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# Spectrum of the Laplacian on graphs of radial functions

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We prove that if  $M$  is a complete, noncompact hypersurface in  $\mathbb{R}^{n+1}$ , which is the graph of a real radial function, then the spectrum of the Laplace operator on  $M$  is the interval  $[0, \infty)$ .

## 1. Introduction

Let  $M$  be a simply connected Riemannian manifold. The Laplace operator  $\Delta : C_0^\infty(M) \rightarrow C_0^\infty(M)$ , defined as  $\Delta = \operatorname{div} \circ \operatorname{grad}$  and acting on  $C_0^\infty(M)$  (the space of smooth functions with compact support), is a second-order elliptic operator and, provided  $M$  is complete, it has a unique extension  $\Delta$  to an unbounded self-adjoint operator on  $L^2(M)$  whose domain is  $\operatorname{Dom}(\Delta) = \{f \in L^2(M) : \Delta f \in L^2(M)\}$ ; see [Grigor'yan 2009, Theorem 11.5]. Since  $-\Delta$  is positive and symmetric, its spectrum is the set of  $\lambda \geq 0$  such that  $\Delta + \lambda I$  does not have a bounded inverse. Sometimes we say “spectrum of  $M$ ” rather than “spectrum of  $-\Delta$ ”, and we denote it by  $\sigma(M)$ . One defines the *essential spectrum*,  $\sigma_{\text{ess}}(M)$ , to be those  $\lambda$  in the spectrum which are either accumulation points of the spectrum or eigenvalues of infinite multiplicity. The *discrete spectrum* is the set  $\sigma_d = \sigma(M) \setminus \sigma_{\text{ess}}(M)$  of all eigenvalues of finite multiplicity which are isolated points of the spectrum.

There is a vast literature on the spectrum of the Laplace operator on complete noncompact manifolds. The first result we mention was published by Tayoshi [1971]. He showed the absence of eigenvalues of  $-\Delta$  for a class of surfaces of revolution, determined by nonnegative radial growth.

Donnelly [1981] showed

$$\sigma_{\text{ess}}(M) = \left[ (n-1)^2 \frac{1}{4} c^2, \infty \right),$$

provided  $M$  is a Hadamard manifold whose sectional curvature approaches  $-c^2$  at infinity. Karp [1984] gave sufficient conditions for a class of manifolds to have

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purely continuous spectrum ( $\sigma_d(M) = \emptyset$ ) under some curvature conditions. Eight years later, Donnelly and Garofalo [1992] obtained results in a similar direction, using the hypothesis of nonnegative radial sectional curvature, without restrictions on the metric.

Cheng and Zhiqin Lu [1992] proved  $\sigma_{\text{ess}}(M) = [0, \infty)$  when  $M$  has nonnegative radial sectional curvature and Li [1994] proved  $\sigma_{\text{ess}}(M) = [0, \infty)$ , provided  $M$  has nonnegative Ricci curvatures and a pole. Zhou [1994] proved  $\sigma_{\text{ess}}(M) = [0, \infty)$  when  $M$  has nonnegative sectional curvatures, generalizing the work of Escobar and Freire [1992].

Kumura [1997] found a result which generalized [Donnelly 1981]. He showed  $\sigma_{\text{ess}}(M) = [\frac{1}{4}c^2, \infty)$  whenever

$$\limsup_{n \rightarrow \infty} \sup_{t > n} |\Delta t - c| = 0,$$

where  $t$  denotes the distance function on  $M$ .

Wang [1997] showed that the spectrum of a complete, noncompact Riemannian manifold with asymptotically nonnegative Ricci curvature is equal to  $[0, \infty)$ .

Zhiqin Lu and Detang Zhou [2011] proved that the  $L^p$  essential spectrum of  $M$  is equal to  $[0, \infty)$  when

$$\liminf_{x \rightarrow \infty} \text{Ric}_M(x) = 0$$

and  $M$  is noncompact and complete. We should mention here that almost all the above works were strongly motivated by the decomposition principle [Donnelly and Li 1979], which states that the essential spectrum of a Riemannian manifold is invariant under compact perturbations of the metric, thus it is a function of the geometry of the ends. In [Monte and Montenegro 2015], it was proved that  $\sigma_{\text{ess}}(M) \supset [(n-1)^2 \frac{1}{4}c^2, \infty)$  for a class of Riemannian manifolds, not necessarily complete, whose metric is given by

$$g_M = dr^2 + \psi^2(rw)g_{\mathbb{S}^{n-1}},$$

using curvature conditions only in a neighborhood of a ray.

