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The H-linked degree-sum parameter for special graph families

Lydia East Kenney and Jeffrey Scott Powell

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For a fixed graph H, a graph G is H-linked if any injection $f : V(H) \to V(G)$ can be extended to an H-subdivision in G. The concept of H-linked generalizes several well-known graph theory concepts such as k-connected, k-linked, and k-ordered. In 2012, Ferrara et al. proved a sharp σ_2 (or degree-sum) bound for a graph to be H-linked. In particular, they proved that any graph G with $n > 20|E(H)|$ vertices and $\sigma_2(G) \geq n + a(H) - 2$ is H-linked, where $a(H)$ is a parameter maximized over certain partitions of $V(H)$. However, they do not discuss the calculation of $a(H)$ in their work. In this paper, we prove the exact value of $a(H)$ in the cases when H is a path, a cycle, a union of stars, a complete graph, and a complete bipartite graph. Several of these results lead to new degree-sum conditions for particular graph classes while others provide alternate proofs of previously known degree-sum conditions.

1. Introduction

We only consider finite, undirected graphs. Let G and H be graphs with vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$, respectively. Let $\mathcal{P}(G)$ denote the set of paths in G. An *H-subdivision* in G is a pair of mappings $f_1 : V(H) \to V(G)$ and $f_2 : E(H) \to \mathcal{P}(G)$ such that:

(i) f_1 is injective.

(ii) For every edge $xy \in E(H)$, the image $f_2(xy)$ in a path in G from $f_1(x)$ to $f_1(y)$ and distinct edges of H map to internally disjoint paths in G.

Note that the existence of an H-subdivision in G means that H is a topological minor of G and, as a result, H is also a minor of G. See Figure 1 for an illustration of an H-subdivision.

A graph G is *H-linked* if any injection $f : V(G) \to V(H)$ can be extended to an H-subdivision. The concept of H-linked was introduced in [Jung 1970], and

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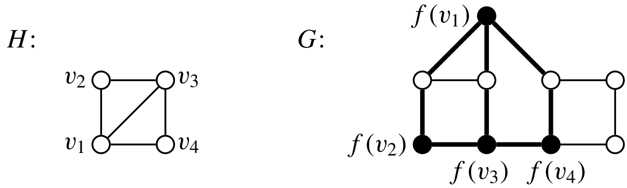

Figure 1. An H-subdivision: the vertices $v_1, v_2, v_3, \ldots, v_4$ of H are mapped via an injection f to vertices in G. The subgraph in G induced by the thick edges and the vertices incident with these edges is an H-subdivision in G.

for appropriate choices of H with $|V(H)| = k$, H-linked generalizes several graph properties including k-connected, k-linked, and k-ordered.

Several recent publications have proven degree conditions for a graph to be H-linked. In [Ferrara et al. 2006; Gould et al. 2006; Kostochka and Yu 2005], sharp minimum degree conditions were proved. Degree-sum conditions were proved in [Kostochka and Yu 2008; Ferrara et al. 2012], and as this paper examines a parameter related to these conditions, we will examine them in further detail. Let $\sigma_2(G)$ denote the minimum degree sum of nonadjacent vertices in G. The minimum degree sum required to guarantee the existence of a property is known as a degree-sum condition or a σ_2 condition. Kostochka and Yu [2008] proved a sharp σ_2 condition for G to be H-linked for every graph H with minimum degree at least two.

Theorem 1.1 [Kostochka and Yu 2008]. *Let G be a graph of order n and let H be a simple graph with k edges and minimum degree at least two. If*

$$\sigma_2(G) \geq \begin{cases} \left\lceil n + \frac{1}{2}(3k-9) \right\rceil, & n > 2.5k - 5.5, \\ \left\lceil n + \frac{1}{2}(3k-8) \right\rceil, & 2k \leq n \leq 2.5k - 5.5, \\ 2n-3, & k \leq 2.5k - 1, \end{cases}$$

then G is H-linked.

