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Jacob Blackwood, Adam Dukehart and Mohammad Javaheri

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Loxodromes on hypersurfaces of revolution

Jacob Blackwood, Adam Dukehart and Mohammad Javaheri

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A loxodrome is a curve that makes a constant angle with the meridians. We use conformal maps and the notion of parallel transport in differential geometry to investigate loxodromes on hypersurfaces of revolution and their spiral behavior near a pole.

1. Introduction

Loxodromes appear historically as mathematical tools in navigation, since they provide efficient navigation routes from one point to another by making a constant *course angle* with the meridians. Even modern technology relies on the ability to calculate loxodromes [Alexander 2004]. Loxodromes are best understood via conformal maps, maps that preserve angles locally. For example, the Mercator projection map is a conformal map under which the meridians and curves of constant latitude (parallels) are mapped to vertical and horizontal lines. A curve making a constant angle with vertical lines as it crosses them is itself a straight line; therefore, the Mercator projection map represents loxodromes as straight lines. As another example of a conformal map, consider the stereographic projection which maps the meridians and parallels to lines through the origin (radial lines) and circles centered at the origin. The curves that make a constant angle with the radial lines are the well-known logarithmic spirals. In other words, the stereographic projection maps loxodromes to logarithmic spirals.

The construction of loxodromes on the sphere and oblate-spheroidal surfaces, which approximate the shape of the earth, has been investigated previously [Bennett 1996; Carlton-Wippern 1992; Smart 1946; Williams 1950]. Spheres and spheroids are examples of surfaces of revolution that we now define. Let $\eta(t) = (u(t), v(t))$, where $t \in (a, b)$, be a curve in the half-plane $H = \{(x, y, 0) \in \mathbb{R}^3 : y > 0\}$. By rotating the *profile curve* $\eta(t)$ around the x-axis in $\mathbb{R}^3$, one obtains a surface of revolution parametrized as

$$x = u(t), \quad y = v(t)\cos\theta, \quad z = v(t)\sin\theta, \quad \text{where } t \in (a, b),\ \theta \in [0, 2\pi).$$

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The *meridians* of the surface are the curves of constant θ, while the *parallels* of the surface are the curves of constant t. A curve on S is called a loxodrome of S if it makes a constant angle with the meridians of S as it crosses them.

In Section 2, we derive the parametric equations of loxodromes on a surface of revolution. We also define a stereographic projection on a given surface of revolution that maps loxodromes to logarithmic spirals in the plane. Finally, we study the distances along loxodromes as well as the spiral behavior of loxodromes near a pole.

In Section 3, we consider the case where the profile curve is a Jordan curve (and so the resulting surface of revolution is a torus). In particular, we study closed loxodromes and their density in the set of all loxodromes. We also show that loxodromes are geodesics in a suitable metric on the surface.

Hypersurfaces of revolution are important and interesting geometric objects, and they have been studied extensively by geometers [Coll and Harrison 2013; do Carmo and Dajczer 1983; Zhang 2012]. In Section 4, we give a definition of loxodromes on hypersurfaces of revolution. As an example, we also find parametric equations of loxodromes on higher-dimensional spheres.

2. The loxodrome equation

Suppose that the profile curve of a surface of revolution S is given by $y = f(x)$, where $f(x)$ is a differentiable function on the interval $(a, b) \subseteq \mathbb{R}$ such that $f(x) > 0$ for all $x \in (a, b)$. Then S is parametrized by the cylindrical map

$$r(x, \theta) = \langle x,\ f(x)\cos\theta,\ f(x)\sin\theta\rangle.$$

Let

$$\gamma(x) = \langle x,\ f(x)\cos\theta(x),\ f(x)\sin\theta(x)\rangle$$

be a loxodrome that makes a constant angle ψ_0 with the meridians. The tangent vector to the meridian $r(x, \theta)$, where θ is constant, is given by

$$\boldsymbol{a} = \frac{\partial}{\partial x} r(x, \theta) = \langle 1,\ f'(x)\cos\theta,\ f'(x)\sin\theta\rangle,$$

while the tangent vector to γ at $x \in (a, b)$ is given by

$$\boldsymbol{b} = \frac{d}{dx}\gamma(x) = \big\langle 1,\ f'(x)\cos\theta - f(x)\theta'(x)\sin\theta,\ f'(x)\sin\theta + f(x)\theta'(x)\cos\theta\big\rangle.$$

