{e}_i A, \mathrm{e}_i A \rangle_{\mathrm{Q}} = 1$ for each i and any $A \in \mathcal{A}^S$. Thus each orbit of an e_i is a subset

[3] Such a condition would be of much interest, and we mention this again as an open problem later.

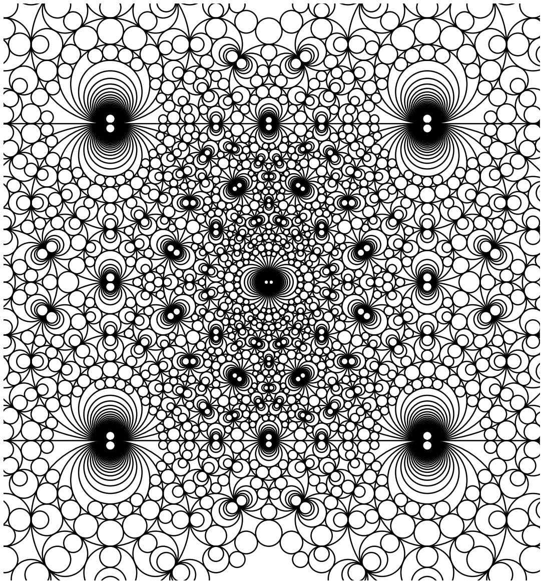

Figure 8. Plot of integer solutions to $f_Q(a, b, c, d) = 1$.

of $\mathcal{I}$. Our next few results explore properties of $\mathcal{I}$. One fact we will use repeatedly is that $(a, b, c, d) \in \mathcal{I}$ means

$$a = b + c + d \pm \sqrt{4(bc + bd + cd) + 1}. \tag{5}$$

Lemma 2. *There is no element of $\mathcal{I}$ with two negative coordinates and two positive coordinates.*

Proof. Assume without loss of generality that a and b are negative and that c and d are positive, and rewrite $f_Q(a, b, c, d) = 1$ as

$$(a - b)^2 + (c - d)^2 = 2(a + b)(c + d) + 1. \tag{6}$$

Then the left side is positive but the right is negative, a contradiction. □

If $(a, b, c, d) \in \mathcal{I}$, then $(-a, -b, -c, -d) \in \mathcal{I}$, and they are the same circle but with opposite orientations. Since orientation changes are already present in the curvature-center matrices, they should not be needed in the labels. Let $\mathcal{I}^+$ be the subset of $\mathcal{I}$ consisting of labels with at least one positive coordinate and at least as many positive coordinates as negative.

Our eventual proof that $\mathcal{I}^+$ will behave as the abstract supergasket will depend on an algorithm to take any element of $\mathcal{I}^+$ and produce a series of transformations that will take us back to some e_i. The next four results show that the S and T transformations map $\mathcal{I}^+$ to itself.

Lemma 3. *Let* $(a, b, c, d) \in \mathcal{I}^+$ *have no negative entries and let* $a = \max\{a, b, c, d\}$. *Then* $b+c+d < a$. *Further*, $a < 3(b+c+d)$ *unless* $(a, b, c, d) = \mathrm{e}_1$.

Proof. If $a \le b+c+d$, then (5) implies $-(bc+bd+cd) > \frac{1}{2}$, a contradiction. If $a \ge 3(b+c+d)$, then (5) yields $b^2+c^2+d^2 \le \frac{1}{2}$, implying $b = c = d = 0$. □

Lemma 4. *For* $(a, b, c, d) \in \mathcal{I}^+$ *with no negative entries and* $a = \max\{a, b, c, d\}$, *unless* $(a, b, c, d) = \mathrm{e}_1$, *we have*

$$(a', b', c', d') := (a, b, c, d)\mathrm{T}_1 \in \mathcal{I}^+ \quad \textit{and} \quad a+b+c+d > a'+b'+c'+d' > 0.$$

