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Equivalence classes of $\mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C})$ orbits in the flag variety of $\mathfrak{gl}(p+q,\mathbb{C})$

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We consider the pair of complex Lie groups

$$(G,K)=\big(\mathrm{GL}(p+q,\mathbb{C}),\mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C})\big)$$

and the finite set $\{\mathcal{Q} : K\text{-orbits on the flag variety } \mathfrak{B}\}$. The moment map μ of the G-action on the cotangent bundle $T^*\mathfrak{B}$ maps each conormal bundle closure $\overline{T^*_{\mathcal{Q}}\mathfrak{B}}$ onto the closure of a single nilpotent K-orbit, $\mathcal{O}_K$. We use combinatorial techniques to describe $\mu^{-1}(\mathcal{O}_K)=\{\mathcal{Q}\in\mathfrak{B} : \mu(T^*_{\mathcal{Q}}\mathfrak{B})=\mathcal{O}_K\}$.

Introduction

We consider the pair (G,K) of complex groups equal to

$$\big(\mathrm{GL}(p+q,\mathbb{C}),\mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C})\big).$$

Such a pair comes from the real Lie group $U(p,q)$, and K is the complexification of the maximal compact subgroup $K_{\mathbb{R}}=U(p)\times U(q)$. We denote by $\mathfrak{g}$ the Lie algebra of G. The group K acts with finitely many orbits both on $\mathcal{N}$, the nilpotent cone of $\mathfrak{g}$, and on $\mathfrak{B}$, the flag variety of $\mathfrak{g}$. The points in the cotangent bundle $T^*\mathfrak{B}$ can be thought of as pairs $(\mathfrak{b},\xi)$ consisting of a Borel subalgebra $\mathfrak{b}=\mathfrak{h}\oplus\mathfrak{n}$ and a covector $\xi\in\mathfrak{n}^*$. The projection $\mu:(\mathfrak{b},\xi)\to\xi$ from the cotangent bundle $T^*\mathfrak{B}$ to $\mathcal{N}$ is the moment map for the G-action on $T^*\mathfrak{B}$. If $\mathcal{Q}$ is a K-orbit on $\mathfrak{B}$, the image $\mu(\overline{T^*_{\mathcal{Q}}\mathfrak{B}})$ lies in $\mathcal{N}$ and it is the closure of a nilpotent K-orbit. We write $\mathcal{O}_K$ for the nilpotent K-orbit. We give a combinatorial algorithmic description, amenable to computer computations, of the set

$$\mu^{-1}(\mathcal{O}_K)=\{\mathcal{Q}\in\mathfrak{B} : \mu(T^*_{\mathcal{Q}}\mathfrak{B})=\mathcal{O}_K\}. \tag{0.1}$$

This is the content of Theorem 4.3. Our approach relies heavily on work by Devra Garfinkle [1993], and on work by Peter Trapa [1999]. Our goal is to keep the presentation accessible to an advanced undergraduate student. Some of our

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arguments can be simplified by using advanced results in representation theory, but we choose instead a combinatorial approach.

We use the combinatorial notion of a *clan* to parametrize K-orbits in $\mathfrak{B}$, as in [Matsuki and Oshima 1990]. For each nilpotent orbit, $\mathcal{O}_K$, we identify a *distinguished clan* $\boldsymbol{c}_{\text{dis}} \in \mu^{-1}(\mathcal{O}_K)$. All other clans in $\mu^{-1}(\mathcal{O}_K)$ are obtained from the distinguished clan in a combinatorial manner. Following [Garfinkle 1993], we attach to each clan $\boldsymbol{c}$ a pair of equally shaped tableaux, one signed and the other numbered. It is known, see [Trapa 1999], that the signed tableau determines $\mu(T^*_{\mathcal{Q}_{\boldsymbol{c}}}\mathfrak{B}) = \mathcal{O}_K$, where $\mathcal{Q}_{\boldsymbol{c}}$ is the K-orbit parametrized by $\boldsymbol{c}$. The resulting map

$$E : \{\text{clans}\} \to \{(T_{\pm}, ST_{\boldsymbol{c}})\}$$

is a bijection. Thus, if we fix $\mathcal{O}_K$ and we let $T_{\pm}^{\text{dis}}$ be the signed tableau that corresponds to $\boldsymbol{c}_{\text{dis}} \in \mu^{-1}(\mathcal{O}_K)$ under E, we have

$$\mu^{-1}(\mathcal{O}_K) = \{\mathcal{Q}_{\boldsymbol{c}} \text{ clans} : E(\boldsymbol{c}) = (T_{\pm}^{\text{dis}}, ST_{\boldsymbol{c}})\}.$$

That is, $\mu^{-1}(\mathcal{O}_K)$ is the set of K-obits on $\mathfrak{B}$ parametrized by clans $\boldsymbol{c}$ having $T_{\pm}^{\text{dis}}$ as the signed tableau in $E(\boldsymbol{c})$. In order to explicitly describe the set $\mu^{-1}(\mathcal{O}_K)$, we use combinatorially defined operators $T_{i,j}$ acting both on clans and on numbered tableaux. The bijection E is compatible with the action of such operators. We conclude that if $\boldsymbol{c} \in \mu^{-1}(\mathcal{O}_K)$, then so is $T_{i,j}\,\boldsymbol{c}$. We argue that any clan in $\mu^{-1}(\mathcal{O}_K)$ can be obtained from the distinguished clan by applying an appropriate sequence of operators $T_{i,j}$. This is the content of Theorem 4.3. If $n = p+q$, and the shape of the tableau is fixed, then the action of operators $T_{i,j}$ on numbered tableaux of that given shape determines $\mu^{-1}(O_{\mathrm{GL}(r,\mathbb{C})\times\mathrm{GL}(s,\mathbb{C})})$ for any (r, s) with $r + s = n$. This implies that the algorithm is in a sense independent of the real form; see Theorem 4.5. When nilpotent K-orbits are parametrized by two-column signed tableaux, we give explicit effective sequences of operators $T_{i,j}$ to generate $\mu^{-1}(\mathcal{O}_K)$. We use this result to describe the clans in $\mu^{-1}(\mathcal{O}_K)$ in special cases. The two column case is discussed in Section 5.

The problem of describing $\mu^{-1}(\mathcal{O}_K)$ when $K = \mathrm{GL}(p, \mathbb{C})\times\mathrm{GL}(q, \mathbb{C})$, considered in this paper, is a particular instance (and an easy one) of a more general question posted by David Vogan.

The paper is organized as follows. We fix notation, and we introduce combinatorial parametrizations of nilpotent orbits and K-orbits in $\mathfrak{B}$ in Section 1. In Section 2, we summarize Garfinkle's algorithm, we describe some of its properties, and we introduce the notion of distinguished clan. We include in Section 3 the definition of operators $T_{i,j}$ at both the tableau and clan level, and we explain some of their properties. We obtain an algorithmic description of $\mu^{-1}(\mathcal{O}_K)$ and prove our main theorem in Section 4. In Section 5, we restrict our attention to nilpotent K-orbits parametrized by two-column signed tableaux and give a detailed description of $\mu^{-1}(\mathcal{O}_K)$ in special cases.

1. Preliminaries

The real form $U(p, q)$. In this section we carefully define the real form of interest. Assume p and q are positive integers with $p \geq q$. Write $n = p + q$, and let

$$I_{p,q} = \begin{pmatrix} I_{p\times p} & 0 \\ 0 & -I_{q\times q} \end{pmatrix},$$

where $I_{p\times p}$, $I_{q\times q}$ are identity matrices. Define

$$G_{\mathbb{R}} = U(p, q) = \{g \in \mathrm{GL}(n, \mathbb{C}) : \bar{g}^T I_{p,q} g = I_{p,q}\}.$$

The map Θ given by

$$\begin{aligned} \Theta : \mathrm{GL}(n, \mathbb{C}) &\to \mathrm{GL}(n, \mathbb{C}), \\ A &\mapsto I_{p,q} A I_{p,q}, \end{aligned}$$

is an involution. We call Θ the Cartan involution. Then,

$$\begin{aligned} \mathrm{GL}(n, \mathbb{C})^{\Theta} &= \big\{A \in \mathrm{GL}(n, \mathbb{C}) : \Theta(A) = A\big\} = K \\ &= \left\{\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} : Z_1 \in \mathrm{GL}(p, \mathbb{C}),\ Z_2 \in \mathrm{GL}(q, \mathbb{C})\right\}. \end{aligned}$$

Similarly, we have

$$U(p, q)^{\Theta} = U(p) \times U(q) = K_{\mathbb{R}}.$$

The differential of Θ, denoted by θ, is an involution at the Lie-algebra level. That is $\theta : \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})$ has $\theta^2 = 1$. The $\pm$-eigenspace decomposition of $\mathfrak{gl}(n, \mathbb{C})$ is

$$\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p},$$

where

$$\begin{aligned} \mathfrak{k} &= \left\{\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} : z_1 \in \mathfrak{gl}(p, \mathbb{C}),\ z_2 \in \mathfrak{gl}(q, \mathbb{C})\right\}, \\ \mathfrak{p} &= \left\{\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} : A \in M(p \times q),\ B \in M(q \times p)\right\}. \end{aligned}$$

Define $\mathfrak{h} \subset \mathfrak{k}$ as the Cartan subalgebra consisting of diagonal matrices of the form $\mathrm{diag}(t_1, t_2, \ldots, t_{p+q})$. This is a maximally abelian subalgebra of $\mathfrak{g}$. The matrices $E_{i,j}$ with all entries zero but for a 1 in the intersection of the i-th row, j-th column satisfy

$$[\mathrm{diag}(t_1, t_2, \ldots, t_{p+q}), E_{i,j}] = (t_i - t_j)\, E_{i,j}.$$

In other words, the $E_{i,j}$ are common eigenvectors of the matrices in $\mathfrak{h}$. They are called root vectors. Their eigenvalues $\epsilon_i - \epsilon_j$, given by

$$(\epsilon_i - \epsilon_j)(\mathrm{diag}(t_1, t_2, \dots, t_{p+q})) = t_i - t_j,$$

are called roots. A root $\epsilon_i - \epsilon_j$ is said to be positive if $i < j$. We set

$$\mathfrak{n} = \bigoplus_{i<j} \mathbb{C} E_{i,j}, \qquad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}, \quad \text{upper triangular matrices.} \tag{1.1}$$

The subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra.

K-orbits on the flag variety of G. The flag variety of G is the variety of Borel subalgebras of $\mathfrak{g}$. We describe this variety geometrically as follows.

Definition 1.2. A flag of G is a sequence of $n+1$ complex vector spaces, $\mathcal{F} = (V_0, V_1, \dots, V_n)$, satisfying the conditions

(1) $\dim V_i = i$;

(2) $\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$.

We define $\mathfrak{B} = \{\text{flags in } \mathbb{C}^n\}$.

The group G acts on $\mathfrak{B}$ via

$$g \cdot \mathcal{F} = (g \cdot V_0, g \cdot V_1, \dots, g \cdot V_n).$$

Let $\{e_1, \dots, e_n\}$ denote the standard basis of $\mathbb{C}^n$, and for each integer $1 \le i \le n$, set $V_i^0 = \langle e_1, \dots, e_i \rangle$. Define $\mathcal{F}_0 = (\{0\}, V_1^0, \dots, V_n^0)$. It is not difficult to see that for any flag, $\mathcal{F}$, there exists a $g \in G$ so that $\mathcal{F} = g \cdot \mathcal{F}_0$. This implies that the action of G on $\mathfrak{B}$ is transitive.

Theorem 1.3. *G acts transitively on $\mathfrak{B}$.*

If $\mathcal{F}_0 = \big(\{0\}, \langle e_1 \rangle, \langle e_1, e_2 \rangle, \dots, \langle e_1, \dots, e_{n-1} \rangle, \mathbb{C}^n\big)$, then $G \cdot \mathcal{F}_0 \cong \mathfrak{B} \cong G/B$, where

$$B = \mathrm{Stab}_G(\mathcal{F}_0) = \begin{pmatrix} e_{11} & e_{12} & \cdots & \cdots & e_{1n} \\ 0 & e_{22} & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & e_{nn} \end{pmatrix}.$$

The following known theorem will play an important role in our work.

Theorem 1.4. *K acts on $\mathfrak{B}$ with finitely many orbits.*

Clan parametrization of K-orbits on the flag variety of G. It will be useful to parametrize K-orbits in $\mathfrak{B}$ in a combinatorial manner. To this end, we use the notion of clans. Clans have been introduced in [Matsuki and Oshima 1990]. We follow the presentation in [Yamamoto 1997].

Definition 1.5. An n-indication is a sequence of symbols $(c_1 \cdots c_n)$ so that

(1) c_i is $+$, $-$, or a natural number;

(2) if $c_i = a \in \mathbb{N}$, then there exists a unique c_j with $c_i = c_j = a$;

(3) $\#\{i : c_i = +\} + \#\{\text{pairs of equal numbers}\} = p$.