See also [Bessa et al. 2010; 2012; 2015; Donnelly and Li 1979; Kleine 1988; 1989; Tayoshi 1971] for geometric conditions implying the discreteness of the spectrum,  $\sigma_{\text{ess}}(M) = \emptyset$ .

In this work we consider complete hypersurfaces which are graphs of radial functions. Our main result is the following theorem.

**Theorem 1.** *Let  $M$  be a complete hypersurface in  $\mathbb{R}^{n+1}$ , which is the graph of a real radial function. Then, the spectrum of the Laplace operator on  $M$  is  $[0, \infty)$ .*

Without loss of generality, we may assume the domain  $\text{Dom } f$  to be connected and symmetric with respect to  $0 \in \mathbb{R}^n$ . From the completeness of  $M$  we further

deduce  $\text{Dom } f$  is an open ball or annulus. The theorem above allows us to construct a bounded hypersurface with the same spectrum of  $\mathbb{R}^{n+1}$  by taking  $M$  to be the graph of the real function  $f(x) = \cos(\tan(\frac{1}{2}\pi|x|))$  defined on the unit open ball.

Throughout the following discussion, for simplicity, we deal with the case where  $f : D \rightarrow \mathbb{R}$  is defined in an open ball. Let  $X : [0, R) \times \Omega \rightarrow D$  be defined by  $X(r, x_1, \dots, x_{n-1}) = rw(x_1, \dots, x_{n-1})$ , where  $0 < R \leq +\infty$  and  $w$  is a coordinate system on  $S^{n-1}$  defined on an open set  $\Omega$  of  $\mathbb{R}^n$ . Note that  $M$  has a natural coordinate system  $Y : [0, R) \times \Omega \rightarrow M$ , given by  $Y(r, x_1, \dots, x_{n-1}) = (rw(x_1, \dots, x_{n-1}), f(r))$ , but we are interested in the spherical coordinate system for  $M$  on  $p = (0, f(0))$ . Consider  $t : [0, R) \rightarrow [0, \infty)$ , given by

$$t(r) = \int_0^r (1 + f'(\tau)^2)^{1/2} d\tau.$$

We claim that  $t$  is a diffeomorphism. Observe that  $t$  is increasing and

$$\lim_{r \rightarrow R} t(r) = +\infty.$$

We denote by  $r : [0, \infty) \rightarrow [0, R)$  the inverse diffeomorphism. By the inverse function theorem,

$$0 < r'(t) = (1 + f'(r)^2)^{-1/2} \leq 1. \tag{1}$$

Finally, the system of spherical coordinates on  $M$ , denoted  $Z : [0, \infty) \times \Omega \rightarrow M$ , is defined by

$$Z(t, x_1, \dots, x_{n-1}) = (r(t)w(x_1, \dots, x_{n-1}), f \circ r(t)).$$

The metric of  $M$  on such a system is given by

$$g_M = dt^2 + r(t)^2 g_{S^{n-1}}.$$

Because of this observation, Theorem 1 is a simple consequence of the theorem below.

**Theorem 2.** *Let  $I \subset \mathbb{R}$  be an unbounded interval and  $M = I \times S^{n-1}$  with metric given by  $g_M = dt^2 + r^2(t)g_{S^{n-1}}$ , where  $0 < r'(t) \leq c$  for all  $t$ . Then, the spectrum of the Laplace operator on  $M$  is  $[0, \infty)$ .*

**Remark.** (1) If  $M$  has a pole at  $p \in M$ , then  $\exp_p : T_p M \rightarrow M$  is a diffeomorphism so that  $M$  isometric to  $T_p M$  with the pullback metric. Therefore, Theorem 2 implies that if  $M$  has a pole  $p$  and  $g_M = dt^2 + r^2(t)g_{S^{n-1}}$  with respect to  $p$  and  $0 < r'(t) < c$ , then  $M$  has spectrum equal to  $[0, \infty)$ .