Proof. Since T_1 only changes a, we know $(a, b, c, d)\mathrm{T}_1$ has at most one negative entry. Thus, if $(a, b, c, d) \ne e_1$, then $(a, b, c, d)\mathrm{T}_1 \in \mathcal{I}^+$. Further, $a'+b'+c'+d' = 3b+3c+3d-a$, so assuming $(a, b, c, d) \ne e_1$, we have $3b+3c+3d-a > a-a = 0$. Using $b+c+d < a$,

$$a'+b'+c'+d'-a-b-c-d = 2b+2c+2d-2a > 0. \qquad \square$$

Lemma 5. *Let* $(a, b, c, d) \in \mathcal{I}^+$ *have exactly one negative entry* a. *Then* $a \ge -\frac{1}{6}(b+c+d)$. *If* $a = -\frac{1}{6}(b+c+d)$ *then* $(a, b, c, d) = (-1, 2, 2, 2)$.

Proof. Assume $a \le -\frac{1}{6}(b+c+d)$. Then (5) implies

$$36 \ge 49(b^2+c^2+d^2) - 46(bc+bd+cd).$$

Assume without loss of generality that $d \ge c \ge b \ge 0$. Using that

$$b^2+c^2+d^2-bc-bd-cd = (b-c)^2+(d-b)(d-c) \ge 0,$$

we have $12 \ge b^2+c^2+d^2$. The only such nonnegative values of b, c, and d that admit an a with $f_{\mathrm{Q}}(a, b, c, d) = 1$ are $b = c = d = 2$. □

Lemma 6. *For* $(a, b, c, d) \in \mathcal{I}^+$ *with exactly one negative entry* a,

$$(a', b', c', d') := (a, b, c, d)\mathrm{S}_1 \in \mathcal{I}^+ \quad \textit{and} \quad a+b+c+d > a'+b'+c'+d' > 0.$$

Proof. Since a is negative, $a' = -a$ is positive. If $(a', b', c', d') \notin \mathcal{I}^+$, then by Lemma 2, each of b', c', and d' are negative. Thus $b+2a = b' < 0$, and similarly $c+2a < 0$ and $d+2a < 0$. Taken together, $b+c+d+6a < 0$, a contradiction.

Since $a < 0$, it follows that $a'+b'+c'+d'-a-b-c-d = 4a > 0$. Finally,

$$a'+b'+c'+d' = 5a+b+c+d > 6a+b+c+d > 0. \qquad \square$$

Now for the main result that establishes the connection between $\mathcal{I}^+$ and the action of $\mathcal{A}^S$.

Lemma 7. *Suppose* $l \in \mathcal{I}^+$. *There exists an element* $A \in \mathcal{A}^S$ *and an* i *such that* $l = \mathrm{e}_i A$.

Proof. Since $l = (a_1, a_2, a_3, a_4) \in \mathcal{I}^+$, either it has no negative entries or exactly one negative entry. Consider the operation

$$l \mapsto \begin{cases} l\mathrm{T}_i & \text{if } l \text{ has no negative entries and } a_i = \max\{a_1, a_2, a_3, a_4\}, \\ l\mathrm{S}_j & \text{if } l \text{ has exactly one negative entry } a_j. \end{cases}$$

By Lemmas 4 and 6, repeated application of this operation will eventually result in e_i for some i and we will have $lA = \mathrm{e}_i$ for some $A \in \mathcal{A}^S$. Since each T_i and S_i are invertible, $l = \mathrm{e}_i A^{-1}$. □

Conversely, for any $A \in \mathcal{A}^S$ and any i, we have $\mathrm{e}_i A \in \mathcal{A}^S$, establishing our bijection.

Theorem 8. *The circles of an Apollonian supergasket are in one-to-one correspondence with $\mathcal{I}^+$.*

If $l = e_i A$ as in Lemma 7, then the three circles $e_j A$ for $j \neq i$ form a configuration with l, and we can call them the "parents" of l. From (6), the elements of $\mathcal{I}$ must have exactly one odd entry, and one can verify the location of this entry is not altered by replacement or inversion. Thus the odd entry provides a quick indicator of which e_i will be obtained by the procedure of Lemma 7.