We define an equivalence relation between two indications. Two indications $(c_1 \cdots c_n)$ and $(c'_1 \cdots c'_n)$ are equivalent if and only if there exists a permutation σ so that

$$c_i = \begin{cases} \sigma(c'_i) & \text{if } c'_i \in \mathbb{N}, \\ + & \text{if } c'_i = +, \\ - & \text{if } c'_i = -. \end{cases}$$

A *clan* is an equivalence class of indications with respect to the equivalence relation.

Define $V_+ = \langle e_1, \dots, e_p \rangle$ and $V_- = \langle e_{p+1}, \dots, e_{p+q} \rangle$.

Proposition 1.6 [Yamamoto 1997, Proposition 2.2.7]. *Let $p+q=n$. Given a flag $\mathcal{F} = (V_0, V_1, \dots, V_n)$ there exists a clan $\mathbf{c} = (c_1 \cdots c_n)$ so that*

(1) $\dim V_i \cap V_+ = \#\{l : c_l = + \text{ for } l \le i\} + \#\{a \in \mathbb{N} : c_s = c_t = a \text{ for } s < t \le i\}$;

(2) $\dim V_i \cap V_- = \#\{l : c_l = - \text{ for } l \le i\} + \#\{a \in \mathbb{N} : c_s = c_t = a \text{ for } s < t \le i\}$;

(3) $\dim V_i - \dim V_i \cap V_+ - \dim V_i \cap V_- = \#\{a \in \mathbb{N} : c_s = c_t = a \text{ for } s \le i < t\}$;

(4) $\dim V_j + \pi_+(V_i) = j + \#\{a \in \mathbb{N} : c_s = c_t = a \text{ for } s \le i < j < t\}$.

Moreover, the set of flags that corresponds to a given clan $\mathbf{c}$, *constitutes a K-orbit in* $\mathfrak{B}$.

The converse of the proposition also holds. Hence, we have the following theorem.

Theorem 1.7 [Yamamoto 1997]. *Clans parametrize K-orbits in $\mathfrak{B}$.*

Example. Assume $G_{\mathbb{R}} = U(2,2)$.

- The clan $(+ + - -)$ corresponds to the flag

$$\mathcal{F}_0 = \big(\{0\} \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset \mathbb{C}^4\big).$$

- The clan $(1\,2\,2\,1)$ corresponds to the flag

$$\mathcal{F} = \big(\{0\} \subset \langle e_1 \rangle \subset \langle e_1, e_2 + e_3 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset \mathbb{C}^4\big).$$

Example. Assume $G_{\mathbb{R}} = U(4,4)$. We attach a flag $\mathcal{F}_{\boldsymbol{c}}$, satisfying (1) through (4) of Proposition 1.6, to the clan $\boldsymbol{c} = (1\,2 + 3\,1 - 2\,3) = (c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7\ c_8)$. Write $\mathcal{F} = (V_0 = \{0\}, V_1, V_2, \ldots, \mathbb{C}^8)$. As $c_1 = c_5 = 1$, we set $V_1 = \langle e_1 + e_5 \rangle$. Note that

$$\begin{cases} \dim V_1 \cap V_+ = 0, \\ \dim V_1 \cap V_- = 0. \end{cases}$$

Similarly, we note that $c_2 = c_7 = 2$ and define $V_2 = \langle e_1 + e_5, e_2 + e_7 \rangle$. Next, as $c_3 = +$, we set $V_3 = \langle e_1 + e_5, e_2 + e_7, e_3 \rangle$. It is easy to check, as $c_1 = c_5$ and $c_2 = c_7$, that $\dim V_3 \cap V_+ = 1$, $\dim V_3 \cap V_- = 0$, and $\dim V_3 - \dim V_3 \cap V_+ - \dim V_3 \cap V_- = 2$.

Continuing in similar manner we get

$$\begin{aligned} \mathcal{F}_{\boldsymbol{c}} = \big(\langle e_1 + e_5 \rangle \subset \langle e_1 + e_5, e_2 + e_7 \rangle \subset \langle e_1 + e_5, e_2 + e_7, e_3 \rangle \\ \subset \langle V_3, e_4 + e_8 \rangle \subset \langle V_4, e_1 - e_5 \rangle \subset \langle V_5, e_6 \rangle \subset \langle V_6, e_2 - e_7 \rangle \subset \mathbb{C}^8\big). \end{aligned}$$

Example. Assume $G_{\mathbb{R}} = U(3,2)$. The flag

$$\big(\{0\} \subset \langle e_1 \rangle \subset \langle e_1, e_2 + e_4 \rangle \subset \langle e_1, e_2 + e_4, e_3 \rangle \subset \langle e_1, e_3, e_4, e_5 \rangle \subset \mathbb{C}^5\big)$$

is parametrized by $(+\,1 + 1\,-)$.

Young diagrams. We introduce some combinatorial tools used in our work.

Definition 1.8. A partition of n is a tuple $[d_1, d_2, \ldots, d_k]$ of positive integers with

(1) $d_1 \geq d_2 \geq \cdots \geq d_k > 0$, and

(2) $\sum d_k = n$.

Given a partition $[d_1, d_2, \ldots, d_k]$, we form a left-justified array of n rows of empty boxes so that the i-th row has length d_i. This is called a Young diagram.

Definition 1.9. A signed tableau is a labeled Young diagram in which boxes are labeled by $+$ and $-$ signs in such a way that the signs alternate along rows. Two signed tableaux are regarded as equal if and only if one can be obtained from the other by interchanging rows of equal length.

Definition 1.10. The signature of a signed tableau is a pair of numbers (i, j), where $i = \#\{+ \text{ signs in the tableau}\}$ and $j = \#\{- \text{ signs in the tableau}\}$.

Definition 1.11. A standard tableau is a labeled Young diagram in which boxes are labeled by numbers that monotonically increase along rows (from left to right) and increase strictly along columns (from top to bottom). We write $b_{i,j}$ for the box in the intersection of the i-th row and j-th column.

Nilpotent G and K-orbits. We think of a nilpotent matrix $X_{n\times n}$ as a linear transformation

$$T_X : \mathbb{C}^n \to \mathbb{C}^n \text{ such that } T^k = 0 \text{ for some } k.$$

Linear algebra tells us that we can write

$$\mathbb{C}^n = V_{p_1} \oplus V_{p_2} \oplus \cdots \oplus V_{p_r}$$

as a sum of vector subspaces with the following properties:

- $T_X : V_{p_i} \to V_{p_i}$.
- Each V_{p_i} admits a basis such that

$$e^i_{p_i} \xrightarrow{T_X} e^i_{p_i-1} \xrightarrow{T_X} \cdots \xrightarrow{T_X} e^i_1 \xrightarrow{T_X} 0.$$

In this basis T_X is represented by its Jordan form J. Moreover, if $Y = g^{-1}Xg$ for some $g \in G$, then the matrix of T_Y with respect to the basis $\{g^{-1}e^i\}$ is also J. We conclude that G acts on the set of nilpotent matrices by conjugation and that this action yields a finite number of orbits.

The Jordan decomposition theorem implies that we can attach to each nilpotent G-orbit, $G \cdot X$, a Young diagram which is completely determined by the Jordan form of X. Indeed, the lengths of the rows of the corresponding Young diagram are given by the size of the Jordan blocks. The following known proposition states that the map from nilpotent G-orbits to Young diagrams is a bijection.

Proposition 1.12 [Collingwood and McGovern 1993]. *There is a one-to-one correspondence between the set of nilpotent orbits and the set of partitions of n. The correspondence sends a nilpotent element X to the partition determined by the block-size of its Jordan form. The orbit* 0 *corresponds to the partition* $[1, 1, \dots, 1]$.

The group K acts by conjugation of the set $\mathcal{N} \cap \mathfrak{p}$ of nilpotent matrices of the form

$$X = \begin{pmatrix} 0 & A_{p\times q} \\ B_{q\times p} & 0 \end{pmatrix}.$$

If we write

$$\mathbb{C}^n = V^+ \oplus V^-, \quad \text{where } V^+ = \langle e_1, \dots, e_p\rangle,\ V^- = \langle e_{p+1}, \dots, e_{p+q}\rangle,$$

then

$$\begin{aligned} X &: V^+ \to V^-, \\ X &: V^- \to V^+. \end{aligned} \tag{1.13}$$

A generalized version of the Jordan decomposition theorem, combined with (1.13), yields a parametrization of K-orbits on $\mathcal{N} \cap \mathfrak{p}$ via Young diagrams with boxes labeled by alternating signs, $+$ and $-$. Our next proposition is well-known and follows from the above discussion.

Proposition 1.14. *There is a one-to-one correspondence between K-orbits in $\mathcal{N}\cap\mathfrak{p}$ and signed tableaux.*

We fix $p \geq q$ with $p+q=n$ and a partition $\lambda = [r_1, r_2, \ldots, r_\ell]$ of n. Such a partition determines a Young diagram of size n. Let $[p_1, p_2, \ldots, p_r]$ be the length of the columns of the Young diagram determined by λ.

Proposition 1.15. *Fix $p \geq q$ with $p+q=n$, and fix $[p_1, p_2, \ldots, p_r]$ integers with $\sum p_i = n$. There is a bijection*

$$\left\{ \begin{array}{l} \text{nilpotent } K\text{-orbits } \mathcal{O}_K \text{ parametrized by} \\ \text{tableaux of column lengths } [p_1, \ldots, p_r] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} (t_1, \ldots, t_s) \text{ integers, } s \leq p_1, \\ t_1 < t_2 < \cdots < t_s \end{array} \right\}.$$

Proof. Assume $\mathcal{O}_K$ is a nilpotent K-orbit parametrized by a signed tableau of shape λ. Note that such a signed tableau is completely determined by its shape and the position of the $-$ signs on the first column of the tableau. The proposition follows by letting $t_1 < t_2 < \cdots < t_s$ denote the positions of the $-$ signs in the first column of the parametrizing tableau. $\square$

2. Garfinkle's algorithm

In this section we describe the algorithm defined in [Garfinkle 1993]. The algorithm assigns to each clan a pair of equally shaped tableaux; one signed, the other numbered. The resulting map has significant representational theoretical meaning. The relevance of the algorithm in our work is explained in the introduction.

Garfinkle's algorithm. Starting with a clan $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)$ form a sequence of pairs

$$\begin{aligned} (i, \epsilon_i) &\quad \text{if } c_i = \epsilon_i, \\ (i, j) &\quad \text{if } c_i = c_j. \end{aligned}$$

Arrange the pairs in order by the largest entry, with the convention that a sign has numerical size 0. Write $\pi_1, \ldots, \pi_r$ for the resulting ordered sequence. Suppose that a smaller, equally shaped pair of tableaux $(T_\pm, ST)$ has been constructed from $\pi_1, \ldots, \pi_{j-1}$. If $\pi_j = (k, \epsilon_k)$, then first add the sign ϵ_k to the topmost row of (a signed tableau in the equivalence class of) $T_\pm$ so that the resulting tableau has signs alternating across rows. Then add the integer k to ST in the unique position so that the two new tableaux have the same shape. If $\pi_j = (k, \ell)$, first add k to ST using the Robinson–Schensted bumping algorithm to get a new tableau ST', and then add a sign ϵ (either $+$ or $-$ as needed) to $T_\pm$ so that the result is a signed tableau $T'_\pm$ of the same shape as ST'. Then add $(\ell, -\epsilon)$ (by the same recipe as the first case) to the first row strictly below the row to which ϵ was added.

Example. Assume $G_\mathbb{R} = U(2, 2)$, and consider the clan $(1 - + 1)$. Attach to $(1 - + 1)$ the sequence $(2, -)(3, +)(1, 4)$.