(2) To the best of our knowledge, this natural result has only been verified in less general settings. For instance, since  $r'(t) > 0$ , then  $r(t)$  is increasing and there are only two possibilities:

- (a)  $\lim_{t \rightarrow \infty} r(t) = \infty$ , or
- (b)  $\lim_{t \rightarrow \infty} r(t) = R$ .

In the first case, since  $r'(t)$  is bounded, we have

$$\lim_{t \rightarrow \infty} \Delta t = \lim_{t \rightarrow \infty} \frac{r'(t)}{r(t)} = 0.$$

By [Kumura 1997, Theorem 1.2], it follows that the spectrum of  $M$  is purely continuous and equal to  $[0, \infty)$ . In the second case, if  $r' \rightarrow 0$  we still have  $r'(t)/r(t) \rightarrow 0$ . Therefore, the main contribution of this paper is the proof of the case where  $r'(t)$  does not converge to zero and  $\lim_{t \rightarrow \infty} r(t) = R < +\infty$ . This is the scenario for the graph of the function  $f(x) = \cos(\tan(\frac{1}{2}\pi|x|))$  presented above.

In the next section we prove Theorem 2. The Appendix is devoted to the Sturm–Liouville theory used in this note.

### 2. Proof of Theorem 2

We concentrate our efforts for the case where  $\lim_{t \rightarrow \infty} r(t) = R$ . Our approach is variational, based on the following lemma.

**Lemma 3** [Davies 1995, Lemma 4.1.2]. *A number  $\lambda \in \mathbb{R}$  lies in the spectrum of a self-adjoint operator  $H$  if and only if there exists a sequence of functions  $f_n \in \text{Dom } H$  with  $\|f_n\| = 1$  such that*

$$\lim_{n \rightarrow \infty} \|Hf_n - \lambda f_n\| = 0.$$

To deduce Theorem 2 from Lemma 3 we will construct, for each  $\lambda > 0$ , a sequence of radial smooth functions  $f_p : M \rightarrow \mathbb{R}$  with compact support such that

$$\|\Delta f_p + \lambda f_p\|_{L^2(M)} \leq \frac{c}{p} \|f_p\|_{L^2(M)} \tag{2}$$

for any natural  $p$ , where  $c$  is a constant which does not depend on  $p$ . It will follow that  $g_p = f_p/\|f_p\|$  has norm one and

$$\lim_{p \rightarrow \infty} \|\Delta g_p + \lambda g_p\|_{L^2(M)} = 0.$$

Therefore, by Lemma 3,  $\lambda$  belongs to the spectrum. To construct the function  $f_p$ , we fix  $t_0 > 0$  and prove that there are  $t_1(\lambda) > t_0$  and a radial function  $u = u(t)$  solution of the problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } [t_0, t_1], \\ u(t_0) = u(t_1) = 0, \\ u > 0 & \text{in } (t_0, t_1). \end{cases} \tag{3}$$

Using Sturm–Liouville theory, we showed that  $u$  can be extended to the whole interval  $[t_0, \infty)$  and it has infinite zeros  $t_0 < t_1 < \dots < t_p < \dots$ . The next step is to consider (for each  $p$ ) a smooth bump function  $h_p$  whose support is the interval  $[t_0, t_{3p}]$ . We then define  $f_p = uh_p$  and show that each  $f_p$  in this sequence satisfies (2). The function  $t \mapsto r^{n-1}(t)$  has a geometric meaning and plays an important role in the proof, thus deserving a special notation. In the sequence of the paper, we let  $v(t) = r^{n-1}(t)$ .

We observe that the first equation in (3) is equivalent to

$$(v(t)u'(t))' + \lambda v(t)u(t) = 0 \tag{4}$$

if  $u = u(t)$  is a radial function. By Theorem 9 in the Appendix, given positive  $t_0$  and  $\lambda$ , (4) has a solution defined on  $[t_0, \infty)$  and satisfying  $u(t_0) = 0$ .

Moreover, Corollary 8 allows us to consider a sequence of zeros  $t_0 < t_1 < \dots$  of  $u$ .

