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Spectrum of the Laplacian on graphs of radial functions

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We prove that if M is a complete, noncompact hypersurface in $\mathbb{R}^{n+1}$, which is the graph of a real radial function, then the spectrum of the Laplace operator on M is the interval $[0, \infty)$.

1. Introduction

Let M be a simply connected Riemannian manifold. The Laplace operator $\Delta : C_0^\infty(M) \to C_0^\infty(M)$, defined as $\Delta = \operatorname{div} \circ \operatorname{grad}$ and acting on $C_0^\infty(M)$ (the space of smooth functions with compact support), is a second-order elliptic operator and, provided M is complete, it has a unique extension Δ to an unbounded self-adjoint operator on $L^2(M)$ whose domain is $\operatorname{Dom}(\Delta) = \{f \in L^2(M) : \Delta f \in L^2(M)\}$; see [Grigor'yan 2009, Theorem 11.5]. Since $-\Delta$ is positive and symmetric, its spectrum is the set of $\lambda \geq 0$ such that $\Delta + \lambda I$ does not have a bounded inverse. Sometimes we say "spectrum of M" rather than "spectrum of $-\Delta$", and we denote it by $\sigma(M)$. One defines the *essential spectrum*, $\sigma_{\text{ess}}(M)$, to be those λ in the spectrum which are either accumulation points of the spectrum or eigenvalues of infinite multiplicity. The *discrete spectrum* is the set $\sigma_d = \sigma(M) \setminus \sigma_{\text{ess}}(M)$ of all eigenvalues of finite multiplicity which are isolated points of the spectrum.

There is a vast literature on the spectrum of the Laplace operator on complete noncompact manifolds. The first result we mention was published by Tayoshi [1971]. He showed the absence of eigenvalues of $-\Delta$ for a class of surfaces of revolution, determined by nonnegative radial growth.

Donnelly [1981] showed

$$\sigma_{\text{ess}}(M) = \big[(n-1)^2 \tfrac{1}{4} c^2, \infty\big),$$

provided M is a Hadamard manifold whose sectional curvature approaches $-c^2$ at infinity. Karp [1984] gave sufficient conditions for a class of manifolds to have

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purely continuous spectrum ($\sigma_d(M)=\varnothing$) under some curvature conditions. Eight years later, Donnelly and Garofalo [1992] obtained results in a similar direction, using the hypothesis of nonnegative radial sectional curvature, without restrictions on the metric.

Cheng and Zhiqin Lu [1992] proved $\sigma_{\text{ess}}(M)=[0,\infty)$ when M has nonnegative radial sectional curvature and Li [1994] proved $\sigma_{\text{ess}}(M)=[0,\infty)$, provided M has nonnegative Ricci curvatures and a pole. Zhou [1994] proved $\sigma_{\text{ess}}(M)=[0,\infty)$ when M has nonnegative sectional curvatures, generalizing the work of Escobar and Freire [1992].

Kumura [1997] found a result which generalized [Donnelly 1981]. He showed $\sigma_{\text{ess}}(M)=\left[\frac{1}{4}c^2,\infty\right)$ whenever

$$\lim_{n\to\infty}\sup_{t>n}|\Delta t-c|=0,$$

where t denotes the distance function on M.

Wang [1997] showed that the spectrum of a complete, noncompact Riemannian manifold with asymptotically nonnegative Ricci curvature is equal to $[0,\infty)$.

Zhiqin Lu and Detang Zhou [2011] proved that the L^p essential spectrum of M is equal to $[0,\infty)$ when

$$\liminf_{x\to\infty}\operatorname{Ric}_M(x)=0$$

and M is noncompact and complete. We should mention here that almost all the above works were strongly motivated by the decomposition principle [Donnelly and Li 1979], which states that the essential spectrum of a Riemannian manifold is invariant under compact perturbations of the metric, thus it is a function of the geometry of the ends. In [Monte and Montenegro 2015], it was proved that $\sigma_{\text{ess}}(M)\supset\left[(n-1)^2\frac{1}{4}c^2,\infty\right)$ for a class of Riemannian manifolds, not necessarily complete, whose metric is given by

$$g_M=dr^2+\psi^2(rw)g_{\mathbb{S}^{n-1}},$$

using curvature conditions only in a neighborhood of a ray.

