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Reeb dynamics of the link of the A_n singularity

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The link of the A_n singularity, $L_{A_n} \subset \mathbb{C}^3$ admits a natural contact structure ξ_0 coming from the set of complex tangencies. The canonical contact form α_0 associated to ξ_0 is degenerate and thus has no isolated Reeb orbits. We show that there is a nondegenerate contact form for a contact structure equivalent to ξ_0 that has two isolated simple periodic Reeb orbits. We compute the Conley–Zehnder index of these simple orbits and their iterates. From these calculations we compute the positive S^1-equivariant symplectic homology groups for (L_{A_n}, ξ_0). In addition, we prove that (L_{A_n}, ξ_0) is contactomorphic to the lens space $L(n+1, n)$, equipped with its canonical contact structure ξ_{std}.

1. Introduction and main results

The classical topological theory of isolated critical points of complex polynomials relates the topology of the link of the singularity to the algebraic properties of the singularity [Milnor 1968]. More generally, the link of an irreducible affine variety $A^n \subset \mathbb{C}^N$ with an isolated singularity at $\mathbb{0}$ is defined by $L_A = A \cap S_\delta^{2N+1}$. For sufficiently small δ, the link L_A is a manifold of real dimension $2n-1$, which is an invariant of the germ of A at $\mathbb{0}$. The links of Brieskorn varieties can sometimes be homeomorphic but not always diffeomorphic to spheres (see [Brieskorn 1966], a preliminary result which further motivated the study of such objects). Recent developments in symplectic and contact geometry have shown that the algebraic properties of a singularity are strongly connected to the contact topology of the link and symplectic topology of (the resolution of) the variety. A wide range of results demonstrating the power of investigating the symplectic and contact perspective of singularities include [Keating 2015; Kwon and van Koert 2016; McLean 2016; Ritter 2010; Seidel 2008b; Ustilovsky 1999].

In this paper we study the contact topology of the link of the A_n singularity, providing a computation of positive S^1-equivariant symplectic homology. This

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is done via our construction of an explicit nondegenerate contact form and the computation of the Conley–Zehnder indices of the associated simple Reeb orbits and their iterates. Our computations show that positive S^1-equivariant symplectic homology is a free $\mathbb{Q}[u]$ module of rank equal to the number of conjugacy classes of the finite subgroup A_n of $\mathrm{SL}(2;\mathbb{C})$. This provides a concrete example of the relationship between the cohomological McKay correspondence and symplectic homology, which is work in progress by McLean and Ritter [≥ 2017]. As a result, the topological nature of the singularity is reflected by qualitative aspects of the Reeb dynamics associated to the link of the A_n singularity.

The link of the A_n singularity is defined by

$$L_{A_n} = f_{A_n}^{-1}(0) \cap S^5 \subset \mathbb{C}^3, \quad f_{A_n} = z_0^{n+1} + 2z_1z_2. \tag{1-1}$$

It admits a natural contact structure coming from the set of complex tangencies,

$$\xi_0 := TL_{A_n} \cap J_0(TL_{A_n}).$$

The contact structure can be expressed as the kernel of the canonically defined contact form,

$$\alpha_0 = \frac{i}{2}\left(\sum_{j=0}^{m}(z_j d\bar{z}_j - \bar{z}_j dz_j)\right)\bigg|_{L_{A_n}}.$$

The contact form α_0 is degenerate and hence not appropriate for computing Floer-theoretic invariants as the periodic orbits of the Reeb vector field defined by

$$\alpha_0(R_{\alpha_0}) = 1, \quad \iota_{R_{\alpha_0}} d\alpha_0 = 0$$

are not isolated.

Our first result is the construction of a nondegenerate contact form α_ϵ such that $(L_{A_n}, \ker\alpha_0)$ and $(L_{A_n}, \ker\alpha_\epsilon)$ are contactomorphic. Define the Hamiltonian on $\mathbb{C}^3$ by

$$H : \mathbb{C}^3 \to \mathbb{R},$$
$$(z_0, z_1, z_2) \mapsto |z|^2 + \epsilon(|z_1|^2 - |z_2|^2),$$

where ϵ is chosen so that $H > 0$ on S^5. We will show

$$\alpha_\epsilon = \frac{1}{H}\left[\frac{(n+1)i}{8}(z_0 d\bar{z}_0 - \bar{z}_0 dz_0) + \frac{i}{4}(z_1 d\bar{z}_1 - \bar{z}_1 dz_1 + z_2 d\bar{z}_2 - \bar{z}_2 dz_2)\right] \tag{1-2}$$

is a nondegenerate contact form. We also find the simple Reeb orbits of R_{α_ϵ} and compute the associated Conley–Zehnder index with respect to the canonical trivialization of $\mathbb{C}^3$ of their iterates.

Theorem 1.1. *The 1-form α_ϵ is a nondegenerate contact form for L_{A_n} such that $(L_{A_n}, \ker\alpha_0)$ and $(L_{A_n}, \ker\alpha_\epsilon)$ are contactomorphic. The Reeb orbits of R_{α_ϵ} are* *defined by*

$$\gamma_+(t) = (0, e^{2i(1+\epsilon)t}, 0), \quad 0 \le t \le \pi/(1+\epsilon),$$
$$\gamma_-(t) = (0, 0, e^{2i(1-\epsilon)t}), \quad 0 \le t \le \pi/(1-\epsilon).$$

The Conley–Zehnder index for $\gamma = \gamma_\pm^N$ *in* $0 \le t \le N\pi/(1\pm\epsilon)$ *is*

$$\mu_{CZ}(\gamma_\pm^N) = 2\left(\left\lfloor \frac{2N}{(n+1)(1\pm\epsilon)} \right\rfloor + \left\lfloor \frac{N(1\mp\epsilon)}{1\pm\epsilon} \right\rfloor - \left\lfloor \frac{2N}{1\pm\epsilon} \right\rfloor \right) + 2N + 1. \quad \text{(1-3)}$$

Remark 1.2. If ϵ is chosen such that $0 < \epsilon \ll 1/N$ then (1-3) can be simplified to

$$\begin{aligned} \mu_{CZ}(\gamma_-^N) &= 2\left\lfloor \frac{2N}{(n+1)(1-\epsilon)} \right\rfloor + 1, \\ \mu_{CZ}(\gamma_+^N) &= 2\left\lfloor \frac{2N}{(n+1)(1+\epsilon)} \right\rfloor + 1. \end{aligned} \quad \text{(1-4)}$$

The proof of Theorem 1.1 is obtained by adapting methods of Ustilovsky [1999] to obtain α_ϵ and to compute the Conley–Zehnder indices. The Conley–Zehnder index is a Maslov index for arcs of symplectic matrices and is defined in Section 2D. These paths of matrices are obtained by linearizing the flow of the Reeb vector field along the Reeb orbit and restricting to ξ_0. To better understand the spread of the Reeb orbits and their iterates in various indices, we have the following example.

Example 1.3. Let $n = 2$ and $0 < \epsilon \ll \frac{1}{10}$. Then

$$\begin{aligned}
\mu_{CZ}(\gamma_-) &= 1, & \mu_{CZ}(\gamma_+) &= 1,\\
\mu_{CZ}(\gamma_-^2) &= 3, & \mu_{CZ}(\gamma_+^2) &= 3,\\
\mu_{CZ}(\gamma_-^3) &= 5, & \mu_{CZ}(\gamma_+^3) &= 3,\\
\mu_{CZ}(\gamma_-^4) &= 5, & \mu_{CZ}(\gamma_+^4) &= 5,\\
\mu_{CZ}(\gamma_-^5) &= 7, & \mu_{CZ}(\gamma_+^5) &= 7,\\
\mu_{CZ}(\gamma_-^6) &= 9, & \mu_{CZ}(\gamma_+^6) &= 7,\\
\mu_{CZ}(\gamma_-^7) &= 9, & \mu_{CZ}(\gamma_+^7) &= 9.
\end{aligned}$$

It is interesting to note that the spread of integers is not uniform between $\mu_{CZ}(\gamma_-^N)$ and $\mu_{CZ}(\gamma_+^N)$, and where these jumps in index occur. However, we see that there are $n = 2$ Reeb orbits with Conley–Zehnder index 1 and $n+1 = 3$ orbits with Conley–Zehnder index $2k+1$ for each $k \ge 1$.

Remark 1.4. Extrapolating this to all values of n and N demonstrates that the numerology of the Conley–Zehnder index realizes the number of free homotopy classes of L_{A_n}. Recall $[\Sigma L_{A_n}] = \pi_0(\Sigma L_{A_n}) = \pi_1(L_{A_n})/\{\text{conjugacy classes}\}$ and $H_1(L_{A_n}, \mathbb{Z}) = \mathbb{Z}_{n+1}$. The information that the $(n+1)$-th iterate of $\gamma_\pm$ is the first contractible Reeb orbit is also encoded in the above formulas. Qualitative aspects of the Reeb dynamics reflect this topological information in the following computation of a Floer-theoretic invariant of the contact structure ξ_0.