For  $p \in \mathbb{N}$ , we choose a smooth bump function  $h = h_p : \mathbb{R} \mapsto \mathbb{R}$  with  $0 \leq h \leq 1$  satisfying

$$\begin{cases} h(t) = 0, & t \in (-\infty, t_0] \cup [t_{3p}, \infty), \\ h(t) = 1, & t \in [t_p, t_{2p}]. \end{cases}$$

Such a function can be defined in the following way: let  $\varphi \in C_0^\infty(\mathbb{R})$  be nonnegative with  $\text{supp } \varphi = [0, 1]$  and  $\int \varphi = 1$ . Let

$$h_p(t) = \int_{-\infty}^t \varphi_p(s) ds,$$

where

$$\varphi_p(t) = \frac{1}{t_p - t_0} \varphi\left(\frac{t - t_0}{t_p - t_0}\right) - \frac{1}{t_{3p} - t_{2p}} \varphi\left(\frac{t - t_{2p}}{t_{3p} - t_{2p}}\right).$$

This construction is useful since it leads to the following estimates:

$$\begin{aligned} \|h'_p\|_\infty &\leq \max\left\{ \frac{\|\varphi\|_\infty}{t_p - t_0}, \frac{\|\varphi\|_\infty}{t_{3p} - t_{2p}} \right\} \leq \frac{C}{p}, \\ \|h''_p\|_\infty &\leq \max\left\{ \frac{\|\varphi'\|_\infty}{(t_p - t_0)^2}, \frac{\|\varphi'\|_\infty}{(t_{3p} - t_{2p})^2} \right\} \leq \frac{C}{p^2}. \end{aligned} \tag{5}$$

Here, we have made use of Corollary 11 in the Appendix.

Consider  $f = f_p = uh_p$ . We are going to prove that such a function satisfies the inequality in (2). Computing  $\Delta f + \lambda f$ , we obtain

$$\Delta f + \lambda f = 2u'h' + uh'' + (n - 1)\frac{r'}{r}h'u.$$

Using the inequalities in (5), together with the fact that  $r$  is increasing and  $r'$  is bounded, we have

$$|\Delta f + \lambda f| \leq \frac{c}{p} (|u'| + |u|) \chi_{[t_0, t_{3p}]}$$

Then,

$$|\Delta f + \lambda f|^2 \leq \frac{c}{p^2} (|u'|^2 + |u|^2) \chi_{[t_0, t_{3p}]},$$

$$\int_M |\Delta f + \lambda f|^2 dM \leq \frac{c}{p^2} \left( \int_{t_0}^{t_{3p}} |u'|^2 v dt + \int_{t_0}^{t_{3p}} |u|^2 v dt \right).$$

Multiplying (4) by  $u$  and using integration by parts we find

$$\int_{t_0}^{t_{3p}} |u'|^2 v(t) dt = \lambda \int_{t_0}^{t_{3p}} |u|^2 v(t) dt,$$

$$\|\Delta f_p + \lambda f_p\|_{L^2(M)} \leq \frac{c}{p} \|u \cdot \chi_{[t_0, t_{3p}]}\|_{L^2(M)} \leq \frac{c}{p} \|u \cdot \chi_{[t_p, t_{2p}]}\|_{L^2(M)} \leq \frac{c}{p} \|f_p\|_{L^2(M)},$$

where the second inequality comes from Lemma 4 below.

**Lemma 4.** *There is a positive constant  $C$  independent on  $p$  such that*

$$\int_{t_0}^{t_{3p}} u^2 v dt \leq C \int_{t_p}^{t_{2p}} u^2 v dt,$$

where  $u$  is solution of (4) and  $t_0 < t_1 < \dots$  are zeros of  $u$ .

This result is a manifestation of the oscillatory behavior of  $u$ . Before justifying its veracity, we state a useful way of estimating  $u$  between two zeros.

**Lemma 5.** *Let  $u$  be a solution of (4), and choose  $t_k, t_{k+1}$  to be consecutive zeros for  $u$ . Define*

$$\alpha_k(t) = a_k \sin\left(\lambda^{1/2} R^{n-1} \int_{t_k}^t v^{-1}(s) ds\right)$$

and

$$\beta_k(t) = b_k \sin\left(\lambda^{1/2} v(t_k) \int_{t_k}^t v^{-1}(s) ds\right),$$

where  $a_k = v(t_k) b_k / (R^{n-1} \lambda^{1/2})$  and  $b_k = u'(t_k) / \lambda^{1/2}$ . Then  $|\alpha_k| \leq |u|$  on  $(t_k, \tilde{t}_k)$  and  $|u| \leq |\beta_k|$  on  $(t_k, t_{k+1})$ , where  $\tilde{t}_k$  is the next zero of  $\alpha_k$  after  $t_k$ .