Note that Theorem 1.1 provides an upper bound on the minimum degree-sum required for any possible H with minimum degree at least two, but it does not supply the optimal bound for every choice of H. A sharp σ_2 bound for this latter case was proved by Ferrara et al. [2012]. Their bound is a function of a parameter of H, called $a(H)$, that is maximized over certain partitions of $V(H)$ into two nonempty sets A and B. We use (A, B) to denote a specific partition of $V(H)$ into these two sets. Let $e(A, B)$ denote the number of edges with one vertex in A and one vertex in B. We will say that these edges "cross the partition". For a vertex v, we let $d_B(v)$ denote the number of neighbors of v in B. For the partition of H given by (A, B), let $\Delta_B(A)$ equal the maximum value of $d_B(v)$ for all $v \in A$.

We are now ready to define $a(H)$. Let

$$a(H) = \max_{\substack{A \cup B = V(H)\\ e(A,B) \geq 1}} \big(e(A, B) + |B| - \Delta_B(A)\big).$$

Using $a(H)$, one can find a sharp $\sigma_2(G)$ condition for G to be H-linked:

Theorem 1.2 [Ferrara et al. 2012]. *Let H be a simple graph and G be a graph on n vertices with $n > 20|E(H)|$. If*

$$\sigma_2(G) \geq n + a(H) - 2,$$

then G is H-linked. This result is sharp.

The same paper also gave a sharp $\sigma_2(G)$ bound for when H is a multigraph. However, in this paper, we restrict our attention to the case when H is a graph. Ferrara et al. [2012] assert that, for particular choices of H, Theorem 1.2 has (as corollaries) the previously proven σ_2 conditions for k-linked and k-ordered. However, no formal proof for these assertions is included and no further examination of the parameter $a(H)$ is presented for any particular H.

In this paper, we prove the value of $a(H)$ when H is a path, cycle, union of stars, complete graph, or complete bipartite graph. Some of these proofs specify new σ_2 conditions while others provide alternate proofs of well-known conditions. One of our aims is to supply some initial results for $a(H)$, as Theorem 1.2 could potentially be a useful tool when routing specific paths between arbitrarily chosen vertices. Additionally, we hope that these initial results for $a(H)$ encourage further study of this unusual parameter. To that end, two examples are given in the conclusion to illustrate some surprising properties of $a(H)$.

To continue, we need some further notation. For a given graph H, let $\mathbb{P}(H)$ be the set of all possible partitions of $V(H)$ into two nonempty sets with at least one edge of H that crosses the partition. For a partition $(A, B) \in \mathbb{P}(H)$, let $a(A, B) = e(A, B) + |B| - \Delta_B(A)$. Thus,

$$a(H) = \max_{(A,B) \in \mathbb{P}(H)} a(A, B).$$

For a partition (A, B), we say that F is an *induced subpartition* of H if F is an induced subgraph of H and the vertices of F are partitioned in the exact same manner in which they were partitioned in H. Note that it is possible for an induced subpartition not to have any edges that cross the partition. See Figure 2 for an illustration of these terms.

Additionally, note that for a partition (A, B), we will often speak of "moving" a vertex from A to B or from B to A. In that language, the labels A and B refer to the two sides of the partition in addition to the sets themselves. For terms and notation not defined here, see [West 1996].

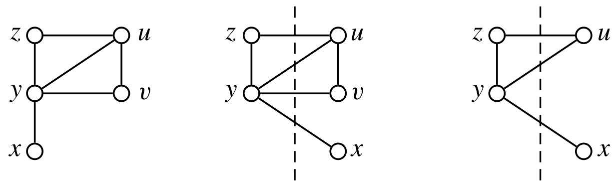

Figure 2. Suppose the graph shown on the left is H. The partition $(A, B) \in \mathbb{P}(H)$ with $A = \{y, z\}$ and $B = \{u, v, x\}$ is illustrated in the center. The vertical line is a visual aid to distinguish between the sets A and B. Note that in this case, $a(A, B) = 4$. The graph on the right is an induced subpartition of the partition (A, B).

2. Lemmas

To start, we prove two lemmas regarding the structure of optimal partitions of H, i.e., partitions $(A, B) \in \mathbb{P}(H)$ for which $a(A, B) = a(H)$. The first lemma notes that certain subpartitions cannot be induced subpartitions of an optimal partition of H.

Let H_1 be the induced subpartition consisting of an induced path of length two with all three vertices in A. Let H_2 be the induced subpartition consisting of an induced path of length three with one edge in A, one edge that crosses the partition, and one edge in B. See Figure 3 for H_1 and H_2.