See also [Bessa et al. 2010; 2012; 2015; Donnelly and Li 1979; Kleine 1988; 1989; Tayoshi 1971] for geometric conditions implying the discreteness of the spectrum, $\sigma_{\text{ess}}(M)=\varnothing$.

In this work we consider complete hypersurfaces which are graphs of radial functions. Our main result is the following theorem.

Theorem 1. *Let M be a complete hypersurface in $\mathbb{R}^{n+1}$, which is the graph of a real radial function. Then, the spectrum of the Laplace operator on M is* $[0,\infty)$.

Without loss of generality, we may assume the domain $\operatorname{Dom} f$ to be connected and symmetric with respect to $0\in\mathbb{R}^n$. From the completeness of M we further deduce Dom f is an open ball or annulus. The theorem above allows us to construct a bounded hypersurface with the same spectrum of $\mathbb{R}^{n+1}$ by taking M to be the graph of the real function $f(x) = \cos\big(\tan\big(\frac{1}{2}\pi|x|\big)\big)$ defined on the unit open ball.

Throughout the following discussion, for simplicity, we deal with the case where $f : D \to \mathbb{R}$ is defined in an open ball. Let $X : [0, R) \times \Omega \to D$ be defined by $X(r, x_1, \ldots, x_{n-1}) = rw(x_1, \ldots, x_{n-1})$, where $0 < R \leq +\infty$ and w is a coordinate system on S^{n-1} defined on an open set Ω of $\mathbb{R}^n$. Note that M has a natural coordinate system $Y : [0, R) \times \Omega \to M$, given by $Y(r, x_1, \ldots, x_{n-1}) = (rw(x_1, \ldots, x_{n-1}), f(r))$, but we are interested in the spherical coordinate system for M on $p = (0, f(0))$. Consider $t : [0, R) \to [0, \infty)$, given by

$$t(r) = \int_0^r \big(1 + f'(\tau)^2\big)^{1/2}\, d\tau.$$

We claim that t is a diffeomorphism. Observe that t is increasing and

$$\lim_{r \to R} t(r) = +\infty.$$

We denote by $r : [0, \infty) \to [0, R)$ the inverse diffeomorphism. By the inverse function theorem,

$$0 < r'(t) = \big(1 + f'(r)^2\big)^{-1/2} \leq 1. \tag{1}$$

Finally, the system of spherical coordinates on M, denoted $Z : [0, \infty) \times \Omega \to M$, is defined by

$$Z(t, x_1, \ldots, x_{n-1}) = \big(r(t)w(x_1, \ldots, x_{n-1}),\, f \circ r(t)\big).$$

The metric of M on such a system is given by

$$g_M = dt^2 + r(t)^2 g_{\mathbb{S}^{n-1}}.$$

Because of this observation, Theorem 1 is a simple consequence of the theorem below.

Theorem 2. *Let $I \subset \mathbb{R}$ be an unbounded interval and $M = I \times \mathbb{S}^{n-1}$ with metric given by $g_M = dt^2 + r^2(t) g_{\mathbb{S}^{n-1}}$, where $0 < r'(t) \leq c$ for all t. Then, the spectrum of the Laplace operator on M is* $[0, \infty)$.

Remark. (1) If M has a pole at $p \in M$, then $\exp_p : T_pM \to M$ is a diffeomorphism so that M isometric to T_pM with the pullback metric. Therefore, Theorem 2 implies that if M has a pole p and $g_M = dt^2 + r^2(t) g_{\mathbb{S}^{n-1}}$ with respect to p and $0 < r'(t) < c$, then M has spectrum equal to $[0, \infty)$.