We associate to $(2,-)(3,+)$ a pair of tableaux, one a signed tableau, the other a standard tableau:

$$\begin{array}{|c|c|}\hline - & + \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}$$

Next, we add $(1,4)$ to obtain

$$\begin{array}{|c|c|}\hline - & + \\ \hline + & \multicolumn{1}{c}{} \\ \cline{1-1} - & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

The algorithm assigns to $(1-+1)$ the signed tableau

$$\begin{array}{|c|c|}\hline - & + \\ \hline + & \multicolumn{1}{c}{} \\ \cline{1-1} - & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Example. Assume $G_{\mathbb{R}} = U(5,4)$, and consider the K-orbit parametrized by the clan $(+1+2332-1)$. Attach to $(+1+2332-1)$ the sequence

$$(1,+)(3,+)(5,6)(4,7)(8,-)(2,9).$$

We associate to $(1,+)(3,+)(5,6)$ a pair of tableaux, one a signed tableau, the other a standard tableau:

$$\begin{array}{|c|c|}\hline + & - \\ \hline + & \multicolumn{1}{c}{} \\ \cline{1-1} + & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Next we add $(4,7)$ to obtain

$$\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline + & \multicolumn{1}{c}{} \\ \cline{1-1} + & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & 5 \\ \hline 6 & \multicolumn{1}{c}{} \\ \cline{1-1} 7 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Our next goal is to include the pair $(8,-)$. This gives

$$\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline + & - \\ \hline + & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & 5 \\ \hline 6 & 8 \\ \hline 7 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

The next step is a little different. When we add the pair (2, 9), we get

$$\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline + & - \\ \hline + & \multicolumn{1}{c}{} \\ \cline{1-1} + & \multicolumn{1}{c}{} \\ \cline{1-1} - & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 8 \\ \hline 6 & \multicolumn{1}{c}{} \\ \cline{1-1} 7 & \multicolumn{1}{c}{} \\ \cline{1-1} 9 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Theorem 2.1 [Trapa 2005; 1999, Theorem 5.6]. (1) *Garfinkle's algorithm defines a bijection between $\{\mathcal{Q} \in K/\mathfrak{B}\}$ and the set of pairs $\{(T_\pm, ST)\}$ consisting of a signed Young tableau and a standard Young tableau of the same shape.*

(2) *If $T_{\pm,\mathcal{Q}}$ is the signed tableau attached via Garfinkle's algorithm to $\mathcal{Q}$, then $T_{\pm,\mathcal{Q}}$ parametrizes $\mu(T^*_{\mathcal{Q}}(\mathfrak{B}))$.*

A distinguished set of K-orbits in $\mathfrak{B}$ that parametrizes nilpotent K-orbits.

Definition 2.2. Fix $p \geq q$ with $p+q=n$, and fix $[p_1, p_2, \ldots, p_r]$ integers with $\sum p_i = n$. Define $\mathcal{S}_{\text{dis}}$ to be the set of clans of length n satisfying the following conditions:

(1) The first p_1 components of the clan (from left to right) are of the form

$$(1 \cdots a_1\, \epsilon_1 \cdots \epsilon_1\, a_1 \cdots 1),$$

where ϵ_1 is either $+$ or $-$.

(2) Components $(c_{\sum_1^{i-1} p_k+1} \cdots c_{\sum_1^i p_k})$ are of the form

$$\left(\sum_1^{i-1} a_k + 1 \cdots \sum_1^{i-1} a_k + a_i \;\; \epsilon_i \cdots \epsilon_i \;\; \sum_1^{i-1} a_k + a_i \cdots \sum_1^{i-1} a_k + 1\right),$$

where ϵ_i is either $+$ or $-$.

(3) $a_1 \geq a_2 \geq \cdots \geq a_r$.

(4) $q = \sum a_i + \sum \delta_{\epsilon_j,-}$ with $\delta_{\epsilon_i,-} = 1$ if $\epsilon_i = -$ and $\delta_{\epsilon_i,-} = 0$ if $\epsilon_i = +$.

An element of $\mathcal{S}_{\text{dis}}$ is called a *distinguished clan*.

Example. The clan $(1\,2+++2\,1\,3\,4-4\,3\,5\,5)$ is a distinguished clan. Observe that $p_1 = 7$, $p_2 = 5$, $p_3 = 2$; $a_1 = a_2 = 2$, $a_3 = 1$, and $q = 6$. The clan $(1\,2\,3\,4\,4\,3\,2\,1)$ is distinguished.

Proposition 2.3. *Fix $p \geq q > 0$ integers so that $p+q=n$. Let $[p_1 \cdots p_r]$ be a sequence of positive integers with $\sum_i p_i = n$. Denote by $\mathcal{O}^{[p_1 \cdots p_r]}$ the nilpotent*

G-orbit parametrized by a tableau with column lengths $p_1,\dots,p_r$. There is a bijection

$$\left\{\begin{matrix}\text{nilpotent } K\text{-orbits } \mathcal{O}_K \text{ such that}\\ G\cdot\mathcal{O}_K=\mathcal{O}^{[p_1\cdots p_r]}\end{matrix}\right\}\longleftrightarrow \mathcal{S}_{\mathrm{dis}}^{[p_1\cdots p_r]}.$$

Proof. Let $\mathcal{O}_K$ be a nilpotent K-orbit. Assume the signed tableau that parametrizes $\mathcal{O}_K$ has columns of lengths $p_1, p_2, \dots, p_r$. By Proposition 1.15, $\mathcal{O}_K$ is completely determined by the position of $-$ signs in the first column of its corresponding signed tableau $T_\pm$. Counting the numbers of the boxes that contain a $-$ sign from top to bottom, list the position of the $-$ signs in the first column as $(t_1, t_2, \dots, t_s)$. Define

$$\begin{aligned}
\ell_1 &= \#\{-\text{ signs in the first column of } T_\pm\},\\
\ell_2 &= \#\{t_i : t_i \le p_2\},\\
&\;\;\vdots\\
\ell_r &= \#\{t_i : t_i \le p_r\}.
\end{aligned}$$

We assign to the nilpotent K-orbit, $\mathcal{O}_K$, a distinguished K-orbit $\mathcal{Q}\subset\mathfrak{B}$. We describe the clan $\boldsymbol{c}_\mathcal{Q}$ that identifies $\mathcal{Q}$ as follows. Write

$$\boldsymbol{c}_\mathcal{Q}=\left(c_1\cdots c_{p_1}c_{p_1+1}\cdots c_{p_1+p_2}c_{p_1+p_2+1}\cdots c_{\sum p_i}\right).$$

The first p_1 entries of $\boldsymbol{c}_\mathcal{Q}$ are given by

$$(c_1\cdots c_{p_1})=\begin{cases}(1\cdots\ell_1+\cdots+\ell_1\cdots 1) & \text{if } p_1\ge 2\ell_1,\\ (1\cdots(p_1-\ell_1)-\cdots-(p_1-\ell_1)\cdots 1) & \text{if } p_1<2\ell_1.\end{cases}$$

Note that $\ell_1=\frac{1}{2}\#\{c_i\in\mathbb{N}\}+\#\{c_i=-\}$.

The next p_2 entries are

$$\left(c_{p_1+1}\cdots c_{p_1+p_2}\right)=\begin{cases}(a_1\cdots a_{\ell_2}-\cdots-a_{\ell_2}\cdots a_1) & \text{if } p_2\ge 2\ell_2,\\ (a_1\cdots a_{p_2-\ell_2}+\cdots+a_{p_2-\ell_2}\cdots a_1) & \text{if } p_2<2\ell_2,\end{cases}\tag{2.4}$$

where the integers a_i are consecutive and

$$a_1=\begin{cases}\ell_1+1 & \text{if } p_1\ge 2\ell_1,\\ p_1-\ell_1+1 & \text{if } p_1<2\ell_1.\end{cases}$$

Note that $\ell_2=\frac{1}{2}\#\{c_i\in\mathbb{N}: p_1+1\le i\le p_1+p_2\}+\#\{c_i=+: p_1+1\le i\le p_1+p_2\}$. Continuing inductively we define the remaining entries in $\boldsymbol{c}_\mathcal{Q}$.

The above construction assigns to $\mathcal{O}_K$ a unique distinguished $\mathbf{c}_\mathcal{Q}$. It is easy to check that Garfinkle's algorithm attaches to $\boldsymbol{c}_\mathcal{Q}$ a pair of tableaux with the signed tableau parametrizing $\mathcal{O}_K$. By Theorem 2.1 , the orbit $\mathcal{Q}$ is such that $\mu(T^*_\mathcal{Q}\mathfrak{B})=\mathcal{O}_K$. The definition of distinguished clan guarantees that the map from nilpotent orbits to distinguished clans is onto. $\square$

Example. Consider the nilpotent orbit $\mathcal{O}_K$ corresponding to

$$\begin{array}{|c|c|c|c|c|c|c|}\hline + & - & + & - & + & - & + \\ \hline - & + & - & + \\ \cline{1-4} + & - & + \\ \cline{1-3} + & - \\ \cline{1-2}\end{array}$$

We have $p_1 = p_2 = 4$, $p_3 = 3$, $p_4 = 2$, $p_5 = p_6 = p_7 = 1$ and $\ell_i = 1$ for all $1 \le i \le 7$. The construction described in the proof of Proposition 2.3 gives $\boldsymbol{c}_{\mathcal{Q}} = \big(1 + + 1\, 2 - - 2\, 3 + 3\, 4\, 4 + - +\big)$. In particular the K-orbit $\mathcal{Q}$ parametrized by clan $\boldsymbol{c}_{\mathcal{Q}}$ belongs to $\mu^{-1}(\mathcal{O}_K)$.

3. The operators $T_{\alpha,\beta}$

We now describe some combinatorial tools that will play an important role in our work. Indeed, given a nilpotent K-orbit $\mathcal{O}_K$, we have defined a distinguished clan $\boldsymbol{c}_{\text{dis}}$ so that $\boldsymbol{c}_{\text{dis}} \in \mu^{-1}(\mathcal{O}_K)$. We will show in Section 4 that each $\boldsymbol{c} \in \mu^{-1}(\mathcal{O}_K)$ can be obtained from $\boldsymbol{c}_{\text{dis}}$ by applying an appropriate sequence of operators $T_{\cdot,\cdot}$. These operators are defined both at the level of standard tableaux and at the level of clans.

$T_{\alpha,\beta}$ on standard tableaux. We follow [Garfinkle 1993, Chapter 3] and we let T be a standard tableau.

Definition 3.1. We say that a root $\alpha_i = \epsilon_i - \epsilon_{i+1}$ is in the τ-invariant of T if the box in T labeled i lies on a row above that containing the box labeled $i+1$.

Example. The τ-invariant of

$$T = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline 4 & 8 \\ \hline 9 & 11 \\ \hline 10 \\ \cline{1-1}\end{array}$$

is $\tau(T) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9\}$.

Definition 3.2. Given $\alpha = \epsilon_i - \epsilon_{i+1}$ and $\beta = \epsilon_{i+1} - \epsilon_{i+2}$, we say that T is in $D_{\alpha,\beta}$, the domain of $T_{\alpha,\beta}$, if $\alpha \notin \tau(T)$ and $\beta \in \tau(T)$. This is the case when either (a) the row containing label $i+2$ is below the row containing label i, which in turn is equal to or below the row that contains $i+1$ or (b) the row containing label $i+1$ is above the row containing label i, which in turn is equal to the row that contains $i+2$. We define

$$T_{\alpha,\beta} : D_{\alpha,\beta} \to D_{\beta,\alpha},$$
$$T \mapsto T_{\alpha,\beta}(T),$$

by switching the labels $i+1$ and $i+2$ in case (a) and by switching the labels i and $i+1$ in case (b).

Remark 3.3. The above definition is extended to the case $\beta = \alpha_{i-1} = \epsilon_{i-1} - \epsilon_i$ in the obvious manner. We often use the abbreviated notation $T_{i,j}$ for T_{α_i,α_j}.

Example. The operator $T_{4,5}$ maps the tableau

$$T = \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline 4 & 8 \\ \hline 9 & 11 \\ \hline 10 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

to the tableau

$$\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline 5 & 8 \\ \hline 9 & 11 \\ \hline 10 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Theorem 3.4 [Vogan 1979]. *Fix λ a partition of n and denote by S_λ the set of standard tableaux of a fixed shape λ. The operators $T_{\alpha,\beta}$ act transitively on S_λ.*

$T_{\alpha,\beta}$ on clans. In this subsection we introduce the notion of τ-invariant on clans and define operations $T_{\alpha,\beta}$ on clans. These notions are not new. The work of Borho, Jantzen and Duflo established the important invariant of an irreducible representation, its τ-invariant. This is a subset of simple roots defined in terms of wall-crossing. As part of an important study of wall-crossing, [Speh and Vogan 1980] and [Vogan 1979] give formulas for the τ-invariant of a representation and related $T_{\alpha,\beta}$ in terms of $\mathbb{Z}_2$-data (in type A, $\mathbb{Z}_2$-data can be interpreted as clan-data). Our combinatorial description of τ-invariant and $T_{\alpha,\beta}$-operations on clans agrees with the work in [Speh and Vogan 1980].