Theorem 1.1 allows us to easily compute positive S^1-equivariant symplectic homology SH_*^{+,S^1}. Symplectic homology is a Floer-type invariant of symplectic manifolds with contact-type boundary; see [Seidel 2008a]. Under additional assumptions, one can prove that the positive S^1-equivariant symplectic homology SH_*^{+,S^1} is in fact an invariant of the contact structure; see [Gutt 2015, Theorems 1.2 and 1.3; Bourgeois and Oancea 2012, Section 4.1.2]. Because of the behavior of the Conley–Zehnder index in Theorem 1.1, we can directly compute $SH_*^{+,S^1}(L_{A_n}, \xi_0)$ and conclude that it is a contact invariant. As a result, the underlying topology of the manifold determines qualitative aspects of any Reeb vector field associated to a contact form defining ξ_0.

Theorem 1.5. *The positive S^1-equivariant symplectic homology of (L_{A_n}, ξ_0) is*

$$SH_*^{+,S^1}(L_{A_n}, \xi_0) = \begin{cases} \mathbb{Q}^n, & * = 1, \\ \mathbb{Q}^{n+1}, & * \geq 3 \text{ and odd}, \\ 0, & * \text{ else}. \end{cases}$$

Proof. To obtain a contact invariant from SH_*^{+,S^1} we need to show in dimension 3 that all contractible Reeb orbits γ satisfy $\mu_{CZ}(\gamma) \geq 3$; see [Gutt 2015, Theorems 1.2 and 1.3; Bourgeois and Oancea 2012, Section 4.1.2]. The first iterate of $\gamma_\pm$ which is contractible is the $(n+1)$-th iterate, and by Theorem 1.1, will always satisfy $\mu_{CZ}(\gamma_\pm) \geq 3$.

If α is a nondegenerate contact form such that the Conley–Zehnder indices of all periodic Reeb orbits are lacunary, meaning they contain no two consecutive numbers, then we can appeal to [Gutt 2015, Theorem 1.1]. This result of Gutt allows us to conclude that over $\mathbb{Q}$-coefficients the differential for $SH^{S^1,+}$ vanishes. In light of Theorem 1.1 we obtain the above result. □

Remark 1.4 yields the following corollary of Theorem 1.5, indicating a Floer-theoretic interpretation of the McKay correspondence [Ito and Reid 1996] via the Reeb dynamics of the link of the A_n singularity. The A_n singularity is the singularity of $f_{A_n}^{-1}(0)$, where f_{A_n} is described as (1-1). This is equivalent to its characterization as the absolutely isolated double point quotient singularity of $\mathbb{C}^2/A_n$, where A_n is the cyclic subgroup of $\mathrm{SL}(2;\mathbb{C})$; see Section 4A. The cyclic group A_n acts on $\mathbb{C}^2$ by $(u, v) \mapsto (e^{2\pi i/(n+1)}u, e^{2\pi i n/(n+1)}v)$.

Corollary 1.6. *The positive S^1-equivariant symplectic homology $SH_*^{+,S^1}(L_{A_n}, \xi_0)$ is a free $\mathbb{Q}[u]$ module of rank equal to the number of conjugacy classes of the finite subgroup A_n of $\mathrm{SL}(2;\mathbb{C})$.*

Remark 1.7. The ongoing work of Nelson [2015; ≥ 2017] and Hutchings and Nelson [2014; ≥ 2017] is needed in order to work under the assumption that a related Floer-theoretic invariant, cylindrical contact homology is a well-defined contact invariant of (L_{A_n}, ξ_0). Once this is complete, the index calculations provided in Theorem 1.1 show that positive S^1-equivariant symplectic homology and cylindrical contact homology agree up to a degree shift.

Bourgeois and Oancea [2012] prove that there are restricted classes of contact manifolds for which one can prove that cylindrical contact homology (with a degree shift) is isomorphic to the positive part of S^1-equivariant symplectic homology when both are defined over $\mathbb{Q}$-coefficients. Their isomorphism relies on having transversality for a generic choice of J, which is presently the case for unit cotangent bundles DT^*L such that $\dim L \geq 5$ or when L is Riemannian manifold which admits no contractible closed geodesics [Bourgeois and Oancea 2015]. Our computations confirm that their results should hold for many more closed contact manifolds.

Our final result is an explicit proof that the singularity (L_{A_n}, ξ_0) and the lens space $(L(n+1, n), \xi_{\text{std}})$ are contactomorphic. The lens space

$$L(n+1,n) = S^3/\big((u,v) \sim (e^{2\pi i/(n+1)}u, e^{2\pi n i/(n+1)}v)\big)$$

admits a contact structure, which is induced by the one on S^3 and can be expressed as the kernel of the contact form

$$\lambda_{\text{std}} = \tfrac{1}{2} i (u d\bar{u} - \bar{u} du + v d\bar{v} - \bar{v} dv).$$

Theorem 1.8. *The link of the A_n singularity $(L_{A_n}, \xi_0 = \ker \alpha_0)$ and the lens space $(L(n+1,n), \xi_{\text{std}} = \ker \lambda_{\text{std}})$ are contactomorphic.*

Theorems 1.5 and 1.8 allow us to reprove the following result of Kwon and van Koert [2016]. Since (L_{A_n}, ξ_0) and $(L(n+1,n), \xi_{\text{std}})$ are contactomorphic and $SH_*^{S^1,+}$ is a contact invariant, $SH_*^{S^1,+}(L(n+1,n), \xi_{\text{std}}) = SH_*^{S^1,+}(L_{A_n}, \xi_0)$.

Theorem 1.9 [Kwon and van Koert 2016, Appendix A]. *The positive S^1-equivariant symplectic homology of $(L(n+1,n), \xi_{\text{std}})$ is*

$$SH_*^{+,S^1}(L(n+1,n), \xi_{\text{std}}) = \begin{cases} \mathbb{Q}^n, & * = 1, \\ \mathbb{Q}^{n+1}, & * \geq 3 \textit{ and odd}, \\ 0, & * \textit{ else}. \end{cases}$$

Their proof relies on the nondegenerate contact form on $(L(n+1,n), \xi_{\text{std}})$. If a_1, a_2 are any rationally independent positive real numbers then

$$\lambda_{a_1,a_2} = \frac{i}{2} \sum_{j=1}^{2} a_j (z_j d\bar{z}_j - \bar{z}_j dz_j)$$

is a nondegenerate contact form for $(L(n+1,n),\xi_{\text{std}})$. The simple Reeb orbits on $L(n+1,n)$ are given by

$$\gamma_1=(e^{it/a_1},0),\quad 0\le t\le(2a_1\pi)/(n+1),$$
$$\gamma_2=(0,e^{it/a_2}),\quad 0\le t\le(2a_2\pi)/(n+1),$$

which descend from the simple isolated Reeb orbits on S^3. Again, the $n+1$ different free homotopy classes associated to this lens space are realized by covers of the isolated Reeb orbits γ_i for $i=1$ or 2. The Conley–Zehnder index for γ_1^N is

$$\mu_{CZ}(\gamma_1^N)=2\left(\left\lfloor\frac{N}{n+1}\right\rfloor+\left\lfloor\frac{Na_1}{(n+1)a_2}\right\rfloor\right)+1, \tag{1-5}$$

with a similar formula holding for γ_2^N.

Outline. The necessary background is given in Section 2. The construction of a nondegenerate contact form and the proof of Theorem 1.1 is given in Section 3. The proof of Theorem 1.8 is given in Section 4.

2. Background

In this section we recall all the necessary symplectic and contact background which is needed to prove Theorems 1.1 and 1.8.

2A. ***Contact structures.*** First we recall some notions from contact geometry.

Definition 2.1. Let M be a manifold of dimension $2n+1$. A *contact structure* is a maximally nonintegrable hyperplane field $\xi=\ker\alpha\subset TM$.

Remark 2.2. The kernel of a 1-form α on M^{2n+1}, $\xi=\ker\alpha$, is a contact structure whenever

$$\alpha\wedge(d\alpha)^n\neq 0,$$

which is equivalent to the condition that $d\alpha$ be nondegenerate on ξ.

Note that the contact structure is unaffected when we multiply the contact form α by any positive or negative function on M. We say that two contact structures $\xi_0=\ker\alpha_0$ and $\xi_1=\ker\alpha_1$ on a manifold M are *contactomorphic* whenever there is a diffeomorphism $\psi:M\to M$ such that ψ sends ξ_0 to ξ_1,

$$\psi_*(\xi_0)=\xi_1.$$

If a diffeomorphism $\psi:M\to M$ is in fact a contactomorphism then there exists a nonzero function $g:M\to\mathbb{R}$ such that $\psi^*\alpha_1=g\alpha_0$. Finding an explicit contactomorphism often proves to be a rather difficult and messy task, but an application of Moser's argument yields Gray's stability theorem, which essentially states that there are no nontrivial deformations of contact structures on a fixed closed manifold.

First we give the statement of Moser's theorem, which says that one cannot vary a symplectic structure by perturbing it within its cohomology class. Recall that a *symplectic structure* on a smooth manifold W^{2n} is a nondegenerate closed 2-form $\omega \in \Omega^2(W)$.

Theorem 2.3 (Moser's theorem, [McDuff and Salamon 1998, Theorem 3.17]). *Let W be a closed manifold and suppose that ω_t is a smooth family of cohomologous symplectic forms on W. Then there is a family of diffeomorphisms Ψ_t of W such that*

$$\Psi_0 = \mathrm{id}, \quad \psi_t^* \omega_t = \omega_0.$$

The aforementioned contact analogue of Moser's theorem is Gray's stability theorem, stated formally below.