(2) To the best of our knowledge, this natural result has only been verified in less general settings. For instance, since $r'(t) > 0$, then $r(t)$ is increasing and there are only two possibilities:

(a) $\lim_{t\to\infty} r(t) = \infty$, or
(b) $\lim_{t\to\infty} r(t) = R$.

In the first case, since $r'(t)$ is bounded, we have

$$\lim_{t\to\infty} \Delta t = \lim_{t\to\infty} \frac{r'(t)}{r(t)} = 0.$$

By [Kumura 1997, Theorem 1.2], it follows that the spectrum of M is purely continuous and equal to $[0, \infty)$. In the second case, if $r' \to 0$ we still have $r'(t)/r(t) \to 0$. Therefore, the main contribution of this paper is the proof of the case where $r'(t)$ does not converge to zero and $\lim_{t\to\infty} r(t) = R < +\infty$. This is the scenario for the graph of the function $f(x) = \cos\bigl(\tan\bigl(\frac{1}{2}\pi|x|\bigr)\bigr)$ presented above.

In the next section we prove Theorem 2. The Appendix is devoted to the Sturm–Liouville theory used in this note.

2. Proof of Theorem 2

We concentrate our efforts for the case where $\lim_{t\to\infty} r(t) = R$. Our approach is variational, based on the following lemma.

Lemma 3 [Davies 1995, Lemma 4.1.2]. *A number $\lambda \in \mathbb{R}$ lies in the spectrum of a self-adjoint operator H if and only if there exists a sequence of functions $f_n \in \operatorname{Dom} H$ with $\|f_n\| = 1$ such that*

$$\lim_{n\to\infty} \|Hf_n - \lambda f_n\| = 0.$$

To deduce Theorem 2 from Lemma 3 we will construct, for each $\lambda > 0$, a sequence of radial smooth functions $f_p : M \to \mathbb{R}$ with compact support such that

$$\|\Delta f_p + \lambda f_p\|_{L^2(M)} \le \frac{c}{p}\|f_p\|_{L^2(M)} \tag{2}$$

for any natural p, where c is a constant which does not depend on p. It will follow that $g_p = f_p/\|f_p\|$ has norm one and

$$\lim_{p\to\infty} \|\Delta g_p + \lambda g_p\|_{L^2(M)} = 0.$$

Therefore, by Lemma 3, λ belongs to the spectrum. To construct the function f_p, we fix $t_0 > 0$ and prove that there are $t_1(\lambda) > t_0$ and a radial function $u = u(t)$ solution of the problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } [t_0, t_1], \\ u(t_0) = u(t_1) = 0, & \\ u > 0 & \text{in } (t_0, t_1). \end{cases} \tag{3}$$

Using Sturm–Liouville theory, we showed that u can be extended to the whole interval $[t_0, \infty)$ and it has infinite zeros $t_0 < t_1 < \cdots < t_p < \cdots$. The next step is to consider (for each p) a smooth bump function h_p whose support is the interval $[t_0, t_{3p}]$. We then define $f_p = uh_p$ and show that each f_p in this sequence satisfies (2). The function $t \mapsto r^{n-1}(t)$ has a geometric meaning and plays an important role in the proof, thus deserving a special notation. In the sequence of the paper, we let $v(t) = r^{n-1}(t)$.

We observe that the first equation in (3) is equivalent to

$$(v(t)u'(t))' + \lambda v(t)u(t) = 0 \tag{4}$$

if $u = u(t)$ is a radial function. By Theorem 9 in the Appendix, given positive t_0 and λ, (4) has a solution defined on $[t_0, \infty)$ and satisfying $u(t_0) = 0$.

Moreover, Corollary 8 allows us to consider a sequence of zeros $t_0 < t_1 < \cdots$ of u.