Definition 3.5. Let $\boldsymbol{c} = (c_1, c_2, \dots, c_n)$ be a clan. We define the τ-invariant of $\boldsymbol{c}$ as

$$\begin{aligned}\{\epsilon_i - \epsilon_{i+1} : \ & (c_i, c_{i+1}) \text{ is a pair of equal signs,}\\ & (c_i, c_{i+1}) \text{ is a pair of equal numbers,}\\ & (c_i, c_{i+1}) = (\pm, a) \text{ so that there is } j < i \text{ with } c_j = a \in \mathbb{N},\\ & (c_i, c_{i+1}) = (a, \pm) \text{ so that there is } j > i+1 \text{ with } c_j = a \in \mathbb{N},\\ & (c_i, c_{i+1}) = (a, b) \text{ so that there are } j < k \text{ with } c_j = b,\, c_k = a \in \mathbb{N}\}.\end{aligned}$$

Remark 3.6. At the Lie-algebra level, each clan determines a Borel subalgebra

$$\mathfrak{b}_{\boldsymbol{c}} = \mathfrak{h}_{\boldsymbol{c}} \oplus \mathfrak{n}_{\boldsymbol{c}} \subset \mathfrak{g}.$$

The parametrization of K-orbits in G/B via clans is arranged to have the following property: there is a unique automorphism of $\mathfrak{g}$ carrying $\mathfrak{b}_{\boldsymbol{c}}$ to the Borel $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ of equation (1.1). Using such an automorphism, one can keep track of the action of θ on $\Delta(\mathfrak{n}_{\boldsymbol{c}})$. In particular if $\alpha \in \Delta(\mathfrak{h}_{\boldsymbol{c}}, \mathfrak{n}_{\boldsymbol{c}})$ corresponds to $\epsilon_i - \epsilon_{i+1}$ via the mentioned automorphism, then $\theta(\alpha)$ corresponds to

$$\begin{cases} \epsilon_i - \epsilon_k & \text{if } c_i \text{ is a sign and } c_{i+1} = c_k \in \mathbb{N}, \\ \epsilon_k - \epsilon_{i+1} & \text{if } c_{i+1} \text{ is a sign and } c_i = c_k \in \mathbb{N}, \\ \epsilon_k - \epsilon_\ell & \text{if } c_i = c_k \in \mathbb{N} \text{ and } c_{i+1} = c_\ell \in \mathbb{N}, \\ \epsilon_i - \epsilon_j & \text{if } c_i, c_j \text{ are signs.} \end{cases}$$

We say that $\alpha \in \Delta(\mathfrak{n}_{\boldsymbol{c}})$ corresponding to $\epsilon_i - \epsilon_{i+1}$ is

$$\begin{cases} \text{imaginary compact} & \text{if } (c_i, c_{i+1}) \text{ is a pair of equal signs,} \\ \text{imaginary noncompact} & \text{if } (c_i, c_{i+1}) \text{ is a pair of distinct signs,} \\ \text{real} & \text{if } (c_i, c_{i+1}) \text{ is a pair of equal numbers,} \\ \text{complex} & \text{otherwise.} \end{cases}$$

We write $\boldsymbol{i}_n$ for imaginary noncompact roots, $\boldsymbol{i}_c$ for imaginary compact roots, and $\boldsymbol{r}$ for real roots. For α, a positive complex root with $\theta(\alpha) > 0$, we write $\mathbb{C}^+$. For α, a positive complex root with $\theta(\alpha) < 0$, we write $\mathbb{C}^-$.

Hence, the τ-invariant of clan $\boldsymbol{c}$ is

$$\tau(\boldsymbol{c}) = \{\text{simple roots } \alpha \in \Delta(\mathfrak{n}_{\boldsymbol{c}}) : \alpha \text{ is } \boldsymbol{i}_c \text{ or } \boldsymbol{r} \text{ or } \mathbb{C}^-\}.$$

In order to define the combinatorial $T_{\alpha,\beta}$-action on clans we introduce a technical definition.

Definition 3.7. Let $\boldsymbol{c}$ be a clan, and write $\mathfrak{b}_{\boldsymbol{c}} = \mathfrak{h}_{\boldsymbol{c}} \oplus \mathfrak{n}_{\boldsymbol{c}}$ for the corresponding Borel subalgebra. Write ϵ for a sign (could be $+$ or $-$). Let $\alpha_i \in \Delta(\mathfrak{n}_{\boldsymbol{c}})$, where α_i corresponds to $\epsilon_i - \epsilon_{i+1}$.

(1) If α_i is imaginary noncompact ($\boldsymbol{i}_n$), we define the Cayley map

$$\mathrm{Cay}_i(c_1 \cdots c_i{=}\epsilon \;\; c_{i+1}{=}-\epsilon \cdots c_n) = (c_1 \cdots c_i{=}1 \;\; c_{i+1}{=}1 \cdots c_n).$$

(2) If α_i is real ($\boldsymbol{r}$), we define the inverse Cayley map

$$\begin{aligned} &\mathrm{Cay}_i^{-1}(c_1 \cdots c_i{=}1 \;\; c_{i+1}{=}1 \cdots c_n) \\ &\qquad = \big\{(c_1 \cdots c_i{=}+ \;\; c_{i+1}{=}- \cdots c_n);\, (c_1 \cdots c_i{=}- \;\; c_{i+1}{=}+ \cdots c_n)\big\}. \end{aligned}$$

(3) If α_i is complex ($\mathbb{C}^+$), the $\theta(\alpha_i)$ corresponds to $\epsilon_j-\epsilon_k$ with $j<k$. We define the cross-action $s_i\times\boldsymbol{c}$ as

$$s_i\times(c_1\cdots c_i{=}\epsilon\ \ c_{i+1}{=}a\cdots a\cdots c_n)=(c_1\cdots c_i{=}a\ \ c_{i+1}{=}\epsilon\cdots a\cdots c_n),$$
$$s_i\times(c_1\cdots a\cdots c_i{=}a\ \ c_{i+1}{=}\epsilon\cdots c_n)=(c_1\cdots a\cdots c_i{=}\epsilon\ \ c_{i+1}{=}a\cdots c_n),$$
$$s_i\times(c_1\cdots c_i{=}a\ \ c_{i+1}{=}b\cdots c_n)=(c_1\cdots c_i{=}b\ \ c_{i+1}{=}a\cdots c_n)$$

for any clan $\boldsymbol{c}$ with the companion of a to the left of the companion of b.

(4) If α_i is complex ($\mathbb{C}^-$), the $\theta(\alpha_i)$ corresponds to $\epsilon_j-\epsilon_k$ with $j>k$. We define the cross-action

$$s_i\times(c_1\cdots a\cdots c_i{=}\epsilon\ \ c_{i+1}{=}a\cdots c_n)=(c_1\cdots a\cdots c_i{=}a\ \ c_{i+1}{=}\epsilon\cdots c_n),$$
$$s_i\times(c_1\cdots c_i{=}a\ \ c_{i+1}{=}\epsilon\cdots a\cdots c_n)=(c_1\cdots c_i{=}\epsilon\ \ c_{i+1}{=}a\cdots c_n),$$
$$s_i\times(c_1\cdots c_i{=}a\ \ c_{i+1}{=}b\cdots c_n)=(c_1\cdots c_i{=}b\ \ c_{i+1}{=}a\cdots c_n)$$

for any clan with the companion of a to the right of the companion of b.

Definition 3.8. Given $\boldsymbol{c}$, a clan, we define $D^{\boldsymbol{c}}_{\alpha,\beta}=\{\text{clans}:\alpha\notin\tau(\boldsymbol{c})\text{ and }\beta\in\tau(\boldsymbol{c})\}$, and we define $T_{\alpha,\beta}:D^{\boldsymbol{c}}_{\alpha,\beta}\to D^{\boldsymbol{c}}_{\beta,\alpha}$ as

$$T_{\alpha,\beta}(\boldsymbol{c})=\begin{cases} s_\alpha\times\boldsymbol{c} & \text{if } \alpha\in\mathbb{C}^+,\ \beta\in\mathbb{C}^- \text{ and } \alpha+\beta\in\{\mathbb{C}^+,\boldsymbol{i}_n\},\\ s_\alpha\times\boldsymbol{c} & \text{if } \alpha\in\mathbb{C}^+,\ \beta\in\boldsymbol{i}_c \text{ and } \alpha+\beta\in\mathbb{C}^+,\\ s_\alpha\times\boldsymbol{c} & \text{if } \alpha\in\mathbb{C}^+,\ \beta\in\boldsymbol{r} \text{ and } \theta(\alpha+\beta)\neq\alpha,\\ s_\beta\times\boldsymbol{c} & \text{if } \alpha\in\mathbb{C}^+,\ \beta\in\mathbb{C}^- \text{ and } \alpha+\beta\in\{\mathbb{C}^-,\boldsymbol{i}_c,\boldsymbol{r}\},\\ s_\beta\times\boldsymbol{c} & \text{if } \alpha\in\boldsymbol{i}_n,\ \beta\in\mathbb{C}^-,\\ \mathrm{Cay}_\alpha\,\boldsymbol{c} & \text{if } \alpha\in\boldsymbol{i}_n,\ \beta\in\boldsymbol{i}_c,\\ \mathrm{Cay}_\beta^{-1}\,\boldsymbol{c}\cap D_{\beta,\alpha} & \text{if } \alpha\in\mathbb{C}^+,\ \beta\in\boldsymbol{r} \text{ and } \theta(\alpha+\beta)=\alpha. \end{cases}$$

Remark 3.9. We verify that $T_{\alpha,\beta}$ in Definition 3.8 is well-defined, i.e., $T_{\alpha,\beta}(\boldsymbol{c})\in D^{\boldsymbol{c}}_{\beta,\alpha}$, by using the formulas given in Definition 3.7 and the definition of τ-invariant of a clan.

Compatibility of $T_{\alpha,\beta}$-actions. We have defined operators $T_{\alpha,\beta}$ both at the level of clans and of standard tableaux. In representation theoretic language these actions correspond to actions on $\mathbb{Z}_2$-data and on primitive ideals. Crucial to our work is the following theorem.

Theorem 3.10 [Garfinkle 1993, Section 4.2]. *Assume $p>q$. Let*

$$E:\{\textit{clans of signature }(p,q)\}\equiv\{\mathfrak{Q}\in K/\mathfrak{B}\}\to\{(T_\pm,ST)\},$$
$$\boldsymbol{c}\mapsto(T^{\boldsymbol{c}}_\pm,ST_{\boldsymbol{c}}),$$

be the bijection between $\{\mathfrak{Q}:K\text{-orbits on }\mathfrak{B}\}$ and pairs of equally shaped tableaux (the first one signed and the second one standard) induced by Garfinkle's algorithm.

Then if $\alpha, \beta \in D_{\alpha,\beta}(\text{clan } \boldsymbol{c})$, *then* $\alpha, \beta \in D_{\alpha,\beta}(ST_{\boldsymbol{c}})$. *Moreover*,

$$E(T_{\alpha,\beta}\boldsymbol{c}) = (T_{\pm}^{\boldsymbol{c}}, T_{\alpha,\beta}(ST_{\boldsymbol{c}})).$$

Remark 3.11. Each clan $\boldsymbol{c}$ determines an orbit $\mathcal{Q} \in \mathfrak{B}$. Via the Beilinson–Bernstein classification, such a $\mathcal{Q}$ determines an irreducible Harish-Chandra module with trivial infinitesimal character, $X(\boldsymbol{c}) = X(\mathcal{Q})$. By [Trapa 2005, Theorem 5.6], $T_{\pm}^{\boldsymbol{c}}$ parametrizes the associated variety of $X(\mathcal{Q})$ (which, under our assumptions, agrees with $\mu(T_{\boldsymbol{c}}^*\mathfrak{B})$. A result by Vogan guarantees that $T_{\alpha,\beta}$ preserves associated variety. Hence it preserves signed tableaux.

4. Characterization of $\mu^{-1}(\mathcal{O}_K)$

In this section we identify K-orbits on $\mathfrak{B}$ with their clan parametrization. Then, we freely write "τ-invariant of $\mathcal{Q}$" meaning the τ-invariant of the associated clan, as given in Section 3. Similarly we write "$T_{\alpha,\beta}$ of an orbit", meaning the corresponding action on clans. Theorem 4.3 gives a combinatorial description of the set $\mu^{-1}(\mathcal{O}_K)$. Theorem 4.5 implies that the combinatoric in Theorem 4.3 is independent of the real form.

Definition 4.1. Given $\boldsymbol{c}, \boldsymbol{c}'$ two clans parametrizing K-orbits $\mathcal{Q}, \mathcal{Q}' \in \mathfrak{B}$, we write $\mathcal{Q} \mapsto \mathcal{Q}'$ if there exist simple adjacent roots α, β with $\alpha \notin \tau(\boldsymbol{c})$, $\beta \in \tau(\boldsymbol{c})$ so that $T_{\alpha,\beta}\boldsymbol{c} = \boldsymbol{c}'$. We say that $\mathcal{Q}$ and $\mathcal{Q}'$ are *τ-linked* if there exists a sequence $(\mathcal{Q}_0, \mathcal{Q}_1, \dots, \mathcal{Q}_r)$ of K-orbits on $\mathfrak{B}$ so that $\mathcal{Q}_0 = \mathcal{Q}$, $\mathcal{Q}_r = \mathcal{Q}'$ and $\mathcal{Q}_0 \mapsto \mathcal{Q}_1 \mapsto \cdots \mapsto \mathcal{Q}_r$.