Since duality D preserves the Q-inner product, the labels (a, b, c, d) of dual circles also satisfy $f_{\mathrm{Q}}(a, b, c, d) = 1$, but the one odd entry of elements of $\mathcal{I}$ means that the elements of $2\mathcal{I}\mathrm{D}$ are all odd integers. Results similar to Lemmas 2, 3, and 5 hold for dual circles, and thus a procedure similar to that of Lemma 7 can return a dual circle to one of the original four dual circles: $\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ or a permutation thereof.

5. Label operations

Having now defined our abstract supergasket as the set $\mathcal{I}^+$, we can begin to put it to use. We are particularly interested in properties shared by all gaskets. As we will see in this section, the labels give a simple way to identify individual circles, but they can also be combined to give simple computations for the configuration operations. As a first example, the next theorem follows directly from analyzing the entries of the S_i.

Theorem 9. *Let $C_1, \ldots, C_4$ be the circles of a Descartes configuration with labels $c_1, \ldots, c_4$. Let C_5 be the replacement of C_4, let C_j' be the inversion of C_j, $2 \leq j \leq 4$, over C_1, and let c_5 and c_j' denote the corresponding labels. Using entrywise operations, $c_5 = 2c_1 + 2c_2 + 2c_3 - c_4$ and $c_j' = 2c_1 + c_j$.*

A key fact is that, using duality and as witnessed by $\mathrm{S}_i\mathrm{D} = \mathrm{D}\mathrm{T}_i$ for each i, replacement can be viewed as inversion and inversion can be viewed as replacement. As an example of an application, for any circle C in the plane and any Descartes configuration with curvature-center matrix M, let I_C be the operation of inversion

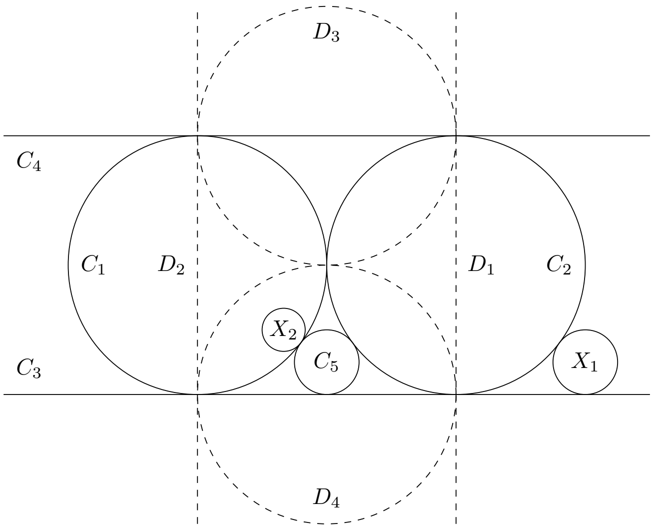

Figure 9. Circle X_1 is the inversion of C_5 over D_1, and X_2 is the inversion of C_5 over C_1.

over C. If, for some i, the intersection of the interior of C with the interior of any circle represented by M or $S_i M$ is empty, then $\mathrm{S}_i I_C M = I_C \mathrm{S}_i M$, and the similar results hold for T_i and for duality D. To see this, recall that the replacement of a circle determines a unique circle tangent to the other three in the original configuration. Inversion preserves tangency, and the unique circle tangent to three of $I_C M$ must be the inversion of the unique circle tangent to the corresponding three of M. Duality is similarly uniquely defined by the points of intersection which preserve their status under inversion. This view can help us to understand the action of an individual S_i or T_i on a given label, since multiplication of a label vector on the right corresponds to "premultiplication" on the left of the matrix M for a configuration.

Theorem 10. *Multiplication of a label vector on the right by* T_i *corresponds to inversion over the* i*-th circle of the original configuration, while multiplication on the right by* S_i *corresponds to inversion over the* i*-th dual circle.*

For example, using (4), the label $(1, 0, 0, 0)$ corresponds to the circle with center $\left(0, \frac{1}{2}\right)$ and curvature 2, called C_1 in Figure 9, and $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$ correspond to C_2, C_3, and C_4, respectively. The dual circles are the D_i. According to Theorem 9, C_5 has label $(2, 2, 2, -1)$. According to Theorem 10, for example, $(2, 2, 2, -1)S_1 = (-2, 6, 6, 3)$ gives circle X_1, which is the inversion of C_5 over D_1, and $(2, 2, 2, -1)T_1 = (4, 2, 2, -1)$ gives circle X_2, which is the inversion of C_5 over C_1.