The constant-angle constraint gives $(\boldsymbol{a}\cdot\boldsymbol{b})^2 = \|\boldsymbol{a}\|^2\,\|\boldsymbol{b}\|^2\cos^2(\psi_0)$, which yields

$$1 + (f'(x))^2 = \big(1 + (f'(x))^2 + (f(x)\theta'(x))^2\big)\cos^2(\psi_0).$$

After solving for $\theta'(x)$ and integrating, one has

$$\theta(x) = \tan(\psi_0)A(x), \tag{2-1}$$

where

$$A(x) = \int_c^x \frac{\sqrt{1+(f'(x))^2}}{f(x)}\,dx + C, \tag{2-2}$$

with constants $c \in (a, b)$ and $C \in \mathbb{R}$.

2.1. ***The conformal stereographic projection.*** The stereographic projection on the sphere has the property that it is conformal and it maps meridians and parallels to lines through the origin and circles centered at the origin respectively. We now describe a map with the same properties on the surface of revolution S with the profile curve $y = f(x)$. Let

$$L(r, s) = \langle \ln(r^2+s^2), \arctan(s/r)\rangle$$

and

$$F(x, \theta) = \langle A^{-1}(x),\, f(A^{-1}(x))\cos\theta,\, f(A^{-1}(x))\sin\theta\rangle,$$

where $A(x)$ is given by (2-2).

We claim that the composition $T = F \circ L$ is a conformal map from an open subset of $\mathbb{R}^2$ to S. The map L is the logarithmic conformal mapping such that L^{-1} maps the horizontal and vertical lines in the (x, θ)-plane to lines through the origin and circles centered at the origin respectively. The map F is also a conformal map that maps the horizontal and vertical lines in the (x, θ)-plane to meridians and parallels on the surface S. To see this, we note that with $g(x) = A^{-1}(x)$, one has

$$F_x = \langle g'(x),\ f'\circ g(x)g'(x)\cos\theta,\ f'\circ g(x)g'(x)\sin\theta\rangle, \tag{2-3}$$

$$F_\theta = \langle 0,\ -f\circ g(x)\sin\theta,\ f\circ g(x)\cos\theta\rangle. \tag{2-4}$$

Therefore,

$$\begin{bmatrix} F_x\cdot F_x & F_x\cdot F_\theta \\ F_\theta\cdot F_x & F_\theta\cdot F_\theta \end{bmatrix} = \begin{bmatrix} (g'(x))^2(1+(f'\circ g(x))^2) & 0 \\ 0 & (f\circ g(x))^2 \end{bmatrix}. \tag{2-5}$$

For F to be a conformal map from the (x, θ)-plane to the surface of revolution S, the matrix in (2-5) must be a multiple of the identity matrix. By (2-2), one has

$$g'(x) = (A^{-1})'(x) = \frac{1}{A'(g(x))} = \frac{f(g(x))}{\sqrt{1+f'(g(x))^2}},$$

which implies that the matrix (2-5) is a multiple of identity; hence F is a conformal map. It follows that T, being a composition of conformal maps, is a conformal map. Moreover, T^{-1} maps the meridians and parallels of S to lines through the origin and circles centered at the origin respectively. Therefore, every loxodrome on S is mapped under T^{-1} to a logarithmic spiral in the (r, s)-plane.

A feature of the logarithmic spiral is its infinite spiraling around the origin. We now study this spiral behavior of loxodromes in more detail.

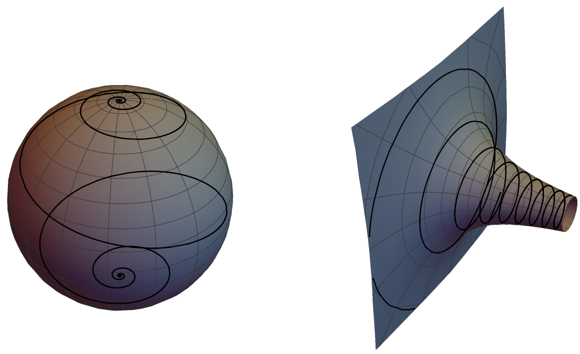

Figure 1. Left: spiral at a point. Right: spiral at infinity.

Theorem 1. *Let $y = f(x)$ be differentiable on the interval $(a, b) \subseteq \mathbb{R}$ and suppose that* $\lim_{x\to b^-} f(x) = 0$ *or* ∞. *Let $\gamma(x) = (x,\ f(x)\cos\theta(x),\ f(x)\sin\theta(x))$ be a loxodrome, where $\theta(x)$ is given by* (2-2) *with* $\tan(\psi_0) \neq 0$. *Then*

$$\lim_{x\to b^-} \theta(x) = \pm\infty, \tag{2-6}$$

where the plus or minus sign is determined by $\operatorname{sign}(\tan(\psi_0))$.