Lemma 4.2. *The τ-linked relation on the set $K/\mathfrak{B}$ is an equivalence relation.*

Proof. The lemma holds since in type A the operators $T_{\alpha,\beta}$ are injective. □

Theorem 4.3. *Let $\mathcal{O}_K$ be a nilpotent K-orbit. Then, $\mathcal{Q}, \mathcal{Q}' \in \mu^{-1}(\mathcal{O}_K)$ if and only if $\mathcal{Q}$ and $\mathcal{Q}'$ are τ-linked.*

Proof. By Theorem 2.1, two orbits $\mathcal{Q}, \mathcal{Q}'$ belong to $\mu^{-1}(\mathcal{O}_K)$ if and only if $E(T_{\mathcal{Q}}^*\mathfrak{B}) = (T_{\pm}^{\mathcal{Q}}, ST_{\mathcal{Q}})$ and $E(T_{\mathcal{Q}'}^*\mathfrak{B}) = (T_{\pm}^{\mathcal{Q}'}, ST_{\mathcal{Q}'})$ have $T_{\pm}^{\mathcal{Q}} = T_{\pm}^{\mathcal{Q}'}$. On the other hand, by Theorem 3.4 there exists a sequence $\{T_{\alpha_i,\beta_i}\}$ so that $ST_{\mathcal{Q}'} = T_{\alpha_r,\beta_r} \circ \cdots \circ T_{\alpha_1,\beta_1} ST_{\mathcal{Q}}$. Now the theorem follows from Theorem 3.10. □

Definition 4.4. Fix a partition $[r_1, r_2, \dots, r_k]$ of $n = p + q$. Define a τ-graph of standard tableaux of shape $[r_1, r_2, \dots, r_k]$ as follows. The vertices of the graph are the standard tableaux of shape $[r_1, r_2, \dots, r_k]$. Two standard tableaux (T_1, T_2) are linked if there is a pair of adjacent simple roots with (α, β) with $\alpha \notin \tau(T_1)$ $\beta \in \tau(T_1)$ and $T_2 = T_{\alpha,\beta}T_1$.

Theorem 4.5. *Fix a partition $[r_1, r_2, \dots, r_k]$ of n. Let (r, t) be any pair of integers so that $r + t = n$. Let $\mathcal{O}_K$ be a nilpotent $\mathrm{GL}(r, \mathbb{C}) \times \mathrm{GL}(s, \mathbb{C})$-orbit with parametrizing tableau of shape $[r_1, r_2, \dots, r_k]$. Let $\boldsymbol{c}$ be the distinguished clan associated to $\mathcal{O}_K$ as* *in Proposition 2.3. Then, $\mu^{-1}(\mathcal{O}_K)$ is completely determined by $\boldsymbol{c}$ and the τ-graph of standard tableaux of shape $[r_1, r_2, \dots, r_k]$.*

Proof. The distinguished clan $\boldsymbol{c}$ parametrizes an orbit $\mathcal{Q}_0 \in \mu^{-1}(\mathcal{O}_K)$. Garfinkle's algorithm attaches to $\mathcal{Q}_0$ a pair $(T_{\pm}^{\boldsymbol{c}}, ST_{\mathbf{c}})$ of shape $[r_1, r_2, \dots, r_k]$. By Theorem 4.3, $\mathcal{Q} \in \mu^{-1}(\mathcal{O}_K)$ if and only if $\mathcal{Q}$ is τ-linked to $\mathcal{Q}_0$. Since Garfinkle's map commutes with the action of operators $T_{\alpha,\beta}$, we conclude that $\mathcal{Q} \in \mu^{-1}(\mathcal{O}_K)$ if and only if the standard tableau associated to $\mathcal{Q}$ via Garfinkle's map belongs to the τ-graph of $ST_{\boldsymbol{c}}$. □

Remark 4.6. The previous theorems imply that the equivalence relation $\mathcal{Q} \simeq \mathcal{Q}'$ if and only if $\mu(T_{\mathcal{Q}}^*\mathfrak{B}) = \mu(T_{\mathcal{Q}'}^*\mathfrak{B})$ is independent of the real form $U(r,t)$ of $GL(n = r+t, \mathbb{C})$.

Remark 4.7. It is important to note that the sequence of operators $\{T_{\alpha_i,\beta_i}\}$ that link two standard tableaux of the same shape is not unique. Our next example illustrates Theorem 4.5. The example concerns tableaux of shape [2, 2, 2, 1, 1]. We show that each standard tableau T of shape [2, 2, 2, 1, 1] can be obtained from

$$\begin{array}{|c|c|}\hline 1 & 6 \\ \hline 2 & 7 \\ \hline 3 & 8 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

by a sequence of $T_{i,j}$. This sequence is not unique. In Section 5, in the setting of two-column standard tableaux, we give explicit effective sequences of operators $T_{i,j}$ to generate $\mu^{-1}(\mathcal{O}_K)$.

Example. We illustrate Theorem 4.5 in an example. First we draw the τ-graph of tableaux of shape [2, 2, 2, 1, 1]. This is a connected graph. In order to fit the diagram, we have divided the graph into halves, shown in Figures 1 and 2. The tableaux on the first row of Figure 2 are indeed obtained by applying $T_{7,6}$ to appropriate tableaux listed in Figure 1.

Next we consider two different real forms, $U(5,3)$ and $U(4,4)$. We set

$$T_1 = \begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline + & - \\ \hline + \\ \cline{1-1} + \\ \cline{1-1}\end{array}, \quad T_2 = \begin{array}{|c|c|}\hline - & + \\ \hline - & + \\ \hline + & - \\ \hline + \\ \cline{1-1} + \\ \cline{1-1}\end{array}, \quad \text{and} \quad T_3 = \begin{array}{|c|c|}\hline - & + \\ \hline + & - \\ \hline + & - \\ \hline + \\ \cline{1-1} - \\ \cline{1-1}\end{array}.$$

We describe $\mu^{-1}(T_1)$, $\mu^{-1}(T_2)$ and $\mu^{-1}(T_3)$.

We start with the standard tableau

$$ST = \begin{array}{|c|c|} \hline 1 & 6 \\ \hline 2 & 7 \\ \hline 3 & 8 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

and we choose a sequence of operators $T_{\cdot,\cdot}$ that generates all standard tableaux of shape $[2, 2, 2, 1, 1]$. Next, we determine $\boldsymbol{c}^i{}_{\mathrm{dis}} \in \mu^{-1}(T_i)$ for $i = 1, 2, 3$. It is useful to observe that $E(\boldsymbol{c}^i{}_{\mathrm{dis}}) = (T_i, ST)$. We show that the chosen sequence of operators $T_{\cdot,\cdot}$ allows us to describe $\mu^{-1}(T_1)$, $\mu^{-1}(T_2)$ and $\mu^{-1}(T_3)$ simultaneously when applied to $\boldsymbol{c}^i_{\mathrm{dis}}$. The example illustrates Theorem 4.5.

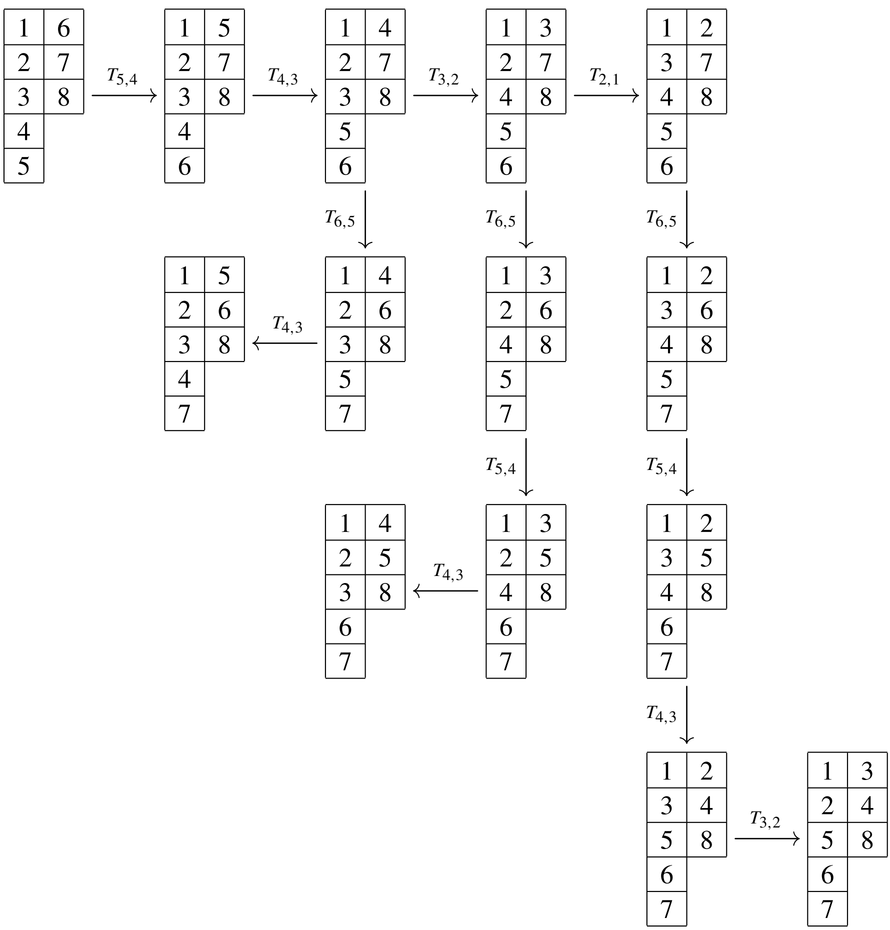

Figure 1

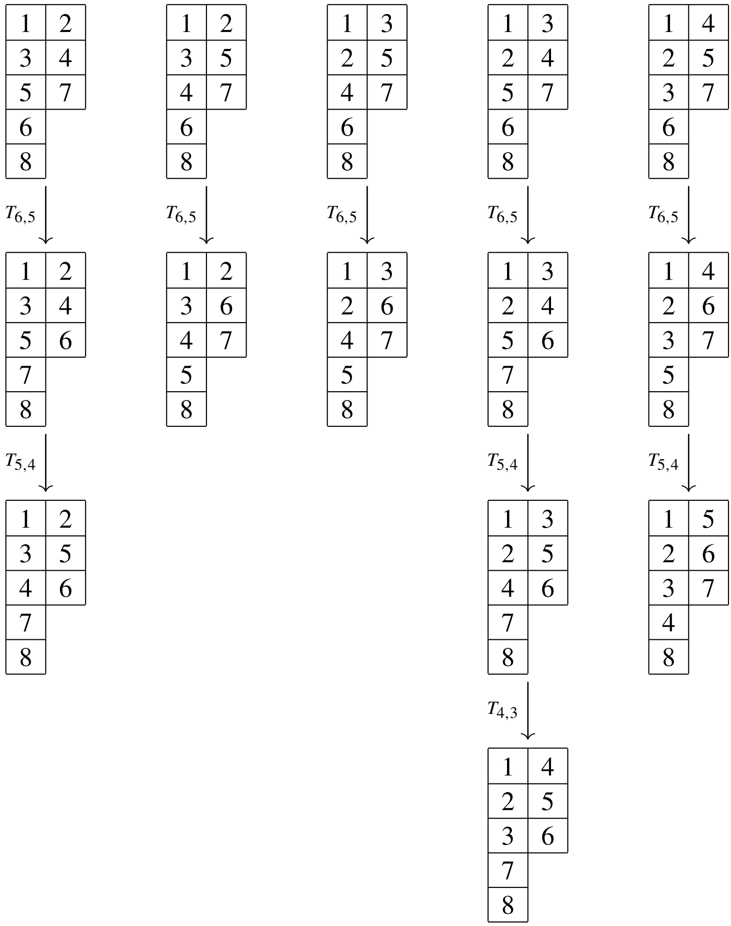

Figure 2

The GL(5, $\mathbb{C}$) × GL(3, $\mathbb{C}$)-orbits in $\mathfrak{B}$ that belong to $\mu^{-1}(T_1)$ are parametrized by the clans

$$\begin{tikzcd}
{+{+}{+}{+}{+}{-}{-}{-}} \arrow[r, "T_{5,4}"] & {+{+}{+}{+}11{-}{-}} \arrow[r, "T_{4,3}"] & {+{+}{+}1{+}1{-}{-}} \arrow[r, "T_{3,2}"] \arrow[d, "T_{6,5}"'] & {+{+}1{+}{+}1{-}{-}} \arrow[r, "T_{2,1}"] \arrow[d, "T_{6,5}"'] & {+1{+}{+}{+}1{-}{-}} \arrow[d, "T_{6,5}"'] \\
 & {+{+}{+}{+}1{-}1{-}} \arrow[r, dotted, leftrightarrow, "T_{4,3}"'] & {+{+}{+}1{+}{-}1{-}} & {+{+}1{+}{+}{-}1{-}} \arrow[d, "T_{5,4}"'] & {+1{+}{+}{+}{-}1{-}} \arrow[d, "T_{5,4}"'] \\
 & & {+{+}{+}1221{-}} & {+{+}1{+}221{-}} \arrow[l, "T_{4,3}"] & {+1{+}{+}221{-}} \arrow[d, "T_{4,3}"'] \\
 & & & & {+1{+}2{+}21{-}} \arrow[d, "T_{3,2}"'] \\
 & & & & {+{+}12{+}21{-}}
\end{tikzcd}$$

$$
\begin{tikzcd}
+1+2+2-1 \arrow[d, "T_{6,5}"'] & +1++22-1 \arrow[d, "T_{6,5}"'] & ++1+22-1 \arrow[d, "T_{6,5}"'] & ++12+2-1 \arrow[d, "T_{6,5}"'] & +++122-1 \arrow[d, "T_{6,5}"'] \\
+1+2+-21 \arrow[d, "T_{5,4}"'] & +1+++--1 & ++1++--1 & ++12+-21 \arrow[d, "T_{5,4}"'] & +++1+--1 \arrow[d, "T_{5,4}"'] \\
+1++2-21 & & & ++1+2-21 \arrow[d, "T_{4,3}"'] & ++++1--1 \\
& & & +++12-21 &
\end{tikzcd}
$$