This first lemma proves that H_1 and H_2 cannot be induced subpartitions in any optimal partition of the graph H.

Lemma 2.1. *Let H be any graph. Suppose $(A, B) \in \mathbb{P}(H)$ with $a(A, B) = a(H)$. Then, H_1 and H_2 are not induced subpartitions in (A, B).*

Proof. Suppose for the sake of contradiction that H_1 is an induced subpartition of (A, B). Let $x, y, z \in A$ be the vertices of H_1 with $d(y) = 2$. Also, let

$$\xi = \begin{cases} 1 & \text{if } \Delta_B(A) = d_B(x) \text{ or } \Delta_B(A) = d_B(z), \\ 0 & \text{otherwise.} \end{cases}$$

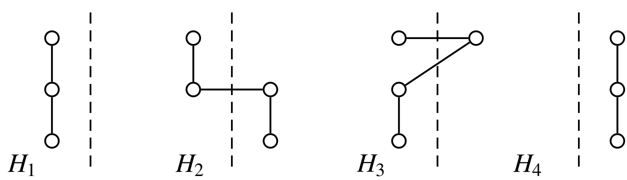

Figure 3. The induced subpartitions H_1, H_2, H_3, and H_4 referenced in Lemmas 2.1, 4.1, and 4.2. The vertical dashed line provides a visual reference to the partition of the vertices into the sets A and B for $(A, B) \in \mathbb{P}(H)$. The vertices to the left of the line in each graph are in A and the vertices on the right are in B.

Consider the partition (A', B') identical to (A, B) except that the vertex y is moved from A to B. Then,

$$\begin{aligned} a(A', B') &= e(A, B) + 2 + |B| + 1 - \Delta_B(A) - \xi \\ &= a(A, B) + 3 - \xi \\ &> a(A, B). \end{aligned}$$

This contradicts our choice of the optimal partition (A, B).

For the sake of contradiction, suppose that H_2 is an induced subpartition of (A, B). Let x, y, z, w be the vertices of H_2 so that $x, y \in A$ and $z, w \in B$, and the edge yz crosses the partition. As H_2 is an induced path of length three, note that $d_G(y) = d_G(z) = 2$. Also, let

$$\xi = \begin{cases} 1 & \text{if } \Delta_B(A) = d_B(x) \text{ or } \Delta_B(A) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the partition (A', B') identical to (A, B) except that the vertex y is moved from A to B and the vertex z is moved from B to A. Then,

$$\begin{aligned} a(A', B') &= e(A, B) + 2 + |B| - \Delta_B(A) - \xi \\ &= a(A, B) + 2 - \xi \\ &> a(A, B). \end{aligned}$$

Once again, this contradicts our choice of the optimal partition (A, B). □

The next lemma is useful for dealing with vertices of degree one in H.

Lemma 2.2. *For a graph H, there exists a partition $(A, B) \in \mathbb{P}(H)$ with $a(A, B) = a(H)$ and the edges incident with vertices of degree one cross the partition.*

Proof. Consider all $(A, B) \in \mathbb{P}(H)$ with $a(A, B) = a(H)$. Among these, choose the partition which has the maximum number of edges incident with degree one vertices which cross the partition. For the sake of contradiction, suppose there is at least one edge incident to a degree one vertex that does not cross the partition. Let x be this degree one vertex and let y be the neighbor of x. Now, let

$$\xi = \begin{cases} 1 & \text{if } d_B(y) \geq \Delta_B(A), \\ 0 & \text{otherwise.} \end{cases}$$

Suppose first that $x \in A$ and the edge xy does not cross the partition. Consider the partition $(A', B') \in \mathbb{P}(H)$, which is identical to (A, B) except that x is moved from A to B. Then, $a(A', B') = a(A, B) + 2 - \xi > a(A, B)$, which contradicts our choice of the optimal partition (A, B).