Proof. Let $c \in (a, b)$. Then

$$\begin{aligned}|\theta(x)-\theta(c)| &= \left|\tan(\psi_0)\int_c^x \frac{\sqrt{1+(f'(x))^2}}{f(x)}\,dx\right| \geq |\tan(\psi_0)|\int_c^x \left|\frac{f'(x)}{f(x)}\right| dx\\ &\geq |\tan(\psi_0)|\,|\ln f(c)-\ln f(x)| \to \infty,\end{aligned}$$

as $x \to b^-$, since $\lim_{x\to b^-} f(x) = 0$ or ∞, and in either case $|\ln f(x)| \to \infty$. □

In the next theorem, we compute distances along loxodromes. We denote the length of a curve γ by $\ell(\gamma)$ and the length of the graph of a function f by $\ell(f)$.

Theorem 2. *Let $y = f(x)$ be a differentiable positive function on the interval $(a, b) \subseteq \mathbb{R}$, and let $\gamma(x) = \langle x,\ f(x)\cos\theta(x),\ f(x)\sin\theta(x)\rangle$ be a loxodrome, where $\theta(x)$ is given by* (2-2). *Then*

$$\ell(\gamma) = |\sec(\psi_0)|\,\ell(f) = |\sec(\psi_0)|\int_a^b \sqrt{1+(f'(x))^2}\,dx. \tag{2-7}$$

Proof. By (2-1), we have

$$\begin{aligned}\|\gamma'(x)\|^2 &= 1+(f'(x)\cos\theta - f(x)\theta'(x)\sin\theta)^2+(f'(x)\sin\theta+f(x)\theta'(x)\cos\theta)^2\\ &= 1+(f'(x))^2+(f(x)\theta'(x))^2\\ &= 1+(f'(x))^2+\tan^2(\psi_0)(1+(f'(x))^2)\\ &= \sec^2(\psi_0)(1+(f'(x))^2),\end{aligned}$$

which implies (2-7). □

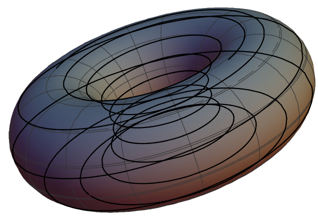

Figure 2. A toric loxodrome.

3. Loxodromes on the torus

Let C be a simple plane curve parametrized by arc length, $C(t) = (x(t), y(t))$, where $x(t)$ and $y(t)$ are differentiable functions of $t \in (a, b)$ and $y(t) > 0$ for all $t \in (a, b)$. Rotating C around the x-axis yields a surface of revolution with the parametrization

$$u(t, \theta) = \langle x(t),\ y(t)\cos\theta,\ y(t)\sin\theta\rangle. \tag{3-1}$$

Suppose that

$$\eta(t) = \langle x(t),\ y(t)\cos\theta(t),\ y(t)\sin\theta(t)\rangle$$

is a loxodrome. A similar calculation to that the previous section implies

$$\theta(t) = \tan(\psi_0)B(t), \tag{3-2}$$

where

$$B(t) = \int \frac{dt}{y(t)}. \tag{3-3}$$

In the next theorem, we discuss closed loxodromes on surfaces of revolution with periodic profile curves.

Theorem 3. *Let $C(t) = \langle x(t), y(t)\rangle$, $y(t) > 0$, be a simple closed differentiable curve parametrized by the arc length and with period $T > 0$, i.e., $C(t+T) = C(t)$ for all $t \in \mathbb{R}$. Let S be the surface of revolution with profile curve $C(t)$. Let $\eta(t)$ be a loxodrome on S, making a constant angle ψ_0 with the meridians of S. Then $\eta(t)$ is a closed curve if and only if*

$$\tan(\psi_0)\cdot\frac{1}{2\pi}\int_0^T \frac{dt}{y(t)} \in \mathbb{Q}. \tag{3-4}$$

In particular, closed loxodromes on S are dense in the set of all loxodromes. In addition, if a loxodrome is not closed, then the loxodrome is dense in S.