The $\mathrm{GL}(5,\mathbb{C}) \times \mathrm{GL}(3,\mathbb{C})$-orbits in $\mathfrak{B}$ that belong to $\mu^{-1}(T_2)$ are parametrized by the clans

$$
\begin{tikzcd}
12+213+3 \arrow[r, "T_{5,4}"] & 12+231+3 \arrow[r, "T_{4,3}"] & 12+321+3 \arrow[r, "T_{3,2}"] \arrow[d, "T_{6,5}"'] & 123+21+3 \arrow[r, "T_{2,1}"] \arrow[d, "T_{6,5}"'] & 132+21+3 \arrow[d, "T_{6,5}"'] \\
& 12+23+13 \arrow[r, dotted, leftrightarrow, "T_{4,3}"'] & 12+32+13 \arrow[d, "T_{5,4}"'] & 123+2+13 \arrow[d, "T_{5,4}"'] & 132+2+13 \arrow[d, "T_{5,4}"'] \\
& & 1223++13 & 1232++13 \arrow[l, "T_{4,3}"] & 1322++13 \arrow[d, "T_{4,3}"'] \\
& & & & 13-+++13 \arrow[d, "T_{3,2}"] \\
& & & & 1-3+++13
\end{tikzcd}
$$

$$
\begin{tikzcd}
13-++1+3 \arrow[d, "T_{6,5}"'] & 1322+1+3 \arrow[d, "T_{6,5}"'] & 1232+1+3 \arrow[d, "T_{6,5}"'] & 1-3++1+3 \arrow[d, "T_{6,5}"'] & 1223+1+3 \arrow[d, "T_{6,5}"'] \\
13-+1++3 \arrow[d, "T_{5,4}"'] & 13221++3 & 12321++3 & 1-3+1++3 \arrow[d, "T_{5,4}"'] & 12231++3 \arrow[d, "T_{5,4}"'] \\
13-1+++3 & & & 1-31+++3 \arrow[d, "T_{4,3}"'] & 12213++3 \\
& & & 1-13+++3 &
\end{tikzcd}
$$

The $\mathrm{GL}(4,\mathbb{C}) \times \mathrm{GL}(4,\mathbb{C})$-orbits in $\mathfrak{B}$ that belong to $\mu^{-1}(T_3)$ are parametrized by the clans

$$
\begin{tikzcd}
12+213-3 \arrow[r, "T_{5,4}"] & 12+231-3 \arrow[r, "T_{4,3}"] & 12+321-3 \arrow[r, "T_{3,2}"] \arrow[d, "T_{6,5}"'] & 123+21-3 \arrow[r, "T_{2,1}"] \arrow[d, "T_{6,5}"'] & 132+21-3 \arrow[d, "T_{6,5}"'] \\
& 12+23-13 \arrow[r, "T_{4,3}"'] & 12+32-13 \arrow[d, "T_{5,4}"'] & 123+2-13 \arrow[d, "T_{5,4}"'] & 132+2-13 \arrow[d, "T_{5,4}"'] \\
& & 12+3-213 & 123+-213 \arrow[l, "T_{4,3}"] & 132+-213 \arrow[d, "T_{4,3}"'] \\
& & & & 13+2-213 \arrow[d, "T_{3,2}"] \\
& & & & 1+32-213
\end{tikzcd}
$$

$$\begin{array}{ccccc}
13{+}2{-}123 & 132{+}{-}123 & 123{+}{-}123 & 1{+}32{-}123 & 12{+}3{-}231\\
{\scriptstyle T_{6,5}}\big\downarrow & {\scriptstyle T_{6,5}}\big\downarrow & {\scriptstyle T_{6,5}}\big\downarrow & {\scriptstyle T_{6,5}}\big\downarrow & {\scriptstyle T_{6,5}}\big\downarrow\\
13{+}21{-}23 & 132{+}1{-}23 & 123{+}1{-}23 & 1{+}321{-}23 & 12{+}32{-}31\\
{\scriptstyle T_{5,4}}\big\downarrow & & & {\scriptstyle T_{5,4}}\big\downarrow & {\scriptstyle T_{5,4}}\big\downarrow\\
13{+}12{-}23 & & & 1{-}312{-}23 & 12{+}23{-}31\\
 & & & {\scriptstyle T_{4,3}}\big\downarrow & \\
 & & & 1{+}132{-}23 &
\end{array}$$

5. The two-column case

Explicit computations of the action of $T_{\alpha,\beta}$-operators on two-column standard tableaux.

Proposition 5.1. *Assume T is a standard tableau of shape $[2^t, 1^{r-t}]$. Further assume that T has its $b_{r,1}$ box labeled $r+\ell$ with $\ell \le t$, and has its $b_{1,2}$ box labeled j. Then, there exists a tableau $\widetilde{T}$ with $\tilde{b}_{r,1}$ labeled $r+\ell-1$ so that one of the following holds*:

(1) *$\ell = 1$ and $T = T_{r,r-1}(\widetilde{T})$.*

(2) *$\ell > 1$ and $T = T_{r+\ell-2,r+\ell-3} \circ T_{r+\ell-1,r+\ell-2}(\widetilde{T})$.*

(3) *$\ell > 1$ and $T = T_{r+\ell-1,r+\ell}(\widetilde{T})$.*

(4) *T has box $b_{\ell,2}$ labeled by an integer $k \ge j+\ell-1$, the box with label $k-1$ is on the first column, and $T = T_{k,k-1}\circ\cdots\circ T_{r+\ell-2,r+\ell-3}\circ T_{r+\ell-1,r+\ell-2}(\widetilde{T})$.*

(5) *T has box $b_{\ell,2}$ labeled by an integer $k \ge j+\ell-1$, the box with label $k-1$ is on the second column, and there is a label s with $j-1 \le s \le k-1$ so that $T = T_{s,s-1}\circ\cdots\circ T_{k-1,k-2}\circ T_{k,k-1}\circ\cdots\circ T_{r+\ell-2,r+\ell-3}\circ T_{r+\ell-1,r+\ell-2}(\widetilde{T})$.*

The proposition is proved by induction on the label of the box $b_{r,1}$ in the intersection of the last row and first column of T. As the standard tableau T has shape $[2^t, 1^{r-t}]$, the box $b_{r,1}$ is labeled by an integer of the form $r+\ell$ for some $\ell \ge 0$. For expository purposes we first prove the proposition when $\ell = 1$ and $\ell = 2$. Lemma 5.2 concerns the case $\ell = 1$. Lemma 5.3 treats the case $\ell = 2$.

Let T_o be the standard tableau of shape $[2^t, 1^{r-t}]$ with box $b_{r,1}$ labeled r and box $b_{t,2}$ labeled $r+t$.

Lemma 5.2. *Assume T is a standard tableau of shape $[2^t, 1^{r-t}]$. Further assume that T has its $b_{r,1}$ box labeled $r+1$. Then, there exists a tableau $\widetilde{T}$ with $\tilde{b}_{r,1}$ labeled r such that either*

(1) *$T = T_{r,r-1}(\widetilde{T})$, or*

(2) *$T = T_{j,j-1}\circ\cdots\circ T_{r-1,r-2}\circ T_{r,r-1}(\widetilde{T})$ for some integer $j < r$.*

Proof. T has $b_{r,1}$ labeled $r+1$. Then $b_{r-1,1}$ is either labeled $r-1$ or is labeled r. There is exactly one such tableau with $b_{r-1,1}$ labeled $r-1$. This is $T_{r,r-1}(T_o)$. Thus $\widetilde{T} = T_o$ and $T = T_{r,r-1}(T_o)$. If the label of $b_{r-1,1}$ is r, then T is of the form

$$T = \begin{array}{|c|c|}
\hline
1 & j \\ \hline
\cdot & r{+}2 \\ \hline
\cdot & \cdot \\ \hline
\cdot & \cdot \\ \hline
\cdot & r{+}t \\ \hline
\cdot \\ \cline{1-1}
\cdot \\ \cline{1-1}
r \\ \cline{1-1}
r{+}1 \\ \cline{1-1}
\end{array}\,.$$

In this case, $T = T_{j,j-1} \circ \cdots \circ T_{r-1,r-2} \circ T_{r,r-1}(T_o)$. □

Lemma 5.3. *Assume T is a standard tableau of shape $[2^t, 1^{r-t}]$. Further assume that T has its $b_{r,1}$ box labeled $r+2$.*

(1) *If $b_{1,2}$ has label r and $r+1$ is the label of $b_{2,2}$, then there exists a tableau $\widetilde{T}$ with $\tilde{b}_{r,1}$ labeled $r+1$ such that $T = T_{r,r-1} \circ T_{r+1,r}(\widetilde{T})$.*

(2) *If $b_{1,2}$ has label $j < r$ and $r+1$ is the label of $b_{2,2}$, then there exists a tableau $\widetilde{T}$ with $\tilde{b}_{r,1}$ labeled $r+1$ such that $T = T_{r+1,r}(\widetilde{T})$.*

(3) *If the label of $b_{r-1,1}$ is $r+1$, then there exists a tableau $\widetilde{T}$ with $\tilde{b}_{r,1}$ labeled $r+1$ such that*

$$T = T_{i,i-1} \circ T_{i+1,i} \circ \cdots \circ T_{r,r-1} \circ T_{r+1,r}(\widetilde{T})$$

for some integer $i < r$.

Proof. Assume first that $r+1$ is the label of $b_{2,2}$. Then $b_{1,2}$ has label j with $j \le r$. When $j \ne r$, we have

$$T = T_{r+1,r}\left(\begin{array}{|c|c|}
\hline
1 & j \\ \hline
\cdot & r{+}2 \\ \hline
\cdot & \cdot \\ \hline
\cdot & \cdot \\ \hline
\cdot & r{+}t \\ \hline
\cdot \\ \cline{1-1}
\cdot \\ \cline{1-1}
r \\ \cline{1-1}
r{+}1 \\ \cline{1-1}
\end{array}\right),$$

When $j = r$, we have $T = T_{r,r-1} \circ T_{r+1,r}(\widetilde{T})$, where $\tilde{b}_{1,2} = r-1$.

We next consider the tableaux T with $b_{r-1,1}$ labeled $r+1$. Observe that T is of the form

$$\begin{array}{|c|c|}
\hline
1 & j \\ \hline
\cdot & k \\ \hline
\cdot & r{+}3 \\ \hline
\cdot & r{+}4 \\ \hline
\cdot & \cdot \\ \hline
\cdot & \cdot \\ \hline
\cdot & r{+}t \\ \hline
\cdot \\ \cline{1-1}
\cdot \\ \cline{1-1}
r{+}1 \\ \cline{1-1}
r{+}2 \\ \cline{1-1}
\end{array}$$

where $k \leq j+1$.

When $k = j+1$, the tableau $T_{r-1,r} \circ \cdots \circ T_{j,j+1} \circ T_{j-1,j}(T)$ has box $b_{r,1}$ labeled $r+2$ and $b_{2,2}$ labeled $r+1$. We have $T_{r-1,r} \circ \cdots \circ T_{j,j+1} \circ T_{j-1,j}(T) = T_{r+1,r}(\widetilde{T})$, with $\widetilde{T}$ a tableau of shape $[2^t, 1^{r-t}]$ having $\tilde{b}_{r,1}$ labeled $r+1$. As the operators $T_{z,z-1}$ are injective (with inverses $T_{z-1,z}$), we have

$$T = T_{j,j-1} \circ T_{j+1,j} \circ \cdots \circ T_{r,r-1} \circ T_{r+1,r}(\widetilde{T}).$$

When $k \neq j+1$, some box in the first column of T has label $k-1$. Then, T is of the form

$$T = \begin{array}{|c|c|}
\hline
1 & j \\ \hline
\cdot & k \\ \hline
\cdot & r{+}3 \\ \hline
\cdot & r{+}4 \\ \hline
\cdot & \cdot \\ \hline
\cdot & \cdot \\ \hline
k{-}1 & \cdot \\ \hline
k{+}1 & \cdot \\ \hline
\cdot & r{+}t \\ \hline
\cdot \\ \cline{1-1}
\cdot \\ \cline{1-1}
\cdot \\ \cline{1-1}
\cdot \\ \cline{1-1}
r{+}1 \\ \cline{1-1}
r{+}2 \\ \cline{1-1}
\end{array}.$$

Hence, $T_{r-1,r} \circ \cdots \circ T_{k,k+1} \circ T_{k-1,k}(T)$ is a tableau with box $b_{2,2}$ labeled $r+1$. By part (2) of this lemma, we have $T_{r-1,r} \circ \cdots \circ T_{k,k+1} \circ T_{k-1,k}(T) = T_{r+1,r}(\widetilde{T})$, where $\widetilde{T}$ is a tableau of shape $[2^t, 1^{r-t}]$ having $\tilde{b}_{r,1}$ labeled $r+1$. We conclude that $T = T_{k,k-1} \circ T_{k+1,k} \circ \cdots \circ T_{r,r-1} \circ T_{r+1,r}(\widetilde{T})$.