To make the exposition more fluid, we postpone the proof until the Appendix.

**Proof of Lemma 4.** Observe that multiplying (4) by  $v(t)u'$  we get

$$(v(t)u')'v(t)u' + \lambda v^2uu' = 0,$$

and so,

$$((v(t)u')^2)' + \lambda v^2(u^2)' = 0.$$



Integrating from  $t_0$  to  $t_k$ , we have

$$v(t_k)^2 u'(t_k)^2 - v(t_0)^2 u'(t_0)^2 = -\lambda \int_{t_0}^{t_k} v^2(s) (u^2(s))' ds.$$

Integrating the right hand side by parts, we find

$$v(t_k)^2 u'(t_k)^2 - v(t_0)^2 u'(t_0)^2 = 2\lambda \int_{t_0}^{t_k} v v' u^2 ds. \tag{6}$$

Since  $r, r' > 0$ , we have  $v, v' > 0$ . Also,  $r(t) < R$  and as a consequence,

$$u'(t_k)^2 > \frac{v(t_0)^2 u'(t_0)^2}{R^{2(n-1)}} \tag{7}$$

for  $k \geq 1$ .

To obtain an estimate in the other direction, we observe that the function  $\beta = \beta_0(t)$  in Lemma 5 satisfies  $\beta'(t_0) = u'(t_0) > 0$  and

$$(v(t)\beta'(t))' + \frac{\lambda v(t_0)^2}{v(t)} \beta(t) = 0. \tag{8}$$

Multiplying by  $v(t)\beta'$  we get, as in the preceding computations,

$$(v(t)^2(\beta')^2)' + \lambda v(t_0)^2(\beta^2)' = 0. \tag{9}$$

Now, if  $\bar{t}_1$  is the next root of  $\beta$  after  $t_0$ , integrating the last equation we find

$$\begin{aligned} v(\bar{t}_1)^2 \beta'(\bar{t}_1)^2 &= v(t_0)^2 \beta'(t_0)^2 \\ &= v(t_0)^2 u'(t_0)^2. \end{aligned} \tag{10}$$

We take  $k = 1$  and estimate the right side of (6) as follows:

$$\begin{aligned} \lambda \int_{t_0}^{t_1} (v^2)' u^2 dt &\leq \lambda \int_{t_0}^{t_1} (v^2)' \beta^2 dt \\ &\leq \lambda \int_{t_0}^{\bar{t}_1} (v^2)' \beta^2 dt \\ &= -\lambda \int_{t_0}^{\bar{t}_1} v^2 (\beta^2)' dt \\ &= -\frac{1}{v(t_0)^2} \int_{t_0}^{\bar{t}_1} v^2 (\lambda v(t_0)^2 \beta^2)' dt. \end{aligned} \tag{11}$$

By (9) we infer

$$\begin{aligned}
 -\frac{1}{v(t_0)^2} \int_{t_0}^{\bar{t}_1} v^2(\lambda v(t_0)^2 \beta^2)' dt &= \frac{1}{v(t_0)^2} \int_{t_0}^{\bar{t}_1} v^2(v^2(\beta')^2)' dt \\
 &= \frac{1}{v(t_0)^2} \int_{t_0}^{\bar{t}_1} (v^4(\beta')^2)' - (v^2)'v^2(\beta')^2 dt \quad (12) \\
 &< \frac{v^4(\bar{t}_1)(\beta')^2(\bar{t}_1) - v^4(t_0)(\beta')^2(t_0)}{v(t_0)^2}.
 \end{aligned}$$

Now, using (10) and that  $\beta'(t_0) = u'(t_0)$ , we find

$$\lambda \int_{t_0}^{t_1} (v^2)'u^2 \leq (v(\bar{t}_1)^2 - v(t_0)^2)u'(t_0)^2 dt.$$

Then, by (6),

$$v(t_1)^2 u'(t_1)^2 - v(t_0)^2 u'(t_0)^2 \leq (v(\bar{t}_1)^2 - v(t_0)^2)u'(t_0)^2.$$