Proof. Since C is parametrized by arc length, we have from (3-2) that

$$\theta(mT+t) - \theta(t) = \tan(\psi_0)\int_t^{mT+t}\frac{dt}{y(t)} = m\tan(\psi_0)\int_0^T\frac{dt}{y(t)}. \tag{3-5}$$

For $\eta(t)$ to be a closed curve, we must have $\theta(mT+t)-\theta(t)=2n\pi$ for some integers m,n with $m\neq 0$. It follows that

$$m\tan(\psi_0)\int_0^T \frac{dt}{y(t)}=2\pi n,$$

which is equivalent to (3-4).

The set of angles ψ_0 for which (3-4) holds is dense in $\mathbb{R}$, and so the set of periodic loxodromes is dense among all loxodromes on S.

Next, suppose that $\eta(t)$ is not closed, and so (3-4) fails. It follows from (3-5) that $\theta(mT+t)-\theta(t)=2\pi m\lambda$, where λ is a fixed irrational number. By Kronecker's approximation theorem, the set $\{2\pi m\lambda \pmod{2\pi} : m\in\mathbb{Z}\}$ is dense in the interval $[0,2\pi)$. Therefore, the set $\{\eta(mT+t) : m\in\mathbb{Z}\}$ is dense in the parallel obtained by rotating $\eta(t)$ around the x-axis. In other words, if η intersects a parallel of S then it is dense in that parallel of S. Since η intersects every parallel of S, we conclude that η is dense in S. □

The flat metric on the torus has the property that every geodesic (paths that are locally of shortest length) is either periodic or dense in the torus. This resembles the property we discussed in Theorem 3 for loxodromes of S. In fact, there exists a metric on S for which the loxodromes are exactly the geodesics. The metric is simply the pullback of the Euclidean flat metric on $\mathbb{R}^2$ by the map R^{-1}, where

$$R(s,\theta)=\big\langle x(B^{-1}(s)),\ y(B^{-1}(s))\cos\theta,\ y(B^{-1}(s))\sin\theta\big\rangle,$$

where $B(t)$ is defined by (3-3).

4. Hypersurfaces of revolution

In this section, we give a definition of loxodromes on hypersurfaces of revolution. Let M be an $(n-2)$-dimensional submanifold of $\mathbb{R}^{n-1}\times\{0\}\subseteq\mathbb{R}^n$, and consider a local parametrization of M

$$\langle \boldsymbol{x},0\rangle=\langle x_1,\dots,x_{n-1},0\rangle:\mathcal{U}\to\mathbb{R}^n, \tag{4-1}$$

where $\mathcal{U}\subseteq\{\langle x_1,\dots,x_{n-1},0\rangle : x_{n-1}>0\}$ is open. There is a natural embedding of $M\times\mathbb{S}^1$ in $\mathbb{R}^n$ defined by

$$\langle \boldsymbol{x},0,\theta\rangle\mapsto R(\boldsymbol{x},\theta)=\langle x_1,\dots,x_{n-2},\ x_{n-1}\cos\theta,\ x_{n-1}\sin\theta\rangle. \tag{4-2}$$

We call this embedded $(n-1)$-dimensional submanifold S of $\mathbb{R}^n$ the hypersurface of revolution and call M the *profile manifold*. By the meridians of S we mean the submanifolds of S given by the images of $R(\boldsymbol{x},\theta)$ for constant θ-values. We denote the meridians of S by M_θ, where $\theta\in\mathbb{R}$.

Let $\gamma : (a, b) \to N$ be a smooth curve so that

$$\gamma(t) = \big\langle x_1(t), \ldots, x_{n-2}(t), x_{n-1}(t)\cos\theta(t), x_{n-1}(t)\sin\theta(t)\big\rangle.$$

To be a loxodrome on S, we require the curve γ to have the property that its relative position to the meridians stays constant. We need to be able to compare the relative position of $\gamma(t)$ to $M_{\theta(t)}$ for different values of t. To do this, one uses the isomorphism $M_\theta \to M$ to bring the position and velocity vectors along γ back on M. One obtains the curve $\eta(t) = R(\gamma(t), -\theta(t)) = \langle \boldsymbol{x}(t), 0\rangle$ on M and the vector field

$$V(t) = R(\gamma'(t), -\theta(t)) = \langle \boldsymbol{x}'(t), x_{n-1}(t)\theta'(t)\rangle \tag{4-3}$$