Suppose now that $x \in B$ and the edge xy does not cross the partition. Consider the partition $(A', B') \in \mathbb{P}(H)$, which is identical to (A, B) except that x is moved from B to A. Then, $a(A', B') = a(A, B)$, which contradicts our choice of the optimal partition $a(A, B)$ which maximizes the number of edges incident with degree one vertices that cross the partition. □

Lemma 2.2 can be used to provide an alternate proof of the σ_2 condition for a graph to be k-linked. A graph G is *k-linked* if, for every list of $2k$ vertices $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$, there exist internally disjoint paths $P_1, \ldots, P_k$ such that each P_i is a path joining s_i and t_i. If H is the union of k independent edges (i.e., k copies of the complete graph K_2), then a graph being H-linked is equivalent to the graph being k-linked. As each vertex in H has degree one, Lemma 2.2 states that there exists an optimal partition of H where all of the edges cross the partition. Thus, $a(H) = 2k - 1$ and Theorem 1.2 gives the σ_2 condition proved previously (and independently) in [Kawarabayashi et al. 2006] and [Gould and Whalen 2006]. Note that the bound on the number of vertices in G given by Theorem 1.2 is higher than the bounds in those references.

The next result follows directly from the first case in the proof of Lemma 2.2. The result differs from Lemma 2.2 in that it applies to every optimal partition of H, whereas Lemma 2.2 applies to only a subset of optimal partitions of H.

Corollary 2.3. *If $(A, B) \in \mathbb{P}(H)$ with $a(A, B) = a(H)$, then the vertices of degree one in A must be incident to edges that cross the partition.*

3. Stars

In this section, we determine the value of $a(H)$ when H is a star or a union of stars. Let $K_{1,k}$ denote a star with one vertex of degree k and k vertices of degree one. By Lemma 2.2, an optimal partition of H exists where all degree one vertices cross the partition. Thus, we have the following:

Corollary 3.1. *If $H = K_{1,k}$ for $k \geq 1$, then $a(H) = k$.*

When $H = K_{1,k}$, G being H-linked is equivalent to G being k-connected. This follows from a theorem by Dirac [1960]. With this fact, Theorem 1.2, and Corollary 3.1, we get the well-known σ_2 condition for a graph G to be k-connected (i.e., $\sigma_2(G) \geq n + k - 2$).

We now determine the value of $a(H)$ when H is a union of stars. For $H = K_{1,k_1} \cup K_{1,k_2} \cup \ldots \cup K_{1,k_m}$, we call the vertex of maximum degree in each star the *hub vertex* or *hub* of that star. Note that, for $K_{1,1}$, either vertex can be considered a hub vertex.

Theorem 3.2. *If $H = K_{1,k_1} \cup K_{1,k_2} \cup \ldots \cup K_{1,k_m}$ with $k_i \geq 1$ for $1 \leq i \leq m$, then*

$$a(H) = 2\sum_{j=1}^{m} k_j - \max\{k_1, k_2, \ldots, k_m\}.$$

Proof. Assume without loss of generality that $k_m \geq k_i$ for all $1 \leq i \leq m-1$. By Lemma 2.2, there exists a partition $(A, B) \in \mathbb{P}(H)$ with $a(A, B) = a(H)$ where all edges incident with vertices of degree one cross the partition. Among all optimal partitions that satisfy that property, choose the partition with the maximum number of hub vertices in A. We will now show that, under the assumptions above, all of the hub vertices are in A.

Claim 3.3. The hub of the star K_{1,k_m} must be in A.

Proof. Let x be the hub of K_{1,k_m} and suppose that $x \in B$. Note that $d_G(x) = k_m$. Consider the partition (A', B') obtained by moving x from B to A and moving its leaves from A to B. Then, noting that $\Delta_B(A) \geq 1$,

$$\begin{aligned} a(A', B') &= a(A, B) + k_m - 1 - (k_m - \Delta_B(A)) \\ &= a(A, B) + \Delta_B(A) - 1 \\ &\geq a(A, B). \end{aligned}$$

However, this contradicts our assumption that (A, B) is an optimal partition of H, which has the maximum number of hubs in A. So, the hub of maximum degree must be in A. □

Assume without loss of generality that the hubs of $K_{1,k_1}, K_{1,k_2}, \ldots, K_{1,k_i}$ are in B (where $i \geq 0$) and the remaining hubs are in A. By the above claim, $i < m$. Now, we have

$$\begin{aligned} a(A, B) &= \sum_{j=1}^{m} k_j + \left(\sum_{t=i+1}^{m} k_t \right) + i - k_m = \sum_{j=1}^{m} k_j + \left(\sum_{t=i+1}^{m-1} k_t \right) + i \\ &\leq \sum_{j=1}^{m} k_j + \sum_{t=1}^{m-1} k_t = 2\left(\sum_{j=1}^{m} k_i \right) - k_m. \end{aligned}$$