Since  $v(t)$  is increasing, it follows that

$$\begin{aligned}
 v(t_1)^2 u'(t_1)^2 &\leq v(\bar{t}_1)^2 u'(t_0)^2 \\
 &\leq v(t_2)^2 u'(t_0)^2.
 \end{aligned} \quad (13)$$

Then,

$$u'(t_1)^2 \leq \frac{v(t_2)^2}{v(t_0)^2} u'(t_0)^2.$$

Using the same argument, one shows by induction that

$$u'(t_k)^2 \leq \frac{v(t_{k+1})^2 v(t_k)^2}{v(t_1)^2 v(t_0)^2} u'(t_0)^2.$$

Since  $r(t) < R$ , we find that

$$u'(t_k)^2 \leq \frac{R^{4(n-1)}}{v(t_0)^2 v(t_1)^2} u'(t_0)^2. \quad (14)$$

Now, using Lemma 5, it's easy to check that

$$\begin{aligned}
 \int_{t_0}^{t_{3p}} u^2 v dt &= \sum_{k=0}^{3p-1} \int_{t_k}^{t_{k+1}} u^2 v(t) dt \\
 &\leq \frac{1}{\lambda} \sum_{k=0}^{3p-1} u'(t_k)^2 \int_{t_k}^{t_{k+1}} \sin^2 \left( \lambda^{1/2} v(t_k) \int_{t_k}^t \frac{ds}{v(s)} \right) v(t) dt.
 \end{aligned} \quad (15)$$

Letting

$$\tau = \lambda^{1/2} v(t_k) \int_{t_k}^t \frac{ds}{v(s)},$$

the change of variables formula shows that

$$\begin{aligned}
 \frac{1}{\lambda} \sum_{k=0}^{3p-1} u'(t_k)^2 \int_{t_k}^{t_{k+1}} \sin^2\left(\lambda^{1/2} v(t_k) \int_{t_k}^t \frac{ds}{v(s)}\right) v(t) dt \\
 = \frac{1}{\lambda^{3/2}} \sum_{k=0}^{3p-1} \frac{u'(t_k)^2}{v(t_k)} \int_0^\pi \sin^2(\tau) v^2(\tau(t)) d\tau \\
 \leq \frac{\pi R^{2(n-1)}}{2\lambda^{3/2} r^{n-1}(t_0)} \sum_{k=0}^{3p-1} u'(t_k)^2 \\
 = C \sum_{k=0}^{3p-1} u'(t_k)^2.
 \end{aligned}
 \tag{16}$$

By (7) and (14), the following inequalities hold:

$$\begin{aligned}
 \sum_{k=0}^{3p-1} u'(t_k)^2 &\leq 3Cp u'(t_0)^2 \\
 &\leq C \sum_{k=p}^{2p-1} u'(t_k)^2.
 \end{aligned}
 \tag{17}$$

We have

$$\int_{t_0}^{t_{3p}} u^2 v dt \leq C \sum_{k=p}^{2p-1} u'(t_k)^2.
 \tag{18}$$

Here, the last inequality comes from (7), for some suitable constant  $C > 0$ . Again by the change of variables formula (this time applied to each  $\alpha_k$ ) and by Lemma 5, one sees that if  $\tilde{t}_k$  is the next zero of  $\alpha_k$  after  $t_k$  we have

$$\begin{aligned}
 \int_{t_p}^{t_{2p}} u^2 v(t) dt &= \sum_{k=p}^{2p-1} \int_{t_k}^{t_{k+1}} u^2 v(t) dt \\
 &\geq \sum_{k=p}^{2p-1} \int_{t_k}^{\tilde{t}_{k+1}} \alpha_k^2 v(t) dt \\
 &\geq C \sum_{k=p}^{2p-1} u'(t_k)^2.
 \end{aligned}
 \tag{19}$$

From (18) we conclude that

$$\int_{t_0}^{t_{3p}} u^2 r^{n-1} dt \leq C \int_{t_p}^{t_{2p}} u^2 r^{n-1} dt$$

for every  $p \in \mathbb{N}$  and for a constant  $C = C(\lambda, R)$ , independent of  $p$ .