Note that our argument above is independent of r and t. $\square$

Proof of Proposition 5.1. The proof is by induction on the label of the box in the intersection of the last row first column of T. Assume T is a standard tableau of shape $[2^t, 1^{r-t}]$. By Lemmas 5.2 and 5.3, the proposition holds when $\ell = 1, 2$. Assume the statement of the proposition holds for any tableau of shape $[2^n, 1^{r-n}]$ with box $b_{r,1}$ labeled $r+m$ with $m < \ell$. We prove that the result holds for a tableau of shape $[2^t, 1^{r-t}]$ with box $b_{r,1}$ labeled $\ell + r$. We have two cases. Either $r+\ell-1$ occurs as a label of a box in the second column of T or $r+\ell-1$ is the label of $b_{r-1,1}$.

Assume that $r+\ell-1$ occurs as label of a box in the second column of T. Such a T is of the form

$$T = \begin{array}{|c|c|}
\hline
1 & j \\ \hline
\cdot & \cdot \\ \hline
\cdot & \cdot \\ \hline
\cdot & r+\ell-1 \\ \hline
\cdot & r+\ell+1 \\ \hline
\cdot & \cdot \\ \hline
k-1 & \cdot \\ \hline
k+1 & \cdot \\ \hline
\cdot & r+t \\ \hline
\cdot \\ \cline{1-1}
\cdot \\ \cline{1-1}
\cdot \\ \cline{1-1}
r+\ell \\ \cline{1-1}
\end{array}\,.$$

Observe that $T_{r+\ell,r+\ell-1}(T) = \widetilde{T}$ is a tableau with $\tilde{b}_{r,1}$ labeled $r+\ell-1$. Since the $T_{\cdot,\cdot}$ are injective, we conclude that $T = T_{r+\ell-1,r+\ell}(\widetilde{T})$.

If $r+\ell-1$ is the label of $b_{r-1,1}$, then T is of the form

$$T = \begin{array}{|c|c|}
\hline
1 & j \\ \hline
\cdot & \cdot \\ \hline
\cdot & \cdot \\ \hline
\cdot & k \\ \hline
\cdot & r+\ell+1 \\ \hline
\cdot & \cdot \\ \hline
k+1 & \cdot \\ \hline
k+2 & \cdot \\ \hline
\cdot & r+t \\ \hline
\cdot \\ \cline{1-1}
\cdot \\ \cline{1-1}
r+\ell-1 \\ \cline{1-1}
r+\ell \\ \cline{1-1}
\end{array}$$

with $k \geq \ell - 1 + j$. Note that $k-1$ can be either in the first or in the second column.

We consider the smaller tableau

$$\widehat{T} = \begin{array}{|c|c|}\hline 1 & j \\ \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \cdot & k \\ \hline \cdot \\ \cline{1-1} \cdot \\ \cline{1-1} r+\ell-1 \\ \cline{1-1}\end{array}\,.$$

By induction hypothesis there exists $\widetilde{\widetilde{T}}$, with the box in the intersection of the last row and first column labeled $r+\ell-2$, so that $\widehat{T}$ is either

- $\widehat{T} = T_{k,k-1} \circ \cdots \circ T_{r+\ell-2,r+\ell-3}\big(\widetilde{\widetilde{T}}\big) = \mathcal{S}_1\big(\widetilde{\widetilde{T}}\big)$,
- $\widehat{T} = T_{s,s-1} \circ \cdots \circ T_{k,k-1} \circ \cdots \circ T_{r+\ell-2,r+\ell-3}\big(\widetilde{\widetilde{T}}\big) = \mathcal{S}_2\big(\widetilde{\widetilde{T}}\big)$ with $j-1 \leq s$, or
- $\widehat{T} = T_{r+\ell-3,r+\ell-4} \circ T_{r+\ell-2,r+\ell-3}\big(\widetilde{\widetilde{T}}\big) = \mathcal{S}_3\big(\widetilde{\widetilde{T}}\big)$.

In each case, $\widetilde{\widetilde{T}}$ has $r+\ell-2$ occurring in the first column. Enlarge $\widetilde{\widetilde{T}}$ to a tableau of shape $[2^t, 1^{r-t}]$ by adding a box with label $r+\ell$ to the first column and $t-\ell$ boxes to the end of the second column with consecutive labels $r+\ell+1$ to $r+t$. Call this new tableau $\widetilde{T}$. It is useful to note that $\widetilde{T}$ has box $\tilde{b}_{r-1,1}$ labeled $r+\ell-2$ and box $\tilde{b}_{\ell,2}$ labeled $r+\ell-1$. It follows that

$$T = \mathcal{S}_i\big(\widetilde{T}\big) \quad \text{with } i \in \{1, 2, 3\}. \tag{5.4}$$

On the other hand, as $\widetilde{T}$ has box $\tilde{b}_{r-1,1}$ labeled $r+\ell-2$ and box $\tilde{b}_{\ell,2}$ labeled $r+\ell-1$,

$$T_{r+\ell-2,r+\ell-1}\big(\widetilde{T}\big) = \widetilde{\widetilde{T}} \quad \text{with } \tilde{\tilde{b}}_{r,1} \text{ labeled } r+\ell-1. \tag{5.5}$$

Combining equations (5.4) and (5.5) we have that T can be obtained from $\widetilde{\widetilde{T}}$ with $\tilde{\tilde{b}}_{r,1}$ labeled $r+\ell-1$ by a sequence of operators $T_{\cdot,\cdot}$ as prescribed by the proposition. □

Example. Consider the standard tableau

$$T = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline 4 & 8 \\ \hline 9 & 11 \\ \hline 10 \\ \cline{1-1}\end{array}\,.$$

We have $r = 6$, $\ell = 4$, and $k = 8$. Observe that $k-1 = 7$, $k-2 = 6$, and $k-3 = 5$ are labels of boxes in the second column of T. Take $s = 5$. Then

$$T = T_{5,4} \circ T_{6,5} \circ T_{7,6} \circ T_{8,7} \circ T_{9,8} \left(\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline 7 & 10 \\ \hline 8 & 11 \\ \hline 9 & \\ \cline{1-1} \end{array} \right).$$

The equivalence class of $\boldsymbol{+++\cdots+--\cdots-}$.

Proposition 5.6. *Let $\mathcal{O}$ be the nilpotent K-orbit parametrized by a two-column tableau with length-sizes (p, q) having all boxes in the first column labeled by $+$. Assume that $\boldsymbol{c}$ is a clan that parametrizes a K-orbit in $\mu^{-1}(\mathcal{O})$. Then*:

(1) $c_1 = +$.

(2) *The first p-entries of $\boldsymbol{c}$ are either $+$ signs or natural numbers.*

(3) *The last q-entries of $\boldsymbol{c}$ are either $-$ signs or natural numbers.*

(4) *If c_k is the last integer entry in $\boldsymbol{c}$, then for all $t > k$ $c_t = -$.*

(5) *If $j \leq p$ and $c_j \in \mathbb{N}$, then there is exactly one $i \geq p+1$ so that $c_j = c_i$.*

(6) *If $i < j$ and (c_i, c_{p+t}) and (c_j, c_{p+s}) are pairs of equal numbers, then $s < t$.*

(7) *If $j < p$ and $c_j \in \mathbb{N}$, then $\#\{c_t \in \mathbb{N}$ with $t \leq j\} \leq \#\{c_t = +$ with $t < j\}$.*

Proof. We first observe that if $\boldsymbol{c} \in \mu^{-1}(\mathcal{O}_K)$, then $c_1 = +$. This is an easy consequence of Garfinkle's algorithm, as otherwise the algorithm would produce a signed tableau having both a $+$ sign and a $-$ sign in the first column. Call c_j the first entry in $\boldsymbol{c}$ (counting from left to right) such that $c_j = a \in \mathbb{N}$. Let c_i be the unique entry of $\boldsymbol{c}$ with $i \neq j$ and $c_i = c_j$. Then we know that each entry $c_t \in \boldsymbol{c}$ with $t < j$ is a $+$ as otherwise the algorithm would not produce a two-column tableau. Similar considerations allow us to conclude that $i \geq p+1$ and that all entries in $\boldsymbol{c}$ with indices larger than i are $-$ signs. Hence, we can write $c_i = c_{p+\ell}$ with $\ell \geq 1$.

Our proof is by induction on ℓ. We first prove that all clans in $\mu^{-1}(\mathcal{O}_K)$ for which the last integer entry (counting from left to right) is c_{p+1} satisfy the proposition. Let $\boldsymbol{c}$ be one such clan. As $q = \#\{-$ signs in $\boldsymbol{c}\} + \#\{$pairs of equal numbers$\}$, we have

$$\boldsymbol{c} = (+\cdots+1+\cdots+1-\cdots-), \quad \text{with } c_j = c_{p+1} = 1.$$

Hence, $\boldsymbol{c}$ satisfies the proposition.

Assume next that clans with last numerical entry in position $p+\ell-1$ satisfy the proposition. We prove that it is so for those clans with last numerical entry in position $p+\ell$. Let $\boldsymbol{c}_\ell$ be a clan that parametrizes an orbit $\mathcal{Q}_{\mathbf{c}_\ell} \in \mu^{-1}(\mathcal{O}_K)$

such that the last numerical entry in $\mathbf{c}_\ell$ is in position $p+\ell$. By Theorem 4.3 and Proposition 5.1, there exists an orbit $\mathcal{Q}_{\boldsymbol{c}_{\ell-1}} \in \mu^{-1}(\mathcal{O}_K)$ which is τ-linked to $\mathcal{Q}_{\boldsymbol{c}_\ell}$. In particular, $\boldsymbol{c}_\ell$ can be obtained from a clan $\boldsymbol{c}_{\ell-1}$, having its last numerical entry in position $p+\ell-1$, by an appropriate sequence of operators $T_{\cdot,\cdot}$ as prescribed by Proposition 5.1. By our induction hypothesis, clan $\boldsymbol{c}_{\ell-1}$ satisfies the proposition; that is:

(a) Each of the first p entries is either a $+$ sign or a natural number with $c_1=+$.

(b) If (c_i,c_j) is a pair of equal numbers, then $i\le p$ and $j\ge p+1$.

(c) After the last numerical entry, the clan consists of $-$ signs.

(d) For each $c_j\in\mathbb{N}$ with $j\le p$, $\#\{c_t\in\mathbb{N} \text{ with } t\le j\}\le \#\{c_t=+ \text{ with } t<j\}$.