For $p \in \mathbb{N}$, we choose a smooth bump function $h = h_p : \mathbb{R} \mapsto \mathbb{R}$ with $0 \le h \le 1$ satisfying

$$\begin{cases} h(t) = 0, & t \in (-\infty, t_0] \cup [t_{3p}, \infty), \\ h(t) = 1, & t \in [t_p, t_{2p}]. \end{cases}$$

Such a function can be defined in the following way: let $\varphi \in C_0^\infty(\mathbb{R})$ be nonnegative with $\operatorname{supp} \varphi = [0, 1]$ and $\int \varphi = 1$. Let

$$h_p(t) = \int_{-\infty}^{t} \varphi_p(s)\, ds,$$

where

$$\varphi_p(t) = \frac{1}{t_p - t_0} \varphi\left(\frac{t - t_0}{t_p - t_0}\right) - \frac{1}{t_{3p} - t_{2p}} \varphi\left(\frac{t - t_{2p}}{t_{3p} - t_{2p}}\right).$$

This construction is useful since it leads to the following estimates:

$$\begin{aligned} \|h_p'\|_\infty &\le \max\left\{\frac{\|\varphi\|_\infty}{t_p - t_0}, \frac{\|\varphi\|_\infty}{t_{3p} - t_{2p}}\right\} \le \frac{C}{p}, \\ \|h_p''\|_\infty &\le \max\left\{\frac{\|\varphi'\|_\infty}{(t_p - t_0)^2}, \frac{\|\varphi'\|_\infty}{(t_{3p} - t_{2p})^2}\right\} \le \frac{C}{p^2}. \end{aligned} \tag{5}$$

Here, we have made use of Corollary 11 in the Appendix.

Consider $f = f_p = uh_p$. We are going to prove that such a function satisfies the inequality in (2). Computing $\Delta f + \lambda f$, we obtain

$$\Delta f + \lambda f = 2u'h' + uh'' + (n-1)\frac{r'}{r}h'u.$$

Using the inequalities in (5), together with the fact that r is increasing and r' is bounded, we have

$$|\Delta f+\lambda f|\le \frac{c}{p}(|u'|+|u|)\chi_{[t_0,t_{3p}]}.$$

Then,

$$|\Delta f+\lambda f|^2\le \frac{c}{p^2}(|u'|^2+|u|^2)\chi_{[t_0,t_{3p}]},$$

$$\int_M |\Delta f+\lambda f|^2\,dM\le \frac{c}{p^2}\biggl(\int_{t_0}^{t_{3p}}|u'|^2 v\,dt+\int_{t_0}^{t_{3p}}|u|^2 v\,dt\biggr).$$

Multiplying (4) by u and using integration by parts we find

$$\int_{t_0}^{t_{3p}}|u'|^2 v(t)\,dt=\lambda\int_{t_0}^{t_{3p}}|u|^2 v(t)\,dt,$$

$$\|\Delta f_p+\lambda f_p\|_{L^2(M)}\le \frac{c}{p}\|u\cdot\chi_{[t_0,t_{3p}]}\|_{L^2(M)}\le \frac{c}{p}\|u\cdot\chi_{[t_p,t_{2p}]}\|_{L^2(M)}\le \frac{c}{p}\|f_p\|_{L^2(M)},$$

where the second inequality comes from Lemma 4 below.

Lemma 4. *There is a positive constant C independent on p such that*

$$\int_{t_0}^{t_{3p}}u^2 v\,dt\le C\int_{t_p}^{t_{2p}}u^2 v\,dt,$$

where u is solution of (4) and $t_0<t_1<\cdots$ are zeros of u.

This result is a manifestation of the oscillatory behavior of u. Before justifying its veracity, we state a useful way of estimating u between two zeros.

Lemma 5. *Let u be a solution of (4), and choose t_k, t_{k+1} to be consecutive zeros for u. Define*

$$\alpha_k(t)=a_k\sin\biggl(\lambda^{1/2}R^{n-1}\int_{t_k}^{t}v^{-1}(s)\,ds\biggr)$$

and

$$\beta_k(t)=b_k\sin\biggl(\lambda^{1/2}v(t_k)\int_{t_k}^{t}v^{-1}(s)\,ds\biggr),$$

where $a_k=v(t_k)b_k/(R^{n-1}\lambda^{1/2})$ and $b_k=u'(t_k)/\lambda^{1/2}$. Then $|\alpha_k|\le|u|$ on $(t_k,\tilde t_k)$ and $|u|\le|\beta_k|$ on (t_k,t_{k+1}), where $\tilde t_k$ is the next zero of α_k after t_k.