So, this gives us an upper bound on $a(A, B)$ for all possible locations of the hubs. For the lower bound, note that the partition $(A', B') \in \mathbb{P}(H)$ where all of the hubs of H are in A' has

$$a(A', B') = 2\left(\sum_{j=1}^{m} k_j \right) - k_m.$$

Therefore, $a(H) = 2\left(\sum_{j=1}^{m} k_j\right) - k_m$, where $k_m = \max\{k_1, k_2, \ldots, k_m\}$. □

Note that Theorem 3.2 can also be used to show that $a(H) = 2k - 1$ when H is the union of k independent edges (which was discussed in the previous section).

4. Cycles and paths

We now move our attention to paths and cycles. Let C_k (for $k \geq 3$) denote a cycle on k vertices and P_k (for $k \geq 2$) denote a path on k vertices.

The following lemmas prove that H_3 (shown in Figure 3) cannot appear as an induced subpartition in any optimal partition of H.

Lemma 4.1. *Let $k \geq 4$. For $H \in \{C_k, P_k\}$, the graph H_3 cannot be an induced subpartition of any partition $(A, B) \in \mathbb{P}(H)$ with $a(A, B) = a(H)$.*

Proof. Suppose for the sake of contradiction that H_3 is an induced subpartition of some partition (A, B) with $a(A, B) = a(H)$. Assume the vertices of H_3 are x, y, z, and w with $x, z, w \in A$ and $y \in B$, and the edges are xy, yz, and zw.

By Corollary 2.3, $d_G(w) \neq 1$ since the edge incident to w does not cross the partition. Let t be a neighbor of w in H. By Lemma 2.1, $t \in B$.

Now, either $d_G(x) = 1$, x has a neighbor in A, or x has a second neighbor in B. If $d_G(x) = 1$ or if x has a neighbor in A, then the partition (A', B') formed from (A, B) by moving x and z to from A to B and y from B to A has $a(A', B') > a(A, B)$. As this contradicts our choice of the optimal partition (A, B), x must have a second neighbor in B.

Let v be the other neighbor of x in B. As a result, $\Delta_B(A) = 2$. Consider the partition (A', B'), which modifies the partition (A, B) by moving w from A to B. Then,

$$\begin{aligned} a(A', B') &= e(A, B) + |B| + 1 - \Delta_B(A) - 0 \\ &= a(A, B) + 1. \end{aligned}$$

Thus, the partition (A', B') has $a(A', B') > a(A, B)$. However, this contradicts the assumption that the partition (A, B) has $a(A, B) = a(H)$.

As all possibilities are exhausted and lead to contradictions, we conclude that H_3 is not an induced subpartition of any partition (A, B) with $a(A, B) = a(H)$. □

This final lemma proves that there exists an optimal partition of H which does not contain H_4 (shown in Figure 3) as an induced subpartition.

Lemma 4.2. *If $H \in \{C_k, P_k\}$ with $k \geq 3$, then there is a partition $(A, B) \in \mathbb{P}(H)$ with $a(A, B) = a(H)$ which does not have H_4 as a subpartition.*

Proof. For the sake of contradiction, assume all partitions (A, B) with $a(A, B) = a(H)$ have H_4 as a subpartition. Consider one such partition (A, B) which contains H_4. Let the vertices of H_4 (all of which are in B) be x, y, and z with the two edges being xy and yz. Consider the partition (A', B') which is identical to (A, B)

except the vertex y is moved from B to A. Then,

$$\begin{aligned} a(A', B') &\geq e(A, B) + 2 + |B| - 1 - \Delta_B(A) - 1 \\ &= a(A, B) + 1 - 1 \\ &= a(A, B). \end{aligned}$$

Note that equality occurs in the first line above only when $\Delta_B(A) = 1$ as the partition (A', B') has $\Delta_{B'}(A') = 2$. Otherwise, $a(A', B') > a(A, B)$. In either case, as $a(A', B') \geq a(A, B)$ and (A', B') does not contain H_4 as a subpartition, we have a contradiction. □

With these lemmas, we are now able to prove the value for $a(H)$ when H is a cycle or path with three or more vertices. Note that by Lemma 2.2, for the single edge P_2, we have $a(P_2) = 1$.