Theorem 2.4 (Gray's stability theorem, [Geiges 2008, Theorem 2.2.2]). *Let ξ_t, $t \in [0, 1]$, be a smooth family of contact structures on a closed manifold V. Then there is an isotopy $(\psi_t)_{t\in[0,1]}$ of V such that*

$$\psi_{t*}(\xi_0) = \xi_t \quad \textit{for each } t \in [0, 1].$$

Next we give the most basic example of a contact structure.

Example 2.5. Consider $\mathbb{R}^{2n+1}$with coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$ and the 1-form

$$\alpha = dz + \sum_{j=1}^{n} x_j dy_j.$$

Then α is a contact form for $\mathbb{R}^{2n+1}$. The contact structure $\xi = \ker \alpha$ is called the standard contact structure on $\mathbb{R}^{2n+1}$.

As in symplectic geometry, a variant of Darboux's theorem holds. This states that locally all contact structures are diffeomorphic to the standard contact structure on $\mathbb{R}^{2n+1}$.

A contact form gives rise to a unique Hamiltonian-like vector field as follows.

Definition 2.6. For any contact manifold $(M, \xi = \ker \alpha)$ the *Reeb vector field* R_α is defined to be the unique vector field determined by α,

$$\iota(R_\alpha) d\alpha = 0, \quad \alpha(R_\alpha) = 1.$$

We define the Reeb flow of R_α by $\varphi_t : M \to M$, $\dot{\varphi}_t = R_\alpha(\varphi_t)$.

The first condition says that R_α points along the unique null direction of the form $d\alpha$ and the second condition normalizes R_α. Because

$$\mathcal{L}_{R_\alpha} \alpha = d\iota_{R_\alpha} \alpha + \iota_{R_\alpha} d\alpha,$$

the flow of R_α preserves the form α and hence the contact structure ξ. Note that if one chooses a different contact form $f\alpha$, the corresponding vector field $R_{f\alpha}$ is very different from R_α, and its flow may have quite different properties.

A *Reeb orbit* γ of period T associated to R_α is defined to be a path $\gamma : \mathbb{R}/T\mathbb{Z} \to M$ given by an integral curve of R_α. That is,

$$\frac{d\gamma}{dt} = R_\alpha \circ \gamma(t), \quad \gamma(0) = \gamma(T).$$

Two Reeb orbits

$$\gamma_1,\ \gamma_0 : \mathbb{R}/T\mathbb{Z} \to M$$

are considered equivalent if they differ by reparametrization, i.e., precomposition with a translation of $\mathbb{R}/T\mathbb{Z}$.

The N-fold cover γ^N is defined to be the composition of $\gamma_\pm$ with $\mathbb{R}/NT\mathbb{Z} \to \mathbb{R}/T\mathbb{Z}$. A *simple Reeb orbit* is one such that $\gamma : \mathbb{R}/T\mathbb{Z} \to M$ is injective.

Remark 2.7. Since Reeb vector fields are autonomous, the terminology "simple Reeb orbit γ" refers to the entire equivalence class of orbits, and likewise for its iterates.

A Reeb orbit γ is said to be *nondegenerate* whenever the linearized return map

$$d(\varphi_T)_{\gamma(0)} : \xi_{\gamma(0)} \to \xi_{\gamma(T)=\gamma(0)}$$

has no eigenvalue equal to 1. A *nondegenerate contact form* is one whose Reeb orbits are all nondegenerate and hence isolated. Note that since the Reeb flow preserves the contact structure, the linearized return map is symplectic.

Next we briefly review the canonical contact form on S^3 and its Reeb dynamics.

Example 2.8 (canonical Reeb dynamics on the 3-sphere). If we define the function $f : \mathbb{R}^4 \to \mathbb{R}$,

$$f(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2,$$

then $S^3 = f^{-1}(1)$. Recall that the canonical contact form on $S^3 \subset \mathbb{R}^4$ is given to be

$$\lambda_0 := -\tfrac{1}{2} df \circ J = (x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)|_{S^3}. \tag{2-1}$$

The Reeb vector field is given by

$$\begin{aligned} R_{\lambda_0} &= \left(x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2} \right) \\ &= (-y_1, x_1, -y_2, x_2). \end{aligned} \tag{2-2}$$

Equivalently we may reformulate these using complex coordinates by identifying $\mathbb{R}^4$ with $\mathbb{C}^2$ via

$$u = x_1 + iy_1, \quad v = x_2 + iy_2.$$

We obtain

$$\lambda_0 = \tfrac{1}{2} i (u d\bar{u} - \bar{u} du + v d\bar{v} - \bar{v} dv)|_{S^3},$$

and

$$\begin{aligned} R_{\lambda_0} &= i\left(u\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{u}} + v\frac{\partial}{\partial v} - \bar{v}\frac{\partial}{\partial \bar{v}}\right) \\ &= (iu, iv). \end{aligned} \tag{2-3}$$

The second expression for R_{λ_0} follows from (2-2) since $iu = (-y_1, x_1)$ and $iv = (-y_2, x_2)$.

To see that the orbits of R_{λ_0} define the fibers of the Hopf fibration, recall that a fiber through a point

$$(u, v) = (x_1 + iy_1, x_2 + iy_2) \in S^3 \subset \mathbb{C}^2$$

can be parameterized as

$$\varphi(t) = (e^{it}u, e^{it}v), \quad t \in \mathbb{R}. \tag{2-4}$$

We compute the time derivative of the fiber

$$\dot{\varphi}(0) = (iu, iv) = (ix_1 - y_1, ix_2 - y_2).$$

Expressed as a real vector field on $\mathbb{R}^4$, which is tangent to S^3, this is the Reeb vector field R_{λ_0} as it appears in (2-3), so the Reeb flow does indeed define the Hopf fibration.

2B. ***Hypersurfaces of contact type.*** Another notion that we need from symplectic and contact geometry is that of a hypersurface of contact type in a symplectic manifold. The following notion of a Liouville vector field allows us to define hypersurfaces of contact type. Liouville vector fields will be used to understand the Reeb dynamics of the nondegenerate contact form α_1 as well as to construct the contactomorphism between (L_{A_n}, ξ_0) and $(L(n+1, n), \xi_{\text{std}})$.

Definition 2.9. A *Liouville vector field Y* on a symplectic manifold (W, ω) is a vector field satisfying

$$\mathcal{L}_Y \omega = \omega.$$

The flow ψ_t of such a vector field is conformal symplectic, i.e., $\psi_t^*(\omega) = e^t \omega$. The flow of these fields is volume expanding, so such fields may only exist locally on compact manifolds.

Whenever there exists a Liouville vector field Y defined in a neighborhood of a compact hypersurface Q of (W, ω), which is transverse to Q, we can define a contact 1-form on Q by

$$\alpha := \iota_Y \omega.$$

Proposition 2.10 [McDuff and Salamon 1998, Proposition 3.58]. *Let (W, ω) be a symplectic manifold and $Q \subset W$ a compact hypersurface. Then the following are equivalent*:

(i) *There exists a contact form α on Q such that $d\alpha = \omega|_Q$.*

(ii) *There exists a Liouville vector field $Y : U \to TW$ defined in a neighborhood U of Q, which is transverse to Q.*

If these conditions are satisfied then Q is said to be of contact type.

We will need the following application of Gray's stability theorem to hypersurfaces of contact type to prove Theorem 1.8 in Section 4.

Lemma 2.11 [Geiges 2008, Lemma 2.1.5]. *Let Y be a Liouville vector field on a symplectic manifold (W, ω). Suppose that M_1 and M_2 are hypersurfaces of contact type in W. Assume that there is a smooth function*

$$h : W \to \mathbb{R} \tag{2-5}$$

such that the time-1 map of the flow of hY is a diffeomorphism from M_1 to M_2. Then this diffeomorphism is in fact a contactomorphism from $(M_1, \ker \iota_Y \omega|_{TM_1})$ to $(M_2, \ker \iota_Y \omega|_{TM_2})$.

2C. *Symplectization.* The symplectization of a contact manifold is an important notion in defining Floer-theoretic theories like symplectic and contact homology. It will also be used in our calculation of the Conley–Zehnder index. Let $(M, \xi = \ker \alpha)$ be a contact manifold. The *symplectization* of $(M, \xi = \ker \alpha)$ is given by the manifold $\mathbb{R} \times M$ and symplectic form

$$\omega = e^t (d\alpha - \alpha \wedge dt) = d(e^t \alpha).$$

Here t is the coordinate on $\mathbb{R}$, and it should be noted that α is interpreted as a 1-form on $\mathbb{R} \times M$, as we identify α with its pullback under the projection $\mathbb{R} \times M \to M$.

Any contact structure ξ may be equipped with a complex structure J such that (ξ, J) is a complex vector bundle. This set is nonempty and contractible. There is a unique canonical extension of the almost complex structure J on ξ to an $\mathbb{R}$-invariant almost complex structure $\tilde{J}$ on $T(\mathbb{R} \times M)$, whose existence is due to the splitting,

$$T(\mathbb{R} \times M) = \mathbb{R}\frac{\partial}{\partial t} \oplus \mathbb{R}R_\alpha \oplus \xi. \tag{2-6}$$

Definition 2.12 (canonical extension of J to $\tilde{J}$ on $T(\mathbb{R} \times M)$). Let $[a, b; v]$ be a tangent vector where $a, b \in \mathbb{R}$ and $v \in \xi$. We can extend $J : \xi \to \xi$ to $\tilde{J} : T(\mathbb{R} \times M) \to T(\mathbb{R} \times M)$ by

$$\tilde{J}[a, b; v] = [-b, a, Jv].$$

Thus $\tilde{J}|_\xi = J$ and $\tilde{J}$ acts on $\mathbb{R}\partial/\partial t \oplus \mathbb{R}R_\alpha$ in the same manner as multiplication by i acts on $\mathbb{C}$, namely $J\partial/\partial t = R_\alpha$.