To make the exposition more fluid, we postpone the proof until the Appendix.

Proof of Lemma 4. Observe that multiplying (4) by $v(t)u'$ we get

$$(v(t)u')'v(t)u'+\lambda v^2uu'=0,$$

and so,

$$\bigl((v(t)u')^2\bigr)'+\lambda v^2(u^2)'=0.$$

Integrating from t_0 to t_k, we have

$$v(t_k)^2u'(t_k)^2 - v(t_0)^2u'(t_0)^2 = -\lambda \int_{t_0}^{t_k} v^2(s)(u^2(s))'\,ds.$$

Integrating the right hand side by parts, we find

$$v(t_k)^2u'(t_k)^2 - v(t_0)^2u'(t_0)^2 = 2\lambda \int_{t_0}^{t_k} vv'u^2\,ds. \tag{6}$$

Since $r, r' > 0$, we have $v, v' > 0$. Also, $r(t) < R$ and as a consequence,

$$u'(t_k)^2 > \frac{v(t_0)^2u'(t_0)^2}{R^{2(n-1)}} \tag{7}$$

for $k \geq 1$.

To obtain an estimate in the other direction, we observe that the function $\beta = \beta_0(t)$ in Lemma 5 satisfies $\beta'(t_0) = u'(t_0) > 0$ and

$$(v(t)\beta'(t))' + \frac{\lambda v(t_0)^2}{v(t)}\beta(t) = 0. \tag{8}$$

Multiplying by $v(t)\beta'$ we get, as in the preceding computations,

$$(v(t)^2(\beta')^2)' + \lambda v(t_0)^2(\beta^2)' = 0. \tag{9}$$

Now, if $\bar{t_1}$ is the next root of β after t_0, integrating the last equation we find

$$\begin{aligned} v(\bar{t_1})^2\beta'(\bar{t_1})^2 &= v(t_0)^2\beta'(t_0)^2 \\ &= v(t_0)^2u'(t_0)^2. \end{aligned} \tag{10}$$

We take $k = 1$ and estimate the right side of (6) as follows:

$$\begin{aligned} \lambda \int_{t_0}^{t_1} (v^2)'u^2\,dt &\leq \lambda \int_{t_0}^{t_1} (v^2)'\beta^2\,dt \\ &\leq \lambda \int_{t_0}^{\bar{t_1}} (v^2)'\beta^2\,dt \\ &= -\lambda \int_{t_0}^{\bar{t_1}} v^2(\beta^2)'\,dt \\ &= -\frac{1}{v(t_0)^2} \int_{t_0}^{\bar{t_1}} v^2(\lambda v(t_0)^2\beta^2)'\,dt. \end{aligned} \tag{11}$$

By (9) we infer

$$\begin{aligned}
-\frac{1}{v(t_0)^2}\int_{t_0}^{\bar{t_1}} v^2(\lambda v(t_0)^2\beta^2)'\,dt &= \frac{1}{v(t_0)^2}\int_{t_0}^{\bar{t_1}} v^2(v^2(\beta')^2)'\,dt \\
&= \frac{1}{v(t_0)^2}\int_{t_0}^{\bar{t_1}} (v^4(\beta')^2)' - (v^2)'v^2(\beta')^2\,dt \qquad (12)\\
&< \frac{v^4(\bar{t_1})(\beta')^2(\bar{t_1}) - v^4(t_0)(\beta')^2(t_0)}{v(t_0)^2}.
\end{aligned}$$

Now, using (10) and that $\beta'(t_0) = u'(t_0)$, we find

$$\lambda\int_{t_0}^{t_1}(v^2)'u^2 \le (v(\bar{t_1})^2 - v(t_0)^2)u'(t_0)^2\,dt.$$