along η. Therefore, to compare the relative positions of $\gamma(t)$ to $M_{\theta(t)}$ at different values of t, we instead compare $V(t)$ along $\eta(t)$ on M at different t-values. This requires a way of comparing the geometry of M at different points along the curve $\eta(t)$, which is exactly what parallel transport along η can do. Let ∇ denote the connection on M induced by the Euclidean metric on $\mathbb{R}^n$. We define a loxodrome $\gamma(t)$, where $t \in (a, b)$, by the equation

$$\nabla_{\eta'(t)} V(t) = 0 \quad \text{for all } t \in (a, b).$$

From (4-3), we have $V(t) = \eta'(t) + x_{n-1}(t)\psi'(t)\vec{N}$, where $\vec{N} = \langle 0, \ldots, 0, 1\rangle$, the unit normal vector to $\mathbb{R}^{n-1} \times \{0\}$. It follows that

$$\begin{aligned} 0 = \nabla_{\eta'(t)} V(t) &= \nabla_{\eta'(t)}\big(\eta'(t) + x_{n-1}(t)\psi'(t)\vec{N}\big) \\ &= \nabla_{\eta'(t)}\eta'(t) + \frac{d}{dt}(x_{n-1}(t)\psi'(t))\vec{N} + x_{n-1}\psi'(t)\nabla_{\eta'(t)}\vec{N} \\ &= \nabla_{\eta'(t)}\eta'(t) + \frac{d}{dt}(x_{n-1}(t)\psi'(t))\vec{N}, \end{aligned}$$

which is equivalent to the pair of equations

$$\begin{cases} \nabla_{\eta'(t)}\eta'(t) = 0, \\ \psi'(t)x_{n-1}(t) = c, \end{cases} \tag{4-4}$$

where c is a constant. The first equation in the coupled system (4-4) is the geodesic equation, and the second equation gives the angle of rotation along the geodesic η. In other words, each loxodrome on S is obtained by rotating a geodesic of M by the angle $\psi(t) = k\int dt/x_{n-1}(t)$, where k is a constant and $x_{n-1}(t)$ is the $(n-1)$-th component of the geodesic. Note that our definition of loxodrome is consistent with the definition of loxodrome on surfaces, since on a surface $\eta(t) = \gamma(t)$ and the second equation in (4-4) is the same as (3-2).

Example 4. The $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ is a hypersurface of revolution with profile manifold $\mathbb{S}^{n-2}$. To obtain the parametric equations of an arbitrary loxodrome on $\mathbb{S}^{n-1}$, we first need to find the parametric equations of an arbitrary geodesic on $\mathbb{S}^{n-2}$. Geodesics on $\mathbb{S}^{n-2}$ are the great circles. Each great circle is the intersection of the sphere with a two-dimensional plane $\mathcal{P}$ that passes through the origin. Choose an orthonormal basis $\{\boldsymbol{u}, \boldsymbol{v}\}$ in $\mathcal{P}$. Then the great circle $\mathbb{S}^2 \cap \mathcal{P}$ can be parametrized as

$$\gamma(\theta) = \boldsymbol{u}\cos\theta + \boldsymbol{v}\sin\theta, \quad 0 \le \theta \le 2\pi.$$

From (4-4), we must have

$$\psi'(\theta) = \frac{c}{u_{n-1}\cos\theta + v_{n-1}\sin\theta} = A \cdot \sec(\theta + \theta_0),$$

where A and θ_0 are constants that depend on u_{n-1}, v_{n-1}. It follows that

$$\psi(\theta) = A \ln|\sec(\theta+\theta_0) + \tan(\theta+\theta_0)| + B,$$

and consequently, the general equation of a loxodrome on $\mathbb{S}^3$ is given by

$$\begin{aligned} x_i(\theta) &= u_i\cos\theta + v_i\sin\theta, \quad 1 \le i \le n-2, \\ x_{n-1}(\theta) &= (u_{n-1}\cos\theta + v_{n-1}\sin\theta)\cos\big(A\ln|\sec(\theta+\theta_0) + \tan(\theta+\theta_0)| + B\big), \\ x_n(\theta) &= (u_{n-1}\cos\theta + v_{n-1}\sin\theta)\sin\big(A\ln|\sec(\theta+\theta_0) + \tan(\theta+\theta_0)| + B\big). \end{aligned}$$

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jm20blac@siena.edu *School of Science, Siena College, 515 Loudon Road, Loudonville, NY 12211, United States*

am06duke@siena.edu *School of Science, Siena College, 515 Loudon Road, Loudonville, NY 12211, United States*

mjavaheri@siena.edu *School of Science, Siena College, 515 Loudon Road, Loudonville, NY 12211, United States*

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