2D. ***The Conley–Zehnder index.*** The Conley–Zehnder index μ_{CZ} is a Maslov index for arcs of symplectic matrices which assigns an integer $\mu_{CZ}(\Phi)$ to every path of symplectic matrices $\Phi : [0, T] \to \mathrm{Sp}(n)$, with $\Phi(0) = \mathbb{1}$. In order to ensure that the Conley–Zehnder index assigns the same integer to homotopic arcs, one must also stipulate that 1 is not an eigenvalue of the endpoint of this path of matrices, i.e., $\det(\mathbb{1} - \Phi(T)) \neq 0$. We define the following set of continuous paths of symplectic matrices that start at the identity and end on a symplectic matrix that does not have 1 as an eigenvalue:

$$\Sigma^*(n) = \{\Phi : [0, T] \to \mathrm{Sp}(2n) : \Phi \text{ is continuous, } \Phi(0) = \mathbb{1}, \text{ and } \det(\mathbb{1} - \Phi(T)) \neq 0\}.$$

The Conley–Zehnder index is a functor satisfying the following properties, and is uniquely determined by the homotopy, loop, and signature properties.

Theorem 2.13 [Robbin and Salamon 1993, Theorem 2.3, Remark 5.4; Gutt 2014, Theorem 2, Propositions 8 and 9]. *There exists a unique functor μ_{CZ} called the* Conley–Zehnder index *that assigns the same integer to all homotopic paths Ψ in $\Sigma^*(n)$,*

$$\mu_{CZ} : \Sigma^*(n) \to \mathbb{Z},$$

such that the following hold:

(1) <u>*Homotopy*</u>: *The Conley–Zehnder index is constant on the connected components of $\Sigma^*(n)$.*

(2) <u>*Naturalization*</u>: *For any paths $\Phi, \Psi : [0, 1] \to \mathrm{Sp}(2n)$,*

$$\mu_{CZ}(\Phi\Psi\Phi^{-1}) = \mu_{CZ}(\Psi).$$

(3) <u>*Zero*</u>: *If $\Psi(t) \in \Sigma^*(n)$ has no eigenvalues on the unit circle for $t > 0$, then $\mu_{CZ}(\Psi) = 0$.*

(4) <u>*Product*</u>: *If $n = n' + n''$, identify $\mathrm{Sp}(2n') \oplus \mathrm{Sp}(2n'')$ with a subgroup of $\mathrm{Sp}(2n)$ in the obvious way. For $\Psi' \in \Sigma^*(n')$ and $\Psi'' \in \Sigma^*(n'')$, we have $\mu_{CZ}(\Psi' \oplus \Psi'') = \mu_{CZ}(\Psi') + \mu_{CZ}(\Psi'')$.*

(5) <u>*Loop*</u>: *If Φ is a loop at $\mathbb{1}$, then $\mu_{CZ}(\Phi\Psi) = \mu_{CZ}(\Psi) + 2\mu(\Phi)$, where μ is the Maslov Index.*

(6) <u>*Signature*</u>: *If $S \in M(2n)$ is a symmetric matrix with $\|S\| < 2\pi$ and $\Psi(t) = \exp(J_0 St)$, then $\mu_{CZ}(\Psi) = \frac{1}{2}\,\mathrm{sgn}(S)$.*

The linearized Reeb flow of γ yields a path of symplectic matrices

$$d(\varphi_t)_{\gamma(0)} : \xi_{\gamma(0)} \to \xi_{\gamma(t)=\gamma(0)}$$

for $t \in [0, T]$, where T is the period of γ.

Thus we can compute the Conley–Zehnder index of $d\varphi_t$, $t \in [0, T]$. This index is typically dependent on the choice of trivialization τ of ξ along γ which was used in linearizing the Reeb flow. However, if $c_1(\xi; \mathbb{Z}) = 0$ we can use the existence of an (almost) complex volume form on the symplectization to obtain a global means of linearizing the flow of the Reeb vector field. The choice of a complex volume form is parametrized by $H^1(\mathbb{R} \times M; \mathbb{Z})$, so an absolute integral grading is only determined up to the choice of volume form. See [Nelson ≥ 2017, §1.1.1].

We define

$$\mu^\tau_{CZ}(\gamma) := \mu_{CZ}(\{d\varphi_t\}|_{t\in[0,T]}).$$

In the case at hand we will be able to work in the ambient space of $(\mathbb{C}^3, J_0)$, and use a canonical trivialization of $\mathbb{C}^3$.

2E. ***The canonical contact structure on Brieskorn manifolds.*** The A_n link is an example of a Brieskorn manifold, which are defined generally by

$$\Sigma(\boldsymbol{a}) = \left\{ (z_0, \dots, z_m) \in \mathbb{C}^{m+1} \,\middle|\, f := \sum_{j=0}^{m} z_j^{a_j} = 0,\, a_j \in \mathbb{Z}_{>0} \text{ and } \sum_{j=0}^{m} |z_j|^2 = 1 \right\}.$$

The link of the A_n singularity after a linear change of variables is $\Sigma(n+1, 2, 2)$ for $n > 3$; see (3-1). Brieskorn gave a necessary and sufficient condition on $\boldsymbol{a}$ for $\Sigma(\boldsymbol{a})$ to be a topological sphere, and means to show when these yield exotic differentiable structures on the topological $(2n-1)$-sphere in [Brieskorn 1966]. A standard calculus argument [Geiges 2008, Lemma 7.1.1] shows that $\Sigma(\boldsymbol{a})$ is always a smooth manifold.

In the mid 1970s, Brieskorn manifolds were found to admit a canonical contact structure, given by their set of complex tangencies,

$$\xi_0 = T\Sigma \cap J_0(T\Sigma),$$

where J_0 is the standard complex structure on $\mathbb{C}^{m+1}$. The contact structure ξ_0 can be expressed as $\xi_0 = \ker \alpha_0$ for the canonical 1-form

$$\alpha_0 := (-d\rho \circ J_0)|_\Sigma = \frac{i}{4}\left(\sum_{j=0}^{m} (z_j d\bar{z}_j - \bar{z}_j dz_j) \right)\Bigg|_\Sigma,$$

where $\rho = (\|z\|^2 - 1)/4$. A proof of this fact may be found in [Geiges 2008, Theorem 7.1.2]. The Reeb dynamics associated to α_0 are difficult to understand. There is a more convenient contact form α_1 constructed by Ustilovsky [1999, Lemma 4.1.2] via the following family.

Proposition 2.14 [Geiges 2008, Proposition 7.1.4]. *The* 1*-form*

$$\alpha_t = \frac{i}{4}\sum_{j=0}^{m} \frac{1}{1-t+t/a_j}(z_j d\bar{z}_j - \bar{z}_j dz_j)$$

is a contact form on $\Sigma(\boldsymbol{a})$ *for each* $t \in [0, 1]$.

Via Gray's stability theorem we obtain the following corollary.

Corollary 2.15. *For all* $t \in (0, 1]$, *the contact manifold* $(\Sigma(\boldsymbol{a}), \ker\alpha_0)$ *is contactomorphic to* $(\Sigma(\boldsymbol{a}), \ker\alpha_t)$.

Next, the Reeb dynamics associated to $\alpha_1 = \frac{1}{4}i\sum_{j=0}^{m} a_j(z_j d\bar{z}_j - \bar{z}_j dz_j)$ are computed.

Remark 2.16. While α_1 is degenerate, one can still easily check that the Reeb vector field associated to α_1 is given by

$$R_{\alpha_1} = 2i\sum_{j=0}^{m} \frac{1}{a_j}\left(z_j\frac{\partial}{\partial z_j} - \bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right) = 2i\left(\frac{z_0}{a_0}, \dots, \frac{z_m}{a_m}\right).$$

Indeed, one computes

$$df(R_{\alpha_1}) = f(\boldsymbol{z}) \quad \text{and} \quad d\rho(R_{\alpha_1}) = 0.$$

This shows that R_{α_1} is tangent to $\Sigma(\boldsymbol{a})$. The defining equations for the Reeb vector field are satisfied since

$$\alpha_1(R_{\alpha_1}) \equiv 1 \quad \text{and} \quad \iota_{R_{\alpha_1}} d\alpha_1 = -d\rho,$$

with the latter form being zero on the $T_p\Sigma(\boldsymbol{a})$. The flow of R_{α_1} is given by

$$\varphi_t(z_0, \dots, z_m) = (e^{2it/a_0}, \dots, e^{2it/a_m}).$$

All the orbits of the Reeb flow are closed, and the flow defines an effective S^1-action on $\Sigma(\boldsymbol{a})$.

In the next section we perturb α_1 to a nondegenerate contact form.

3. Proof of Theorem 1.1

3A. ***Constructing a nondegenerate contact form.*** Here, we adapt a method used by Ustilovsky [1999, §4] to obtain a nondegenerate contact form α_ϵ on L_{A_n} whose kernel is contactomorphic to ξ_0. Ustilovsky's methods yielded a nondegenerate contact form on Brieskorn manifolds of the form $\Sigma(p, 2, \dots, 2)$, which are diffeomorphic to S^{4m+1}.