Then, by (6),

$$v(t_1)^2u'(t_1)^2 - v(t_0)^2u'(t_0)^2 \le \big(v(\bar{t_1})^2 - v(t_0)^2\big)u'(t_0)^2.$$

Since $v(t)$ is increasing, it follows that

$$\begin{aligned}
v(t_1)^2u'(t_1)^2 &\le v(\bar{t_1})^2u'(t_0)^2 \\
&\le v(t_2)^2u'(t_0)^2.
\end{aligned} \qquad (13)$$

Then,

$$u'(t_1)^2 \le \frac{v(t_2)^2}{v(t_0)^2}u'(t_0)^2.$$

Using the same argument, one shows by induction that

$$u'(t_k)^2 \le \frac{v(t_{k+1})^2v(t_k)^2}{v(t_1)^2v(t_0)^2}u'(t_0)^2.$$

Since $r(t) < R$, we find that

$$u'(t_k)^2 \le \frac{R^{4(n-1)}}{v(t_0)^2v(t_1)^2}u'(t_0)^2. \qquad (14)$$

Now, using Lemma 5, it's easy to check that

$$\begin{aligned}
\int_{t_0}^{t_{3p}} u^2v\,dt &= \sum_{k=0}^{3p-1}\int_{t_k}^{t_{k+1}} u^2v(t)\,dt \\
&\le \frac{1}{\lambda}\sum_{k=0}^{3p-1} u'(t_k)^2\int_{t_k}^{t_{k+1}} \sin^2\left(\lambda^{1/2}v(t_k)\int_{t_k}^{t}\frac{ds}{v(s)}\right)v(t)\,dt.
\end{aligned} \qquad (15)$$

Letting

$$\tau = \lambda^{1/2}v(t_k)\int_{t_k}^{t}\frac{ds}{v(s)},$$

the change of variables formula shows that

$$\begin{aligned}\frac{1}{\lambda}\sum_{k=0}^{3p-1} u'(t_k)^2 \int_{t_k}^{t_{k+1}} \sin^2\Big(\lambda^{1/2} v(t_k)\int_{t_k}^{t}\frac{ds}{v(s)}\Big) v(t)\,dt \\ = \frac{1}{\lambda^{3/2}}\sum_{k=0}^{3p-1}\frac{u'(t_k)^2}{v(t_k)}\int_0^{\pi}\sin^2(\tau)v^2(\tau(t))\,d\tau \\ \le \frac{\pi R^{2(n-1)}}{2\lambda^{3/2} r^{n-1}(t_0)}\sum_{k=0}^{3p-1} u'(t_k)^2 \\ = C\sum_{k=0}^{3p-1} u'(t_k)^2.\end{aligned} \tag{16}$$

By (7) and (14), the following inequalities hold:

$$\begin{aligned}\sum_{k=0}^{3p-1} u'(t_k)^2 &\le 3Cpu'(t_0)^2 \\ &\le C\sum_{k=p}^{2p-1} u'(t_k)^2.\end{aligned} \tag{17}$$

We have

$$\int_{t_0}^{t_{3p}} u^2 v\,dt \le C\sum_{k=p}^{2p-1} u'(t_k)^2. \tag{18}$$

Here, the last inequality comes from (7), for some suitable constant $C > 0$. Again by the change of variables formula (this time applied to each α_k) and by Lemma 5, one sees that if $\tilde{t}_k$ is the next zero of α_k after t_k we have

$$\begin{aligned}\int_{t_p}^{t_{2p}} u^2 v(t)\,dt &= \sum_{k=p}^{2p-1}\int_{t_k}^{t_{k+1}} u^2 v(t)\,dt \\ &\ge \sum_{k=p}^{2p-1}\int_{t_k}^{\tilde{t}_{k+1}} \alpha_k^2 v(t)\,dt \\ &\ge C\sum_{k=p}^{2p-1} u'(t_k)^2.\end{aligned} \tag{19}$$

From (18) we conclude that

$$\int_{t_0}^{t_{3p}} u^2 r^{n-1}\,dt \le C\int_{t_p}^{t_{2p}} u^2 r^{n-1}\,dt$$

for every $p \in \mathbb{N}$ and for a constant $C = C(\lambda, R)$, independent of p.