### Appendix: Elements of Sturm–Liouville theory

For the convenience of the reader, we present some facts about Sturm–Liouville problems used in the previous section. Our motivation relies on the study of

$$(v(t)u')' + \lambda v(t)u = 0 \quad t \geq t_0 > 0, \quad (20)$$

where  $v(t) = r^{n-1}(t)$  for fixed  $n \in \mathbb{N}$ . In the following we assume the function  $r(t)$  to be positive; moreover:

- (I)  $0 < r'(t) \leq c$ .
- (II)  $\lim_{t \rightarrow \infty} r(t) = R < +\infty$ .

We start with a classical terminology.

**Definition 6.** Equation (20) is said to be oscillatory if any of its solutions has arbitrarily large zeros.

The following theorem is a practical criterion for oscillation.

**Theorem 7.** Let  $v(t)$  be a positive continuous function on  $[t_0, \infty)$  and  $\lambda > 0$ . Then, the equation

$$(v(t)u')' + \lambda v(t)u = 0$$

for  $t \geq t_0$  is oscillatory, provided  $\int_{t_0}^{\infty} v(t) dt = +\infty$  and  $\int_{t_0}^t v(t) dt \leq Ct^a$ , for some positive constants  $C$  and  $a$ .

The proof is discussed in [do Carmo and Zhou 1999, Theorem 2.1]. Since  $\lim_{t \rightarrow \infty} r(t) = R$ , we easily have the following.

**Corollary 8.** Equation (20) is oscillatory.

**Theorem 9.** For positive  $v$ , any solution  $u$  of (20) on a interval  $[t_0, t_0 + \delta]$  with initial values  $u(t_0) = x_0$  and  $u'(t_0) = x_1$  can be extended to  $[t_0, \infty)$ .

Again, the proof is presented in [do Carmo and Zhou 1999, Theorem 2.2].

The next propositions appear in the literature as Sturm comparison theorems; see [Hartman 1982, Theorem 3.1]. These are standard results, but for the sake of self-containment we decided to present their proofs. They emerge as useful ways to compare solutions for ordinary differential equations, as we did in Section 2.

**Proposition 10.** Let  $x, y$  be nontrivial solutions for

$$\begin{cases} (p(t)x')' + q(t)x = 0, \\ (p_1(t)y')' + q_1(t)y = 0, \end{cases}$$

where  $p(t) \geq p_1(t) > 0$  and  $q_1(t) \geq q(t)$  for every  $t \in I$ . If  $t_1 < t_2$  are consecutive zeros of  $x$ , then either  $y$  has a zero on  $J = (t_1, t_2)$  or there is a  $d \in \mathbb{R}$  for which  $y = dx$  on  $J$ .

*Proof.* As a starting point, note that if  $y(t_i) = 0$ , then by uniqueness we have  $y = dx$  for  $d = y'(t_i)/x'(t_i)$ . Uniqueness also implies that the set of zeroes of  $x$  does not have a cluster point, so the interval  $J$  is well-defined. Therefore, it is enough to consider the case where  $x$  and  $y$  are linearly independent. Observe that if  $y$  does not have a zero on  $J$ , then

$$\left( x \frac{(p(t)x'y - p_1(t)xy')}{y} \right)' = (q_1 - q)x^2 + (p - p_1)(x')^2 + \frac{p_1(x'y - xy')^2}{y^2}.$$

Integrating from  $t_1$  to  $t_2$ , we have

$$\int_{t_1}^{t_2} (q_1 - q)x^2 dt + \int_{t_1}^{t_2} (p - p_1)(x')^2 dt + \int_{t_1}^{t_2} p_1 \frac{(x'y - xy')^2}{y^2} dt = 0.$$

Then, if  $y$  is not multiple of  $x$ , the Wronskian  $(xy' - x'y)$  is nonzero on  $J$  and we get a contradiction with the last equation. □

As a consequence, we obtain a universal estimate from below to the distance between two consecutive zeros of a solution of (20).