We define the change of coordinates to go from $\Sigma(n+1,2,2)$ with defining function $f = z_0^{n+1} + z_1^2 + z_2^2$ to L_{A_n} with defining function $f_{A_n} = w_0^{n+1} + 2w_1w_2$:

$$\Psi(w_0, w_1, w_2) = \Big(\underbrace{w_0}_{:=z_0}, \underbrace{\tfrac{\sqrt{2}}{2}(w_1+w_2)}_{:=z_1}, \underbrace{\tfrac{\sqrt{2}}{2}(-iw_1+iw_2)}_{:=z_2} \Big). \tag{3-1}$$

We obtain

$$\Psi^* f(z_0, z_1, z_2) = w_0^{n+1} + 2w_1w_2. \tag{3-2}$$

Then the pull-back of

$$\frac{\alpha_1}{2} = \frac{i}{8}\sum_{j=0}^{m} a_j(z_j d\bar{z}_j - \bar{z}_j dz_j)$$

is given by

$$\frac{\Psi^*\alpha_1}{2} = \frac{(n+1)i}{8}(w_0 d\overline{w}_0 - \overline{w}_0 dw_0) + \frac{i}{4}(w_1 d\overline{w}_1 - \overline{w}_1 dw_1 + w_2 d\overline{w}_2 - \overline{w}_2 dw_2).$$

We now construct the Hamiltonian function

$$H(w) = |w|^2 + \epsilon(|w_1|^2 - |w_2|^2).$$

We choose $0 < \epsilon < 1$ such that $H(w)$ is positive on S^5, and define the contact form

$$\alpha_\epsilon = \Psi^*\alpha_1/(2H). \tag{3-3}$$

Remark 3.1. The above shows that $(\Sigma(n+1,2,2), \ker\alpha_1)$ is contactomorphic to $(\Psi(\Sigma(n+1,2,2)), \ker\alpha_\epsilon)$. Moreover $L_{A_n} = \Psi(\Sigma(n+1,2,2))$, where L_{A_n} was defined in (1-1).

Proposition 3.2. *The Reeb vector field for α_ϵ is*

$$\begin{aligned} R_{\alpha_\epsilon} &= \frac{4i}{n+1} w_0 \frac{\partial}{\partial w_0} - \frac{4i}{n+1}\overline{w}_0 \frac{\partial}{\partial \overline{w}_0} + 2i(1+\epsilon)\left(w_1 \frac{\partial}{\partial w_1} - \overline{w}_1 \frac{\partial}{\partial \overline{w}_1}\right) \\ &\qquad + 2i(1-\epsilon)\left(w_2 \frac{\partial}{\partial w_2} - \overline{w}_2 \frac{\partial}{\partial \overline{w}_{2j}}\right) \\ &= \left(\frac{4i}{n+1} w_0, 2i(1+\epsilon) w_1, 2i(1-\epsilon) w_2 \right). \end{aligned} \tag{3-4}$$

Remark 3.3. The second formulation of the Reeb vector field is equivalent to the first in the above proposition via the standard identification of $\mathbb{R}^4$ with $\mathbb{C}^2$, as explained in Example 2.8, (2-3).

Before proving Proposition 3.2 we need the following lemma.

Lemma 3.4. *On $\mathbb{C}^3$, the vector field*

$$X(w) = \frac{1}{2}\left(\sum_{j=0}^{2} w_j \frac{\partial}{\partial w_j} + \overline{w}_j \frac{\partial}{\partial \overline{w}_j}\right) \tag{3-5}$$

is a Liouville vector field for the symplectic form

$$\omega_1 = \frac{d(\Psi^*\alpha_1)}{2} = \frac{i(n+1)}{4} dw_0 \wedge d\overline{w}_0 + \frac{i}{2}\sum_{j=1}^{2} dw_j \wedge d\overline{w}_j.$$

The Hamiltonian vector field X_H of H with respect to ω_1 is $-R_{\alpha_\epsilon}$, as in (3-4).

Proof. Recall that the condition to be a Liouville vector field is $\mathcal{L}_X\omega_1 = \omega_1$. We show this with Cartan's formula, given as

$$\begin{aligned}\mathcal{L}_X\omega_1 &= \iota_X d\omega_1 + d(\iota_X\omega_1)\\ &= d(\iota_X\omega_1).\end{aligned}$$

We do the explicit calculation for the first term and the rest easily follows:

$$\begin{aligned} d\left(\frac{i(n+1)}{4} d\omega_0 \wedge d\bar{\omega}_0\left(\frac{1}{2}\left(w_0\frac{\partial}{\partial w_0} + \overline{w}_0 \frac{\partial}{\partial \overline{w}_0}\right), \cdot\right)\right) \\ &= d\left(\frac{i(n+1)}{8} w_0 d\overline{w}_0 - \overline{w}_0 dw_0\right)\\ &= \frac{i(n+1)}{8}(dw_0\wedge d\overline{w}_0 - d\overline{w}_0 \wedge dw_0)\\ &= \frac{i(n+1)}{4} dw_0 \wedge d\overline{w}_0,\end{aligned}$$

so $X(w)$ is indeed a Liouville vector field for ω_1.

Next we prove that $\omega_1(-R_{\alpha_\epsilon}, \cdot) = dH(\cdot)$. First we calculate dH:

$$dH = \left(\sum_{j=0}^{2} w_j d\overline{w}_j + \overline{w}_j dw_j\right) + \epsilon(w_1 d\overline{w}_1 + \overline{w}_1 dw_1 - w_2 d\overline{w}_2 - \overline{w}_2 dw_2).$$

Then we compare the coefficients of dH to the coefficients of $\omega_1(-R_{\alpha_\epsilon}, \cdot)$ associated to each term, $(dw_i \wedge d\overline{w}_i)$. The $(dw_0 \wedge d\overline{w}_0)$ term is

$$\begin{aligned}\frac{i(n+1)}{4} dw_0 \wedge d\overline{w}_0\left(-\frac{4i}{n+1} w_0 \frac{\partial}{\partial w_0} + \frac{4i}{n+1}\overline{w}_0 \frac{\partial}{\partial \overline{w}_0}, \cdot\right) \\ &= \frac{i(n+1)}{4}\left(-\frac{4i}{n+1} w_0 d\overline{w}_0 - \frac{4i}{n+1}\overline{w}_0 dw_0\right)\\ &= w_0 d\overline{w}_0 + \overline{w}_0 dw_0.\end{aligned}$$

The $(dw_1 \wedge d\overline{w}_1)$ term is

$$\tfrac{1}{2} i\, dw_1 \wedge d\overline{w}_1 \left(-2i(1+\epsilon) w_1 \frac{\partial}{\partial w_1} + 2i(1+\epsilon)\overline{w}_1 \frac{\partial}{\partial \overline{w}_1} \right)$$
$$= \tfrac{1}{2} i(-2i(1+\epsilon)w_1 d\overline{w}_1 - 2i(1+\epsilon)\overline{w}_1 dw_1)$$
$$= (1+\epsilon)w_1 d\overline{w}_1 + (1+\epsilon)\overline{w}_1 dw_1.$$

The $(dw_2 \wedge d\overline{w}_2)$ term is obtained in a similar way. Summing the terms yields $\omega_1(-R_{\alpha_\epsilon}, \cdot\,) = dH(\cdot\,)$. □

Proof of Proposition 3.2. First we show that $X_H = -R_{\alpha_\epsilon}$ is tangent to the link $\Psi(\Sigma(n+1, 2, 2))$. We compute

$$\begin{aligned}(\Psi_* df)(R_{\alpha_\epsilon}) &= ((n+1)w_0^n dw_0 + 2w_1 dw_2 + 2w_2 dw_1)(R_{\alpha_\epsilon}) \\ &= 4i w_0^{n+1} + 4i(1-\epsilon)w_1 w_2 + 4i(1+\epsilon)w_1 w_2 \\ &= 4i(\Psi^* f) \\ &= 0.\end{aligned}$$

The last equality is because $\Psi^* f$ is constant along $\Psi(\Sigma(n+1, 2, 2))$. Now we have to show that $\frac{1}{2}\Psi^*\alpha_1(X_H) = -H$. We have

$$\begin{aligned}\tfrac{1}{2}\Psi^*\alpha_1(\cdot\,) &= \iota_X \omega_1(\cdot\,) = \omega_1(X(w), \cdot\,) = -\omega(\cdot\,, X(w)), \\ \tfrac{1}{2}\Psi^*\alpha_1(X_H) &= -\omega(X_H, X(w)) = -dH(X(w)) \\ &= -|w|^2 - \epsilon(|w_1|^2 - |w_2|^2) \\ &= -H.\end{aligned}$$

From these, we conclude

$$\begin{aligned}\alpha_\epsilon(X_H) &= -\frac{1}{H} H = -1, \\ d\alpha_\epsilon(X_H, \cdot\,) &= -\frac{1}{2H^2}(dH \wedge \Psi^*\alpha_1)(X_H, \cdot\,) + \frac{1}{2H} d\Psi^*\alpha_1(X_H, \cdot\,) \\ &= -\frac{1}{2H^2} dH(X_H)\Psi^*\alpha_1(\cdot\,) + \frac{1}{2H^2}\Psi^*\alpha_1(X_H) dH(\cdot\,) + \frac{1}{H}\omega(X_H, \cdot\,) \\ &= -\frac{1}{2H^2}\omega_1(X_H, X_H)\Psi^*\alpha_1(\cdot\,) - \frac{1}{H} dH(\cdot\,) + \frac{1}{H} dH(\cdot\,) \\ &= 0.\end{aligned}$$

By Lemma 3.4, we know $-X_H = R_{\alpha_\epsilon}$ so the result follows. □

3B. ***Isolated Reeb orbits.*** In this short section, we prove the following proposition.