Appendix: Elements of Sturm–Liouville theory

For the convenience of the reader, we present some facts about Sturm–Liouville problems used in the previous section. Our motivation relies on the study of

$$(v(t)u')' + \lambda v(t)u = 0 \quad t \geq t_0 > 0, \tag{20}$$

where $v(t) = r^{n-1}(t)$ for fixed $n \in \mathbb{N}$. In the following we assume the function $r(t)$ to be positive; moreover:

(I) $0 < r'(t) \leq c$.

(II) $\lim_{t\to\infty} r(t) = R < +\infty$.

We start with a classical terminology.

Definition 6. Equation (20) is said to be oscillatory if any of its solutions has arbitrarily large zeros.

The following theorem is a practical criterion for oscillation.

Theorem 7. *Let $v(t)$ be a positive continuous function on $[t_0, \infty)$ and $\lambda > 0$. Then, the equation*

$$(v(t)u')' + \lambda v(t)u = 0$$

for $t \geq t_0$ is oscillatory, provided $\int_{t_0}^{\infty} v(t)\,dt = +\infty$ and $\int_{t_0}^{t} v(t)\,dt \leq Ct^a$, for some positive constants C and a.

The proof is discussed in [do Carmo and Zhou 1999, Theorem 2.1]. Since $\lim_{t\to\infty} r(t) = R$, we easily have the following.

Corollary 8. *Equation (20) is oscillatory.*

Theorem 9. *For positive v, any solution u of (20) on a interval $[t_0, t_0 + \delta]$ with initial values $u(t_0) = x_0$ and $u'(t_0) = x_1$ can be extended to $[t_0, \infty)$.*

Again, the proof is presented in [do Carmo and Zhou 1999, Theorem 2.2].

The next propositions appear in the literature as Sturm comparison theorems; see [Hartman 1982, Theorem 3.1]. These are standard results, but for the sake of self-containment we decided to present their proofs. They emerge as useful ways to compare solutions for ordinary differential equations, as we did in Section 2.

Proposition 10. *Let x, y be nontrivial solutions for*

$$\begin{cases} (p(t)x')' + q(t)x = 0, \\ (p_1(t)y')' + q_1(t)y = 0, \end{cases}$$

where $p(t) \geq p_1(t) > 0$ and $q_1(t) \geq q(t)$ for every $t \in I$. If $t_1 < t_2$ are consecutive zeros of x, then either y has a zero on $J = (t_1, t_2)$ or there is a $d \in \mathbb{R}$ for which $y = dx$ on J.

Proof. As a starting point, note that if $y(t_i) = 0$, then by uniqueness we have $y = dx$ for $d = y'(t_i)/x'(t_i)$. Uniqueness also implies that the set of zeroes of x does not have a cluster point, so the interval J is well-defined. Therefore, it is enough to consider the case where x and y are linearly independent. Observe that if y does not have a zero on J, then

$$\left(x\frac{(p(t)x'y - p_1(t)xy')}{y}\right)' = (q_1 - q)x^2 + (p - p_1)(x')^2 + \frac{p_1(x'y - xy')^2}{y^2}.$$

Integrating from t_1 to t_2, we have

$$\int_{t_1}^{t_2}(q_1 - q)x^2\,dt + \int_{t_1}^{t_2}(p - p_1)(x')^2\,dt + \int_{t_1}^{t_2} p_1\frac{(x'y - xy')^2}{y^2}\,dt = 0.$$

Then, if y is not multiple of x, the Wronskian $(xy' - x'y)$ is nonzero on J and we get a contradiction with the last equation. □

As a consequence, we obtain a universal estimate from below to the distance between two consecutive zeros of a solution of (20).