**Corollary 11.** *Let  $\{t_p\}_{p=1}^\infty$  be an increasing sequence of zeros of  $u$ . There is a universal constant  $C > 0$  such that  $t_{p+1} - t_p > C$  for any  $p \in \mathbb{N}$ .*

*Proof.* Given  $p \in \mathbb{N}$ , define  $\varphi(t) = \sin(2^{(n-1)/2}\lambda^{1/2}(t - t_p))$ . Then,  $\varphi$  has a zero at  $t = t_p$  and

$$\left(\frac{1}{2}R\right)^{n-1}\varphi'' + \lambda R^{n-1}\varphi = 0.$$

Now,  $\left(\frac{1}{2}R\right)^{n-1} < v(t) < R^{n-1}$  for  $t$  sufficiently large, lets say for  $t > c_0$ . As a consequence, if  $p$  is sufficiently large, we can apply Proposition 10 for  $u$  and  $\varphi$  to conclude that the next zero of  $\varphi$  is on  $(t_p, t_{p+1})$ .

Since the next zero of  $\varphi$  after  $t_p$  is on  $t = t_p + \pi/(2^{(n-1)/2}\lambda)$ , we have

$$t_{p+1} - t_p > \frac{\pi}{2^{(n-1)/2}\lambda}$$

for  $t_p > c_0$ , from which the corollary follows. □

**Proposition 12.** *Let  $x, y$  be nontrivial solutions for*

$$\begin{cases} (p(t)x')' + q(t)x = 0, \\ (p_1(t)y')' + q_1(t)y = 0, \end{cases}$$

*on an interval  $[a, b]$ , where  $p \geq p_1 > 0$ ,  $q_1 > q$  and  $x(a) = 0$ . Suppose that  $c \in (a, b]$  is such that  $x(c) \neq 0$ ,  $y(c) \neq 0$  and  $x$  has the same number of zeros as  $y$*

on  $(a, c)$ . Then

$$\frac{p(c)x'(c)}{x(c)} \geq \frac{p_1(c)y'(c)}{y(c)}.$$

*Proof.* We only deal with the case where  $y$  is different from  $dx$ , otherwise there is nothing to prove. Let  $a = a_0, \dots, a_n$  be the zeros of  $x$  on  $[a, c)$  and  $b_0, \dots, b_{n-1}$  be the zeros of  $y$  on  $(a, c)$ . By Proposition 10, we have

$$a_i < b_i < a_{i+1}$$

for  $i = 0, \dots, n - 1$ . Consequently,  $y$  has no zero on  $(a_n, c)$ . Now, we can use the same idea from the proof of Proposition 10 to conclude that

$$\left( (px'y - p_1xy') \frac{x}{y} \right)' \geq 0$$

on  $(a_n, c)$ . Integrating both sides from  $a_n$  to  $c$  and using that  $x(a_n) = 0$ , we get

$$(px'y - p_1xy')(c) \frac{x(c)}{y(c)} \geq 0,$$

and since we can always assume that  $x(c)y(c) > 0$ , we find

$$\frac{p(c)x'(c)}{x(c)} \geq \frac{p_1y'(c)}{y(c)}. \quad \square$$

**Proof of Lemma 5.** Observe that  $\alpha_k(t_k) = 0$ ,  $\alpha'_k(t_k) = u'_k(t_k)$  and

$$(v(t)\alpha'_k)' + \lambda \frac{R^{2(n-1)}}{v(t)} \alpha_k = 0.$$

Since

$$\frac{R^{2(n-1)}}{v(t)} \geq R^{n-1} \geq v(t)$$

for all  $t \geq t_k$ , we can apply Proposition 12 to  $u$  and  $\alpha_k$  and establish that

$$\frac{u'(t)}{u(t)} \geq \frac{\alpha'_k(t)}{\alpha_k(t)}, \quad t \in (t_k, \tilde{t}_k).$$

So, taking  $\epsilon > 0$  and integrating the inequality above from  $t_k + \epsilon$  to  $t$ , we get

$$\begin{aligned} \log \left( \frac{|u(t)|}{|u(t_k + \epsilon)|} \right) &\geq \log \left( \frac{|\alpha_k(t)|}{|\alpha_k(t_k + \epsilon)|} \right), \\ \frac{|u(t)|}{|\alpha_k(t)|} &\geq \frac{|u(t_k + \epsilon)|}{|\alpha_k(t_k + \epsilon)|}. \end{aligned}$$

Sending  $\epsilon \rightarrow 0$  and using that  $u'(t_k) = \alpha'_k(t_k) \neq 0$ , we find  $|\alpha_k| \leq |u|$ .

The proof of the other inequality follows the same ideas and is omitted.

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
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