Proposition 3.5. *The only simple periodic Reeb orbits of R_{α_ϵ} are nondegenerate and defined by*

$$\gamma_+(t) = (0, e^{2i(1+\epsilon)t}, 0), \quad 0 \le t \le \pi/(1+\epsilon),$$
$$\gamma_-(t) = (0, 0, e^{2i(1-\epsilon)t}), \quad 0 \le t \le \pi/(1+\epsilon).$$

Proof. The flow of

$$R_{\alpha_\epsilon} = \left(\frac{4i}{n+1} w_0, 2i(1+\epsilon)w_1, 2i(1-\epsilon)w_2\right)$$

is given by

$$\varphi_t(w_0, w_1, w_2) = \left(e^{4it/(n+1)} w_0, e^{2i(1+\epsilon)t} w_1, e^{2i(1-\epsilon)t} w_2\right).$$

Since ϵ is small and irrational, the only possible periodic trajectories are

$$\gamma_0(t) = (e^{4i/(n+1)t}, 0, 0),$$
$$\gamma_+(t) = (0, e^{2i(1+\epsilon)t}, 0),$$
$$\gamma_-(t) = (0, 0, e^{2i(1-\epsilon)t}).$$

It is important to note that the first trajectory does not lie in $\Psi(\Sigma(n+1, 2, 2))$, but rather on the total space $\mathbb{C}^3$. This is because the point $\gamma_0(0) = (1, 0, 0)$ is not a zero of $f_{A_n} = w_0^{n+1} + 2w_1 w_2$.

Next we need to check that the linearized return maps $d\phi|_\xi$ associated to γ_+ and γ_- have no eigenvalues equal to 1. We consider the first orbit γ_+ of period $\pi/(1+\epsilon)$, as a similar argument applies to the return flow associated to γ_-. The differential of its total return map is

$$d\varphi_T = \left.\begin{pmatrix} e^{4iT/(n+1)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2i(1-\epsilon)T} \end{pmatrix}\right|_{T=\pi/(1+\epsilon)}.$$

Since ϵ is a small irrational number, the total return map only has one eigenvalue which is 1. The eigenvector associated to the eigenvalue which is 1 is in the direction of the Reeb orbit γ^+, but since we are restricting the return map to ξ, we can conclude that γ_+ is nondegenerate. □

3C. ***Computation of the Conley–Zehnder index.*** To compute the Conley–Zehnder indices of the Reeb orbits in Theorem 1.1 we use the same method as shown in [Ustilovsky 1999], extending the Reeb flow to give rise to a symplectomorphism of $\mathbb{C}^3 \setminus \{\mathbb{0}\}$. This permits us to do the computations in $\mathbb{C}^3$, equipped with the symplectic form

$$\omega_1 = \frac{d(\Psi^*\alpha_1)}{2} = \frac{i(n+1)}{4} dw_0 \wedge d\overline{w}_0 + \frac{i}{2}\sum_{j=1}^{2} dw_j \wedge d\overline{w}_j.$$

We may equip the contact structure ξ_0 with the symplectic form $\omega = d\alpha_1$ instead of $d\alpha_\epsilon$ when computing the Conley–Zehnder indices. This is because $\ker \alpha_\epsilon = \ker \alpha_1 = \xi_0$, as $\alpha_\epsilon = (1/H)\alpha_1$ with $H > 0$ and because $\omega|_\xi = H d\alpha_\epsilon|_\xi$ and H is constant along Reeb trajectories.

Our first proposition shows that we can construct a standard symplectic basis for the symplectic complement

$$\xi^\omega = \{v \in \mathbb{C}^3 : \omega(v, w) = 0 \text{ for all } w \in \xi\}$$

of ξ in $\mathbb{C}^3$. As a result, $c_1(\xi^\omega) = 0$. Since $c_1(\mathbb{C}^3) = 0$, we know $c_1(\xi) = 0$. Thus we may compute the Conley–Zehnder indices in the ambient space $\mathbb{C}^3$ and use additivity of the Conley–Zehnder index under direct sums of symplectic paths to compute it in ξ.

Proposition 3.6. *There exists a standard symplectic basis for the symplectic complement ξ^ω with respect to $\omega = d\alpha_1$.*

Proof. Notice that $\xi^\omega = \operatorname{span}(X_1, Y_1, X_2, Y_2)$, where

$$X_1 = (\overline{w}_0^n, \overline{w}_1, \overline{w}_2), \quad Y_1 = iX_1,$$
$$X_2 = R_\epsilon, \quad Y_2 = w.$$

We make this into a symplectic standard basis for ξ^ω via a Gram–Schmidt process. The new basis is given by

$$\tilde{X}_1 = \frac{X_1}{\sqrt{\omega(X_1, Y_2)}}, \qquad \tilde{Y}_1 = \frac{Y_1}{\sqrt{\omega(X_1, Y_1)}} = i\tilde{X}_1,$$

$$\tilde{X}_2 = X_2, \qquad \tilde{Y}_2 = Y_2 - \frac{\omega(X_1, Y_2)Y_1 - \omega(Y_1, Y_2)X_1}{\omega(X_1, Y_1)}$$

$$= Y_2 - \frac{n-1}{2} w_0^{n+1} w(X_1, Y_1) X_1.$$

This is a standard basis for the symplectic vector space ξ^ω; i.e., the form ω in this basis is given by

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \\ & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}. \qquad \square$$

Now we are ready to prove the Conley–Zehnder index formula in Theorem 1.1.

Proposition 3.7. *The Conley–Zehnder index for $\gamma = \gamma_\pm^N$ in $0 \le t \le N\pi/(1\pm\epsilon)$ is*

$$\mu_{CZ}(\gamma_\pm^N) = 2\left(\left\lfloor \frac{2N}{(n+1)(1\pm\epsilon)} \right\rfloor + \left\lfloor \frac{N(1\mp\epsilon)}{1\pm\epsilon} \right\rfloor - \left\lfloor \frac{2N}{1\pm\epsilon} \right\rfloor\right) + 2N + 1. \quad (3\text{-}6)$$

Proof. The Reeb flow φ which we introduced in the previous section can be extended to a flow on $\mathbb{C}^3$, which we also denote by φ. The action of the extended Reeb flow on $\mathbb{C}^3$ is given by

$$\begin{aligned} d\varphi_t(w)\tilde{X}_1 &= e^{4it}\tilde{X}_1(\varphi_t(w)), & d\varphi_t(w)\tilde{Y}_1 &= e^{4it}\tilde{Y}_1(\varphi_t(w)), \\ d\varphi_t(w)\tilde{X}_2 &= \tilde{X}_2(\varphi_t(w)), & d\varphi_t(w)\tilde{Y}_2 &= \tilde{Y}_2(\varphi_t(w)). \end{aligned}$$

Define

$$\Phi := d\varphi_t|_{\mathbb{C}^3} = \operatorname{diag}(e^{4i/(n+1)t}, e^{2i(1+\epsilon)t}, e^{2i(1-\epsilon)t}).$$

We can now use the additivity of the Conley–Zehnder index under direct sums of symplectic paths, Theorem 2.13(4) to get

$$\mu_{CZ}(\gamma_\pm) = \mu_{CZ}(\Phi) - \mu_{CZ}(\Phi_{\xi^\omega}),$$

where

$$\Phi_{\xi^\omega} := d\varphi_t|_{\xi^\omega} = \operatorname{diag}(e^{4it}, 1). \quad (3\text{-}7)$$

The right-hand side of (3-7) is easily computed via the crossing form; see [Robbin and Salamon 1993, Remark 5.4]. In particular we have

$$\mu_{CZ}\big(\{e^{it}\}|_{t\in[0,T]}\big) = \begin{cases} T/\pi, & T \in 2\pi\mathbb{Z}, \\ 2\lfloor T/2\pi \rfloor + 1, & \text{otherwise}. \end{cases}$$

Thus for $\{\Phi(t)\} = \{e^{4it/(n+1)} \oplus e^{2it(1+\epsilon)} \oplus e^{2it(1-\epsilon)}\}$ with $0 \le t \le T$ we obtain

$$\begin{aligned} \mu_{CZ}(\Phi) = &\begin{cases} 4T/((n+1)\pi), & T \in \tfrac{1}{2}(n+1)\pi\mathbb{Z}, \\ 2\big\lfloor 2T/((n+1)\pi)\big\rfloor + 1, & T \notin \tfrac{1}{2}(n+1)\pi\mathbb{Z}, \end{cases} \\ &+ \begin{cases} 2T(1+\epsilon)/\pi, & T \in \pi/(1+\epsilon)\mathbb{Z}, \\ 2\big\lfloor T(1+\epsilon)/\pi \big\rfloor + 1, & T \notin \pi/(1+\epsilon)\mathbb{Z}, \end{cases} \\ &+ \begin{cases} 2T(1-\epsilon)/\pi, & T \in \pi/(1-\epsilon)\mathbb{Z}, \\ 2\big\lfloor T(1-\epsilon)/\pi \big\rfloor + 1, & T \notin \pi/(1-\epsilon)\mathbb{Z}. \end{cases} \end{aligned}$$