Corollary 11. *Let $\{t_p\}_{p=1}^{\infty}$ be an increasing sequence of zeros of u. There is a universal constant $C > 0$ such that $t_{p+1} - t_p > C$ for any $p \in \mathbb{N}$.*

Proof. Given $p \in \mathbb{N}$, define $\varphi(t) = \sin(2^{(n-1)/2}\lambda^{1/2}(t - t_p))$. Then, φ has a zero at $t = t_p$ and

$$\left(\tfrac{1}{2}R\right)^{n-1}\varphi'' + \lambda R^{n-1}\varphi = 0.$$

Now, $\left(\frac{1}{2}R\right)^{n-1} < v(t) < R^{n-1}$ for t sufficiently large, lets say for $t > c_0$. As a consequence, if p is sufficiently large, we can apply Proposition 10 for u and φ to conclude that the next zero of φ is on (t_p, t_{p+1}).

Since the next zero of φ after t_p is on $t = t_p + \pi/(2^{(n-1)/2}\lambda)$, we have

$$t_{p+1} - t_p > \frac{\pi}{2^{(n-1)/2}\lambda}$$

for $t_p > c_0$, from which the corollary follows. □

Proposition 12. *Let x, y be nontrivial solutions for*

$$\begin{cases}(p(t)x')' + q(t)x = 0,\\ (p_1(t)y')' + q_1(t)y = 0,\end{cases}$$

on an interval $[a, b]$, where $p \geq p_1 > 0$, $q_1 > q$ and $x(a) = 0$. Suppose that $c \in (a, b]$ is such that $x(c) \neq 0$, $y(c) \neq 0$ and x has the same number of zeros as y

on (a, c). *Then*

$$\frac{p(c)x'(c)}{x(c)} \geq \frac{p_1(c)y'(c)}{y(c)}.$$

Proof. We only deal with the case where y is different from dx, otherwise there is nothing to prove. Let $a = a_0, \dots, a_n$ be the zeros of x on $[a, c)$ and $b_0, \dots, b_{n-1}$ be the zeros of y on (a, c). By Proposition 10, we have

$$a_i < b_i < a_{i+1}$$

for $i = 0, \dots, n-1$. Consequently, y has no zero on (a_n, c). Now, we can use the same idea from the proof of Proposition 10 to conclude that

$$\Big((px'y - p_1xy')\frac{x}{y}\Big)' \geq 0$$

on (a_n, c). Integrating both sides from a_n to c and using that $x(a_n) = 0$, we get

$$(px'y - p_1xy')(c)\frac{x(c)}{y(c)} \geq 0,$$

and since we can always assume that $x(c)y(c) > 0$, we find

$$\frac{p(c)x'(c)}{x(c)} \geq \frac{p_1y'(c)}{y(c)}.$$

□

Proof of Lemma 5. Observe that $\alpha_k(t_k) = 0$, $\alpha_k'(t_k) = u_k'(t_k)$ and

$$(v(t)\alpha_k')' + \lambda\frac{R^{2(n-1)}}{v(t)}\alpha_k = 0.$$

Since

$$\frac{R^{2(n-1)}}{v(t)} \geq R^{n-1} \geq v(t)$$

for all $t \geq t_k$, we can apply Proposition 12 to u and α_k and establish that

$$\frac{u'(t)}{u(t)} \geq \frac{\alpha_k'(t)}{\alpha_k(t)}, \qquad t \in (t_k, \tilde{t}_k).$$

So, taking $\epsilon > 0$ and integrating the inequality above from $t_k + \epsilon$ to t, we get

$$\log\bigg(\frac{|u(t)|}{|u(t_k+\epsilon)|}\bigg) \geq \log\bigg(\frac{|\alpha_k(t)|}{|\alpha_k(t_k+\epsilon)|}\bigg),$$
$$\frac{|u(t)|}{|\alpha_k(t)|} \geq \frac{|u(t_k+\epsilon)|}{|\alpha_k(t_k+\epsilon)|}.$$

Sending $\epsilon \to 0$ and using that $u'(t_k) = \alpha_k'(t_k) \neq 0$, we find $|\alpha_k| \leq |u|$.

The proof of the other inequality follows the same ideas and is omitted.

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