6. An inner product

Curvature-center coordinate vectors take on another meaning when viewed in $\mathbb{R}^4$ with the indefinite inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{G}}$ given by the matrix

$$\mathrm{G} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

For circles C_1 and C_2 that are not lines, let d be the distance between their centers and let r_1 and r_2 be their respective radii. If C_1 and C_2 intersect at an angle θ, then $d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta$. Define a quantity A for C_1 and C_2 as $2Ar_1r_2 = d^2 - r_1^2 - r_2^2$ [Kotlov et al. 1997]. A then generalizes the intersection angle to any pair of circles. Moreover, if v_1 and v_2 are the curvature-center coordinate vectors of C_1 and C_2, respectively, then $A = \langle v_1, v_2\rangle_{\mathrm{G}} = v_1 \mathrm{G} v_2^{\mathrm{T}}$. For two circles C_1 and C_2 (including lines), letting $\langle C_1, C_2\rangle_{\mathrm{G}}$ be the G-inner product of their curvature-center vectors, we get the following characterization:

$\langle C_1, C_2\rangle_{\mathrm{G}}$	C_1 and C_2
-1	are internally tangent
1	are externally tangent
0	are mutually orthogonal
$-\cos\alpha$	intersect at angle α
< -1	are disjoint, one inside the other
> 1	are disjoint, outside each other

In general, given four circles $C_1, \ldots, C_4$ with curvature-center coordinate vectors $v_1, \ldots, v_4$, Jerzy Kocik [2007] defines their *configuration matrix* $F = F(C_1, \ldots, C_4)$ to be the Gram matrix of the vectors $v_1, \ldots, v_4$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{G}}$; that is, $F_{ij} = \langle v_i, v_j\rangle_{\mathrm{G}}$. Thus if M is the curvature-center matrix for $C_1, \ldots, C_4$, then $F = M\mathrm{G}M^{\mathrm{T}}$.

For a (Descartes) configuration, the configuration matrix F is $-\mathrm{Q}$. In that case, F is invertible, thus so is M, and $F = M\mathrm{G}M^{\mathrm{T}}$ if and only if $M^{\mathrm{T}}F^{-1}M = \mathrm{G}^{-1}$. The inverses of F and G are also related to previously defined matrices: $\mathrm{G}^{-1} = -\frac{1}{4}\mathrm{W}$ and $F^{-1} = -\frac{1}{4}\mathrm{Q}$.

From the above, if M is the curvature-center matrix of a Descartes configuration, $M\mathrm{G}M^{\mathrm{T}} = -\mathrm{Q}$. Thus for labels u and v, we have $\langle u, v\rangle_{\mathrm{Q}} = -\langle uM, vM\rangle_{\mathrm{G}}$, so that Q-inner products of our label vectors also give the geometric relationships between the circles they represent. For example, letting $\langle C_1, C_2\rangle_{\mathrm{Q}}$ be the Q-inner product of the labels of circles C_1 and C_2, we have the following theorem.

Theorem 11. *Circles C_1 and C_2 are externally tangent, mutually orthogonal, or internally tangent if and only if $\langle C_1, C_2\rangle_{\mathrm{Q}}$ is -1, 0, or 1, respectively.*

Viewing the circles as vectors suggests additional constructions, including one that resembles a Householder transformation:[4] Let C be any circle in a superpacking and let c be its label. For other labels d, consider the map $d \mapsto d(\mathrm{I} - 2Qc^{\mathrm{T}}c)$ (with labels used as vectors). Since C is internally tangent to itself, $\langle C, C\rangle_{\mathrm{Q}} = cQc^{\mathrm{T}} = 1$ and this map is an involution. Moreover, for any circle C' tangent to C, from Theorem 11 we have $\langle C, C'\rangle_{\mathrm{Q}} = -1$, so that $c' \mapsto c' + 2c$. From Theorem 9, this map inverts the circles tangent to C over C. Finally, every other circle in the supergasket can be obtained via replacement and/or duality and we saw earlier that those operations commute with inversion over C.