Likewise for Φ_{ξ^ω} with $0 \le t \le T$ we obtain

$$\mu_{CZ}(\Phi_{\xi^\omega}) = \begin{cases} 4T/\pi, & T \in \pi/2\mathbb{Z}, \\ 2\lfloor 2T/\pi \rfloor + 1, & T \notin \pi/2\mathbb{Z}. \end{cases}$$

Hence we get that the Conley–Zehnder index for $\gamma_\pm^N$ in $0 \le t \le N\pi/(1\pm\epsilon)$ is given by

$$\mu_{CZ}(\gamma_\pm^N) = 2\left(\left\lfloor \frac{2N}{(n+1)(1\pm\epsilon)} \right\rfloor + \left\lfloor \frac{N(1\mp\epsilon)}{1\pm\epsilon} \right\rfloor - \left\lfloor \frac{2N}{1\pm\epsilon} \right\rfloor\right) + 2N + 1. \quad (3\text{-}8)$$

This completes the proof. □

4. Proof of Theorem 1.8

This section proves that (L_{A_n}, ξ_0) and $(L(n+1, n), \xi_{\text{std}})$ are contactomorphic. This is done by constructing a 1-parameter family of contact manifolds via a canonically defined Liouville vector field and applying Gray's stability theorem.

4A. ***Contact geometry of*** $(L(n+1, n), \xi_{\text{std}})$**.** The lens space $L(n+1, n)$ is obtained via the quotient of S^3 by the binary cyclic subgroup $A_n \subset \mathrm{SL}(2, \mathbb{C})$. The subgroup A_n is given by the action of $\mathbb{Z}_{n+1}$ on $\mathbb{C}^2$ defined by

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} e^{2\pi i/(n+1)} & 0 \\ 0 & e^{2n\pi i/(n+1)} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

The following exercise shows that $L(n+1, n)$ is homeomorphic to L_{A_n}. This construction will be needed later in another proof, so we explain it here to set up the notation.

The origin is the only fixed point of the A_n action on $\mathbb{C}^2$ and hence is an isolated quotient singularity of $\mathbb{C}^2/A_n$. We can represent $\mathbb{C}^2/A_n$ as a hypersurface of $\mathbb{C}^3$ as follows. Consider the monomials

$$z_0 := uv, \quad z_1 := \tfrac{1}{\sqrt{2}} i u^{n+1}, \quad z_2 := \tfrac{1}{\sqrt{2}} i v^{n+1}.$$

These are invariant under the action of A_n and satisfy the equation $z_0^{n+1} + 2z_1 z_2 = 0$. Recall that

$$f_{A_n}(z_0, z_1, z_2) = z_0^{n+1} + 2z_1 z_2 \quad \text{and} \quad L_{A_n} = S^5 \cap \{f_{A_n}^{-1}(0)\}.$$

Moreover,

$$\begin{aligned} \tilde{\varphi} &: \mathbb{C}^2 \to \mathbb{C}^3, \\ (u, v) &\mapsto (uv, \tfrac{1}{\sqrt{2}} i u^{n+1}, \tfrac{1}{\sqrt{2}} i v^{n+1}), \end{aligned} \tag{4-1}$$

descends to the map

$$\varphi : \mathbb{C}^2/A_n \to \mathbb{C}^3,$$

which sends $\varphi(\mathbb{C}^2/A_n)$ homeomorphically onto the hypersurface $f_{A_n}^{-1}(0)$.

Rescaling away from the origin of $\mathbb{C}^3$ yields a homeomorphism between $\varphi(S^3/A_n)$ and L_{A_n}. As 3-manifolds which are homeomorphic are also diffeomorphic [Moise 1952], we obtain the following proposition.

Proposition 4.1. *$L(n+1, n)$ is diffeomorphic to L_{A_n}.*

Remark 4.2. In order to prove that two manifolds are contactomorphic, one must either construct an explicit diffeomorphism or make use of Gray's stability theorem. Sadly, φ is not a diffeomorphism onto its image when $u = 0$ or $v = 0$. As the above diffeomorphism is only known to exist abstractly, we will need to appeal the latter method to prove that (L_{A_n}, ξ_0) and $(L(n+1, n), \xi_{\text{std}})$ are contactomorphic. As a result, this proof is rather involved.

Our application of Gray's stability theorem uses the flow of a Liouville vector field to construct a 1-parameter family of contactomorphisms. First we prove that $L(n+1, n)$ is a contact manifold whose contact structure descends from the quotient of S^3.

Consider the standard symplectic form on $\mathbb{C}^2$ given by

$$\begin{aligned} \omega_{\mathbb{C}^2} &= d\lambda_{\mathbb{C}^2}, \\ \lambda_{\mathbb{C}^2} &= \tfrac{1}{2} i(u d\bar{u} - \bar{u} du + v d\bar{v} - \bar{v} dv). \end{aligned} \tag{4-2}$$

The following proposition shows that λ_0 restricts to a contact form on $L(n+1, n)$. We define $\ker\lambda = \xi_{\text{std}}$ on $L(n+1, n)$.

Proposition 4.3. *The vector field*

$$Y_0 = \frac{1}{2}\left(u\frac{\partial}{\partial u} + \bar{u}\frac{\partial}{\bar{u}} + v\frac{\partial}{\partial v} + \bar{v}\frac{\partial}{\bar{v}}\right)$$

is a Liouville vector field on $(\mathbb{C}^2/A_n, \omega_{\mathbb{C}^2})$ *away from the origin and transverse to* $L(n+1, n)$.

Proof. We have that $\mathbb{C}^2/A_n$ is a smooth manifold away from the origin because 0 is the only fixed point by the action of A_n. Write

$$S^3/A_n = \{(u, v) \in \mathbb{C}^2/A_n : |u|^2 + |v|^2 = 1\}.$$

Then $L(n+1, n) = S^3/A_n$ is a regular level set of $g(u, v) = |u|^2 + |v|^2$ Choose a Riemannian metric on $\mathbb{C}^2/A_n$ and note that

$$Y_0 = \tfrac{1}{4}\nabla g.$$

Thus Y_0 is transverse to $L(n+1, n)$. Since

$$\mathcal{L}_{Y_0}\omega_{\mathbb{C}^2} = d(i_{Y_0} d\lambda_{\mathbb{C}^2}) = \omega_{\mathbb{C}^2},$$

we may conclude that Y_0 is indeed a Liouville vector field on $(\mathbb{C}^2/A_n, \omega_{\mathbb{C}^2})$ away from the origin. Thus by Proposition 2.10, $L(n+1, n)$ is a hypersurface of contact type in $\mathbb{C}^2/A_n$. □

4B. ***The proof that (L_{A_n}, ξ_0) and $(L(n+1, n), \xi_{\text{std}})$ are contactomorphic.*** First we set up L_{A_n} and $\varphi(L(n+1, n))$ as hypersurfaces of contact type in $\{f_{A_n}^{-1}(0)\} \setminus \{\mathbb{0}\}$. Define $\rho : \mathbb{C}^3 \to \mathbb{R}$ by

$$\rho(z) = \tfrac{1}{4}|z|^2 - 1 = \tfrac{1}{4} z_0\bar{z}_0 + \cdots + z_2\bar{z}_2 - 1.$$

The standard symplectic structure on $\mathbb{C}^3$ is given by

$$\omega_{\mathbb{C}^3} = \tfrac{1}{2} i (dz_0 \wedge d\bar{z}_0 + \cdots + dz_2 \wedge d\bar{z}_2).$$

Moreover,

$$Y = \nabla\rho = \frac{1}{2} \sum_{j=0}^{2} z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \tag{4-3}$$

is a Liouville vector field for $(\mathbb{C}^3, \omega_{\mathbb{C}^3})$. We define

$$\lambda_{\mathbb{C}^3} = \iota_Y \omega_{\mathbb{C}^3}.$$

A standard calculation analogous to the proof of Proposition 4.3 shows that Y is a Liouville vector field on $(\{f_{A_n}^{-1}(0)\} \setminus \{\mathbb{0}\}, \omega_{\mathbb{C}^3})$.

Remark 4.4. Both $\varphi(L(n+1, n))$ and L_{A_n} are hypersurfaces of contact type in $(\{f_{A_n}^{-1}(0)\} \setminus \{\mathbb{0}\}, \omega_{\mathbb{C}^3})$. Note that $\varphi(L(n+1, n))$ is in fact transverse to the Liouville vector field Y because

$$\begin{aligned}\varphi(L(n+1, n)) &= \varphi(\{|u|^2 + |v|^2 = 1\}/A_n)\\ &= \varphi(\{|u|^4 + 2|u|^2|v|^2 + |v|^4 = 1\}/A_n)\\ &= \big\{2|z_0|^2 + 4^{1/(n+1)}|z_1|^{4/(n+1)} + 4^{1/(n+1)}|z_2|^{4/(n+1)} = 1\big\} \cap f_{A_n}^{-1}(0).\end{aligned}$$

We will want $\varphi(L(n+1, n))$ and L_{A_n} to be disjoint in $\{f_{A_n}^{-1}(0)\}$. This is easily accomplished by rescaling r in the definition of the link.