In order to show that $\boldsymbol{c}_\ell$ also satisfies the proposition, we study the effect of the sequence of operators $T_{\cdot,\cdot}$ on $\boldsymbol{c}_{\ell-1}$. The sequence of relevant operators $T_{\cdot,\cdot}$ is that of Proposition 5.1. The first operator in the sequence is $T_{p+\ell-1,p+\ell-2}$. Since $\boldsymbol{c}_{\ell-1}\in D_{p+\ell-1,p+\ell-2}$ and it satisfies the proposition, its entries $c_{p+\ell-2}, c_{p+\ell-1}, c_{p+\ell}$ are of the form $(\cdots a\cdots b\cdots\,|\cdots b\,a\,-)$ or $(\cdots a\cdots +\,|\,a\,-)$. Thus, $T_{p+\ell-1,p+\ell-2}(\boldsymbol{c}_{\ell-1})$ gives $(\cdots a\cdots b\cdots\,|\cdots b-a)$ or $(\cdots a\cdots +\,|\,-a)$. All such new clans satisfy the proposition. The action of $T_{p+\ell-2,p+\ell-3}$ on one such new clan depends on its $c_{p+\ell-3}$ entry. We have the following possibilities:

$$\begin{array}{lll} (\cdots a\cdots b\cdots\,|\cdots -b-a), & (\cdots a\cdots ++\,|-a), & (\cdots a\cdots b\cdots +\,|\,b-a),\\ (\cdots a\cdots b\cdots c\cdots\,|\cdots c\,b-a), & (\cdots a\cdots b\,|\,b-a), & (\cdots a\cdots b+\,|-b\cdots a). \end{array}$$

Thus, $T_{p+\ell-2,p+\ell-3}$ applied to the clans above gives

$$\begin{array}{lll} (\cdots a\cdots b\cdots\,|\cdots b--a), & (\cdots a\cdots +b\,|\,b\,a), & (\cdots a\cdots b\cdots +\,|-b\,a),\\ (\cdots a\cdots b\cdots c\cdots\,|\cdots c-b\,a), & (\cdots a\cdots +\,|--a), & (\cdots a\cdots +b\,|-b\cdots a). \end{array}$$

The clans so produced clearly satisfy the proposition. When studying the consecutive action of $T_{\cdot,\cdot}$, as prescribed by Proposition 5.1, we need to also consider clans containing the patterns

$$(\cdots ++a\cdots\,|\cdots a),\quad (\cdots +a+b\cdots\,|\cdots b-a),\quad (\cdots +a+c\,|\,c\cdots\,a),$$
$$(\cdots a\,b+c\,|\,c-b\cdots a),\quad (\cdots ++a\,|\cdots a).$$

In these cases, $T_{i,i+1}$ maps the above clans to new clans containing the patterns

$$(\cdots +a+\cdots\,|\cdots a),\quad (\cdots ++a\,b\cdots\,|\cdots b-a),\quad (\cdots ++a\,c\,|\,c-a),$$
$$(\cdots a+b\,c\,|\,c-b\cdots a),\quad (\cdots +a+\,|\cdots a).$$

Conditions (1) through (6) of the proposition are clearly satisfied by these new clans. The only nonobvious conclusion is that the clans

$$\boldsymbol{c}' = T_{\cdot,\cdot}\big(\boldsymbol{c}{=}(\cdots + + a \cdots \mid \cdots a \cdots)\big) = (\cdots + a + \cdots \mid \cdots a \cdots)$$

and

$$\boldsymbol{c}' = T_{\cdot,\cdot}\big(\boldsymbol{c}{=}\cdots + + a \mid \cdots a \cdots)\big) = (\cdots + a + \mid \cdots a \cdots)$$

satisfy condition (7). Let $A = \#\{+$ signs in $\boldsymbol{c}$ that occur to the left of $a\}$, and let $B = \#\{c_t \in \boldsymbol{c}$: integer entry to the left or at the position of $a\}$. By the induction hypothesis we have $B \le A$. If $B < A$, then $\boldsymbol{c}'$ satisfies (7). We assume that $A = B$ and derive a contradiction. Write the first p-entries of $\boldsymbol{c}$ as $[+\,\gamma\ +\ +\,a\cdots]$. Let A_γ denote the number of $+$ signs in γ and let B_γ denote the number of integers in γ. We have $A = A_\gamma + 3 = B = B_\gamma + 1$. Hence,

$$B_\gamma = A_\gamma + 2. \tag{5.7}$$

If the last numerical entry in γ is c_{t_γ} then, as $\mathbf{c}$ satisfies (7) by the induction hypothesis,

$$B_\gamma \le \#\{+ \text{ signs to the left of } c_{t_\gamma}\}. \tag{5.8}$$

On the other hand,

$$\begin{aligned} A_\gamma = {} & \#\{+ \text{ signs in } \boldsymbol{c} \text{ to the left of } c_{t_\gamma}\} \\ & + \#\{+ \text{ signs in } \gamma \text{ occurring to the right of } c_{t_\gamma}\} - 1. \end{aligned} \tag{5.9}$$

Combining the identities in (5.7) and (5.9) with the inequality (5.8), we obtain

$$\begin{aligned} \#\{+ \text{ signs to the left of } c_{t_\gamma}\} + \#\{+ \text{ signs in } \gamma \text{ occurring to the right of } c_{t_\gamma}\} + 1 \\ \le \#\{+ \text{ signs to the left of } c_{t_\gamma}\}. \end{aligned} \tag{5.10}$$

As inequality (5.10) cannot hold, we conclude that $A < B$. □

Corollary 5.11. *Let $\mathcal{O}_K$ be the nilpotent K-orbit parametrized by a two-column tableau with length-sizes (p, q) having all boxes in the first column labeled by $+$. Assume that $\mathbf{c}$ is a clan that parametrizes a K-orbit in $\mu^{-1}(\mathcal{O}_K)$. Then,*

$$0 \le \#\{\textit{ pairs of equal numbers in } \boldsymbol{c}\} \le \min\big\{\big[\tfrac{1}{2}p\big], q\big\}.$$

Proof. Garfinkle's algorithm assigns to $\boldsymbol{c}$ a signed tableau and a standard tableau. The algorithm is such that each pair of equal numbers in $\boldsymbol{c}$ produces a $-$ sign in the corresponding signed tableau. Hence, under our assumptions

$$\#\{\text{pairs of equal numbers in } \boldsymbol{c}\} \le q.$$

On the other hand, part (7) of Proposition 5.6 implies

$$\#\{\text{pairs of equal numbers in } \boldsymbol{c}\} \le \big[\tfrac{1}{2}p\big].$$

The corollary follows. □

On $\mu^{-1}(\mathcal{O}_K)$ for orbits $\mathcal{O}_K$ parametrized by a two-column signed tableau. A bijection between the set of nilpotent K-orbits and a set consisting of distinguished clans is exhibited in Proposition 2.3. In this subsection we give the explicit parametrization of nilpotent K-orbits in terms of clans in the two-column case. We introduce some notation. We consider two-column tableaux with column lengths (r,t) with $r+t=p+q=n$. Set

$$L_1 = \#\{-\text{ signs in the first column}\}, \tag{5.12}$$

$$L_2 = \#\{+\text{ signs in the second column}\}. \tag{5.13}$$

Proposition 5.14. *Let $\mathcal{O}_K$ be a nilpotent K-orbit. Assume that the signed tableau parametrizing $\mathcal{O}$ has two columns. Then $\mu^{-1}(\mathcal{O}_K)$ contains the K-orbit $Q_{\boldsymbol{c}}$ in $\mathcal{B}$ for exactly one of the following*:

(1) $\boldsymbol{c}=\big(1 2\cdots r-L_1-\cdots-r-L_1\cdots 1\, r+1\cdots r+t-L_2+\cdots+r+t-L_2\cdots r+1\big)$, *with* $L_1\geq\left[\frac{r}{2}\right]$, $L_2\geq\left[\frac{t}{2}\right]$.

(2) $\boldsymbol{c}=\big(1\,2\cdots r-L_1-\cdots-r-L_1\cdots 1\, r+1\cdots r+L_2-\cdots-r+L_2\cdots r+1\big)$, *with* $L_1\geq\left[\frac{r}{2}\right]$, $L_2\leq\left[\frac{t}{2}\right]$.

(3) $\boldsymbol{c}=\big(1\,2\cdots L_1+\cdots+L_1\cdots 1\, r+1\cdots r+t-L_2+\cdots+r+t-L_2\cdots r+1\big)$, *with* $L_1\leq\left[\frac{r}{2}\right]$, $L_2\geq\left[\frac{t}{2}\right]$.

(4) $\boldsymbol{c}=\big(1\,2\cdots L_1+\cdots+L_1\cdots 1\, r+1\cdots r+L_2-\cdots-r+L_2\cdots r+1\big)$, *with* $L_1\leq\left[\frac{r}{2}\right]$, $L_2\leq\left[\frac{t}{2}\right]$.

Proof. The proposition follows from Proposition 2.3 and Garfinkle's algorithm. □

Proposition 5.15. *Keep the notation just introduced. Assume $\boldsymbol{c}\in\mu^{-1}(\mathcal{O}_K)$, and let $N_{\boldsymbol{c}}=\#\{$pairs of equal numbers in $\boldsymbol{c}\}$. Then one has the following*:

(1) *If $L_1\geq\left[\frac{r}{2}\right]$, $L_2\geq\left[\frac{t}{2}\right]$, and*

$$M=\min\big\{\big[\tfrac{1}{2}\max\{2L_1-r,2L_2-t\}\big],\min\{2L_1-r,2L_2-t\}\big\},$$

then for each integer k with

$$n-(L_1+L_2)\leq k\leq n-(L_1+L_2)+M,$$

there exists a clan $\boldsymbol{c}_k\in\mu^{-1}(\mathcal{O}_K)$ so that $N_{\boldsymbol{c}_k}=k$.

(2) *If $L_1\leq\left[\frac{r}{2}\right]$, $L_2\leq\left[\frac{t}{2}\right]$, and*

$$M=\min\big\{\big[\tfrac{1}{2}\max\{r-2L_1,t-2L_2\}\big],\min\{r-2L_1,t-2L_2\}\big\},$$

then for each integer k with

$$L_1+L_2\leq k\leq (L_1+L_2)+M,$$

there exists a clan $\boldsymbol{c}_k\in\mu^{-1}(\mathcal{O}_K)$ so that $N_{\boldsymbol{c}_k}=k$.

(3) *If* $L_1 \le \left[\frac{r}{2}\right]$ *and* $L_2 \ge \left[\frac{t}{2}\right]$, *then for each integer* k *with*

$$t - L_2 \le k \le t - L_2 + L_1,$$

there exists a clan $\boldsymbol{c}_k \in \mu^{-1}(\mathcal{O}_K)$ *so that* $N_{\boldsymbol{c}_k} = k$.

(4) *If* $L_1 \ge \left[\frac{r}{2}\right]$ *and* $L_2 \le \left[\frac{t}{2}\right]$, *then for each integer* k *with*

$$L_2 \le k \le r - L_1 + L_2,$$

there exists a clan $\boldsymbol{c}_k \in \mu^{-1}(\mathcal{O}_K)$ *so that* $N_{\boldsymbol{c}_k} = k$.

Proof. We prove that (2) holds. Statements (1), (3), and (4) can be proved using similar arguments. By Proposition 5.6 it is enough to show that clans of the form

$$\big(a_1\, b_1\, b_2 \cdots b_{L_2}\, a_2 \cdots a_{L_1} + \cdots + - - \cdots - a_{L_1} \cdots a_1\, b_{L_2} \cdots b_1\big) \tag{5.16}$$

are in $\mu^{-1}(\mathcal{O}_K)$.

We start by observing that Proposition 5.14 guarantees that $\mu^{-1}(\mathcal{O}_K)$ contains the clan

$$\boldsymbol{c} = \big(a_1\, a_2 \cdots a_{L_1} + \cdots + a_{L_1} \cdots a_1\, b_1 \cdots b_{L_2} - \cdots - b_{L_2} \cdots b_1\big).$$

By Theorem 4.3, the proposition is settled once an appropriate sequence of operators $T_{\cdot,\cdot}$, when applied to $\boldsymbol{c}$, produces clans of the desired shape.

Clan $\boldsymbol{c}$ is in the domain of $T_{r,r-1}$. Hence, by Theorem 4.3, $T_{r,r-1}\boldsymbol{c} \in \mu^{-1}(\mathcal{O}_K)$. Similarly, we argue that $T_{2,1} \circ T_{3,2} \circ \cdots \circ T_{r,r-1}(\boldsymbol{c}) \in \mu^{-1}(\mathcal{O}_K)$. That is,

$$\begin{aligned}
\boldsymbol{c}' &= \big(a_1 b_1 a_2 \cdots a_{L_1} + \cdots + a_{L_1} \cdots a_1 b_2 \cdots b_{L_2} - \cdots - b_{L_2} \cdots b_1\big),\\
\boldsymbol{c}'' &= \big(a_1\, b_1\, b_2 \cdots b_{L_2}\, a_2 \cdots a_{L_1} + \cdots + a_{L_1} \cdots a_1 - \cdots - b_{L_2} \cdots b_1\big)
\end{aligned}$$

are clans in $\mu^{-1}(\mathcal{O}_K)$. The next operator in the sequence is $T_{r+L_2,r+L_2+1}$, which when applied to $\boldsymbol{c}''$ gives

$$\boldsymbol{c}''' = \big(a_1\, b_1\, b_2 \cdots b_{L_2}\, a_2 \cdots a_{L_1} + \cdots + - a_{L_1} \cdots a_2 - a_1 - \cdots - b_{L_2} \cdots b_1\big).$$

Next, we compute $T_{r-L_1+L_2,r-L_1+L_2-1} \circ \cdots \circ T_{r+L_2,r+L_2+1}(\boldsymbol{c}'')$ to obtain

$$\boldsymbol{c}^{iv} = \big(a_1\, b_1\, b_2 \cdots b_{L_2}\, a_2 \cdots a_{L_1} + \cdots + - a_{L_1} \cdots a_2 - \cdots - a_1\, b_{L_2} \cdots b_1\big).$$

Note that now, at “the center” of the clan we have the $+ + \cdots + -$ pattern. Further applications of similar operators yield the clan in (5.16). $\square$

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