Theorem 12. *If c and d are circles in the abstract superpacking, then $d(\mathrm{I} - 2Qc^{\mathrm{T}}c)$ is the inversion of d over c.*

Note that by computing $(\mathrm{I} - 2Qe_i^{\mathrm{T}}e_i)$ for $i \in \{1, \ldots, 4\}$, Theorem 12 also provides another justification for part of Theorem 10.

7. Curvatures

We return now to the fascinating problem mentioned at the start: given four original integer curvatures, which other curvatures can or will occur? Certain conditions modulo 24 are known [Graham et al. 2003], and recent progress has been made in the form of a positive density theorem [Bourgain and Fuchs 2011] and a local-global theorem [Bourgain and Kontorovich 2014]. Our labels can provide an analysis similar to the proof of the positive density theorem, which involves looking at the curvatures of circles tangent to a given circle.

In the proof of the positive density theorem, if a, b, c, and d are the curvatures of the first four circles, then the set of curvatures of the circles tangent to the circle C_1 of curvature a involves the quadratic form $f(x, y) = Ax^2 + 2Bxy + Cy^2$, where $A = a + b$, $B = \frac{1}{2}(a + b + d - c)$, and $C = a + d$. In particular, the set of curvatures of the circles tangent to C_1 is shown to contain the set $\{f(x, y) - a : \gcd(x, y) = 1\}$. For our approach, notice that the Ford circles are the circles tangent to one of the four original circles in the Ford gasket (the x-axis). Our labels extend Northshield's [2015] in that the abstract Ford circles are (s, t, u, v), where $\gcd(s, t, u) = 1$ and $s^2 + t^2 + u^2 = (s + t + u)^2$. In particular, using Northshield's ideas, the abstract Ford circle labels can be parametrized as

$$\big(x(x+y),\ y(x+y),\ x^2+xy+y^2-1,\ -xy\big)$$

with $\gcd(x, y) = 1$. Thus, if a, b, c, and d are the initial curvatures of a gasket, then

$$\big(x^2+xy+y^2-1,\ x(x+y),\ -xy,\ y(x+y)\big)$$

[4] A Householder transformation of a vector is the result of multiplication by a matrix of the form $I - vv^{\mathrm{T}}$, where I is an identity matrix and v is a column vector of the appropriate size.

has curvature

$$a(x^2+xy+y^2-1)+b(x(x+y))+c(-xy)+d(y(x+y)) = f(x,y)-a.$$

Equation (6) also gives some information about the set of curvatures of the Ford supergasket since $2(a+b)$ is the curvature of the circle (a, b, c, d). In particular, given a desired curvature κ, the equations

$$2(a+b)=\kappa, \quad a-b=y_1, \quad c-d=y_2, \quad \text{and} \quad c+d=y_3$$

provide a connection to the solutions of the equation $y_1^2+y_2^2=\kappa y_3+1$. Recalling Fermat's result that any number of the form pq^2, where the prime factorization of p consists of primes that are congruent to 1 modulo 4, can be written as the sum of two perfect squares gives a quick way to see that every integer occurs as a curvature in the Ford supergasket.

Ideally, we could characterize the subset of supergasket labels that form a gasket and find a parametrization using that characterization. Suppose $f_{\mathrm{Q}}(a, b, c, d) = 1$ and d is odd. Then $4(ab+ac+bc)+1$ is a perfect square, say m^2, so $4(ab+ac+bc) = m^2-1$ and m must be odd. Thus $ab+ac+bc = n(n-1)$ for some integer n. Conversely, if $ab+ac+bc = n(n-1)$, then $4(ab+ac+bc)+1$ is a perfect square. Perhaps there exists a simple characterization of the n that occur in this fashion.

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