Definition 4.5. Define

$$L_{A_n}^r = f_{A_n}^{-1}(0) \cap S_r^5,$$

with the assumption that r has been chosen so that $\varphi(L(n+1, n))$ and $L_{A_n}^r$ are disjoint in $\{f_{A_n}^{-1}(0)\}$ and so that the flow of the Liouville vector field Y "hits" $\varphi(L(n+1, n))$ before $L_{A_n}^r$.

The first result is the following lemma, which provides a 1-parameter family of diffeomorphic manifolds starting on $\varphi(L(n+1, n))$ and ending on $L_{A_n}^r$. First we set up some notation. Let

$$\psi_t : \mathbb{R} \times X \to X$$

be the flow of Y and $\psi_t(z) = \gamma_z(t)$ the unique integral curve passing through $z \in \varphi(L(n+1, n))$ at time $t = 0$. For any integral curve γ of Y we consider the initial value problem

$$\gamma'(t) = Y(\gamma(t)) \quad \text{and} \quad \gamma(0) = z \in \varphi(L(n+1, n)). \tag{4-4}$$

By means of the implicit function theorem and the properties of the Liouville vector field Y we can prove the following claim.

Lemma 4.6. *For every γ_z, there exists a $\tau(z) \in \mathbb{R}_{>0}$ such that $\gamma_z(\tau(z)) \in L^r_{A_n}$. The choice of $\tau(z)$ varies smoothly for each $z \in \varphi(L(n+1, n))$.*

Proof. In order to apply the implicit function theorem, we must show for all (t, z) with $\rho \circ \gamma = 0$ that

$$\frac{\partial(\rho \circ \gamma)}{\partial t} \neq 0.$$

Note that $\rho \circ \gamma$ is smooth. By the chain rule,

$$\left.\frac{\partial(\rho \circ \gamma)}{\partial t}\right|_{(s,p)} = \operatorname{grad} \rho|_{\gamma(s,p)} \cdot \dot{\gamma}|_{(s,p)},$$

where $\dot{\gamma}|_{(s,p)} = \partial\gamma/\partial t|_{(s,p)}$.

If $\operatorname{grad} \rho|_{\gamma(s,p)} \cdot \dot{\gamma}|_{(s,p)} = 0$, then $\operatorname{grad} \rho$ is not transverse along $\{(\rho \circ \gamma)(s, p) = 0\}$ or $\dot{\gamma}|_{(s,p)} = 0$, since $\operatorname{grad} \rho \neq 0$. By construction, $\operatorname{grad} \rho = \nabla\rho$ is a Liouville vector field transverse to $L^r_{A_n}$. Furthermore, the conformal symplectic nature of a Liouville vector field implies that for any integral curve γ satisfying the initial value problem given by (4-4), $\dot{\gamma}|_{(s,p)} \neq 0$. Thus we see that the conditions for the implicit function theorem are satisfied and our claim is proven. □

Remark 4.7. The time $\tau(z)$ can be normalized to 1 for each z, yielding a 1-parameter family of diffeomorphic contact manifolds (M_t, ζ_t) for $0 \leq t \leq 1$ given by

$$M_t = \psi_t\big(\varphi(L(n+1, n))\big), \quad \zeta_t = TM_t \cap J_{\mathbb{C}^3}(TM_t),$$

where

$$M_0 = \psi_0\big(\varphi(L(n+1, n))\big) = \varphi(L(n+1, n)), \quad M_1 = \psi_1\big(\varphi(L(n+1, n))\big) = L_{A_n}.$$

Moreover, we can relate the standard contact structure on $L(n+1, n)$ under the image of φ. To avoid excessive parentheses, we use S^3/A_n in place of $L(n+1, n)$ in this lemma.

Lemma 4.8. *On $\varphi(S^3/A_n)$,*

$$\varphi_*(\xi_{\text{std}}) = T(\varphi(S^3/A_n)) \cap J_{\mathbb{C}^3}\big(T(\varphi(S^3/A_n))\big).$$

Proof. Since $A_n \subset \mathrm{SL}(2, \mathbb{C})$, we have

$$\tilde{\varphi}(J_{\mathbb{C}^2} TS^3) = J_{\mathbb{C}^3}(T\tilde{\varphi}(S^3)).$$

Examining $\varphi_*(\xi_{\text{std}})$ yields

$$\begin{aligned}\varphi_*(T(S^3/A_n) \cap J_{\mathbb{C}^2} T(S^3/A_n)) &= \tilde{\varphi}_*(TS^3 \cap J_{\mathbb{C}^2}(TS^3)) = \tilde{\varphi}_*(TS^3) \cap \tilde{\varphi}_*(J_{\mathbb{C}^2}(TS^3)) \\ &= \tilde{\varphi}_*(TS^3) \cap J_{\mathbb{C}^3}\tilde{\varphi}_*(TS^3) = T\tilde{\varphi}(S^3) \cap J_{\mathbb{C}^3}(T\tilde{\varphi}(S^3)) \\ &= T(\varphi(S^3/A_n)) \cap J_{\mathbb{C}^3}(T\varphi(S^3/A_n)).\end{aligned}$$

□

Lemmas 4.6 and 4.8 in conjunction with Remark 4.7 and Lemma 2.11 yield the following proposition.

Proposition 4.9. *The image of the lens space* $(\varphi(L(n+1,n)), \varphi_*\xi_{\text{std}})$ *is contactomorphic to* (L_{A_n}, ξ_0).

It remains to show that $(\varphi(L(n+1,n)), \varphi_*\xi_{\text{std}})$ and $(L(n+1,n), \xi_{\text{std}})$ are contactomorphic. To accomplish this, we use Moser's lemma to prove the following lemma.

Lemma 4.10. *The manifolds* $(\mathbb{C}^2 \setminus \{\mathbb{0}\}, d\lambda_{\mathbb{C}^2})$ *and* $(\mathbb{C}^2 \setminus \{\mathbb{0}\}, d\tilde{\varphi}^*\lambda_{\mathbb{C}^3})$ *are contactomorphic.*

Proof. Consider the family of 2-forms

$$\omega_t = (1-t)\omega_{\mathbb{C}^2} + t\tilde{\varphi}^*\omega_{\mathbb{C}^3}$$

for $0 \le t \le 1$. Then ω_t is exact because Y_0 and Y are Liouville vector fields for $\mathbb{C}^2 \setminus \mathbb{0}$ equipped with the symplectic forms $\omega_{\mathbb{C}^2}$ and $\omega_{\mathbb{C}^3}$ respectively; thus $d\lambda_t = \omega_t$ for

$$\lambda_t = (1-t)\lambda_{\mathbb{C}^2} + t\tilde{\varphi}^*(\lambda_{\mathbb{C}^3})$$

for $0 \le t \le 1$. We claim that λ_t is a family of contact forms for each $t \in [0, 1]$.

We compute

$$\begin{aligned}\frac{2}{i}\tilde{\varphi}^* d\lambda_{\mathbb{C}^3} &= d(uv)\wedge d(\overline{uv}) + d(u^{n+1})\wedge d(\bar{u}^{n+1}) + d(v^{n+1})\wedge d(\bar{v}^{n+1}) \\ &= ((n+1)^2|u|^{2n}+|v|^2)du\wedge d\bar{u} + 2\Re(u\bar{v}dv\wedge d\bar{u}) \\ &\qquad\qquad + ((n+1)^2|v|^{2n}+|u|^2)dv\wedge d\bar{v}.\end{aligned}$$

Since ω_t is exact for each $t \in [0, 1]$, we know $d(\omega_t) = 0$ for each $t \in [0, 1]$. Moreover, a simple calculation reveals that $\omega_t \wedge \omega_t$ is a volume form on $\mathbb{C}^2$ for each $t \in [0, 1]$. Thus we may conclude that ω_t is a symplectic form for each $t \in [0, 1]$. Applying Moser's argument, Theorem 2.3, yields the desired result. □

This yields the desired corollary.

Corollary 4.11. *The manifolds* $(L(n+1,n), \ker\lambda_{\mathbb{C}^2})$ *and* $(L(n+1,n), \ker\varphi^*\lambda_{\mathbb{C}^3})$ *are contactomorphic.*

Proof. Let $\phi : (\mathbb{C}^2 \setminus \{\mathbb{0}\}, d\lambda_{\mathbb{C}^2})$ and $(\mathbb{C}^2 \setminus \{\mathbb{0}\}, d\tilde{\varphi}^*\lambda_{\mathbb{C}^3})$ be the symplectomorphism, which exists by Lemma 4.10. It induces the desired contactomorphism. On $\mathbb{C}^2 \setminus \{\mathbb{0}\}$,

$$\phi^* d(\varphi^*\lambda_{\mathbb{C}^3}) = d\lambda_{\mathbb{C}^2};$$

thus,

$$d\phi^*(\varphi^*\lambda_{\mathbb{C}^3}) = d\lambda_{\mathbb{C}^2}.$$

So on $L(n+1,n)$,

$$\phi_*(\xi_{\text{std}}) = \phi_*(\ker\lambda_{\mathbb{C}^2}) = \ker\varphi_*\lambda_{\mathbb{C}^3} = \varphi_*\xi_{\text{std}}.$$

Proposition 4.9 and Corollary 4.11 complete the proof of Theorem 1.8. □

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