Theorem 4.3. *For $k \geq 3$, we have $a(C_k) = \left\lceil \frac{1}{2}(3k - 5) \right\rceil$ and $a(P_k) = \left\lceil \frac{1}{2}(3k - 6) \right\rceil$.*

Proof. Let $H \in \{P_k, C_k\}$ and assume that $V(H) = \{1, 2, 3, \ldots, k\}$ with the vertices numbered based on an arbitrary orientation of H. If $k = 3$, then it is straightforward to show that $a(C_3) = a(P_3) = 2 = \left\lceil \frac{1}{2}(3(3) - 5) \right\rceil = \left\lceil \frac{1}{2}(3(3) - 6) \right\rceil$. If $k = 4$, then it is also straightforward to show that $a(C_4) = 4 = \left\lceil \frac{1}{2}(3(4) - 5) \right\rceil$ and $a(P_4) = 3 = \left\lceil \frac{1}{2}(3(4) - 6) \right\rceil$. So, assume $k \geq 5$. Consider a partition $(A, B) \in \mathbb{P}(H)$ with $a(A, B) = a(H)$. By Lemma 4.2, we may assume H_4 is not an induced subpartition of (A, B). It follows from Lemma 2.1, Lemma 4.1, Corollary 2.3, and the fact that $k \geq 5$ that the partition (A, B) cannot have any edge with both endpoints in A. Consequently, $\Delta_B(A) = 2$.

Assume for the sake of contradiction that the partition (A, B) has at least two edges with both endpoints in B. Among the edges with both endpoints in B, choose the two edges with the fewest edges of H between them based on the orientation of H. Let $(i, i+1)$ and $(j, j+1)$ with $j > i$ be two edges with both endpoints in B. Note that $i + 1 \neq j$ since H_4 is not an induced subpartition. In particular, vertex $i + 2$ must be in A and by Lemma 2.1, $i + 3$ must be in B as otherwise H_2 would be an induced subpartition. Lemma 2.1, Lemma 4.2, and our choice of j imply that $j = i + t$ for some positive odd integer t and the vertices $i + 1, i + 3, \ldots, i + t$ are in B while the vertices $i + 2, i + 4, \ldots, i + t - 1$ are in A.

Consider the partition (A', B') formed by starting with (A, B) and moving vertices $i + 1, i + 3, \ldots i + t$ from B to A and moving $i + 2, i + 4, \ldots, i + t - 1$ from A to B. Then,

$$\begin{aligned} a(A', B') &= e(A, B) + 2 + |B| - 1 - \Delta_A(B) \\ &= a(A, B) + 1 \\ &> a(A, B). \end{aligned}$$

However, this contradicts our choice of the partition (A, B). Thus, as no edge of the partition can have both endpoints in A, all of the edges of (A, B) must cross the partition with the possible exception of exactly one edge which must have both endpoints in B.

If $H = C_k$ with k even, then (A, B) can have no edge with both endpoints in B as one edge in B would force the existence of another edge with either both endpoints in B or both endpoints in A. Thus, (A, B) must either be the partition with $B = \{1, 3, \dots k-1\}$ and $A = \{2, 4, \dots, k\}$ or the same partition with the vertices in A and B swapped. Consequently, $a(A, B) = a(C_k) = k + \frac{1}{2}k - 2 = \frac{1}{2}(3k-4)$.

If $H = C_k$ with k odd, then (A, B) must have exactly one edge with both endpoints in B as all edges cannot cross the partition. Thus, (A, B) must be the partition with $A = \{1, 3, \dots, k-2\}$ and $B = \{2, 4, \dots, k-1, k\}$ or a vertex relabeling of this partition. Consequently, $a(A, B) = a(C_k) = k + \lceil \frac{1}{2}k \rceil - 2 = \lceil \frac{1}{2}(3k-5) \rceil$.

If $H = P_k$ with k odd, then (A, B) must have all edges crossing the partition. Thus, (A, B) must be the partition with $A = \{2, 4, \dots, k-1\}$ and $B = \{1, 3, \dots, k\}$. If $H = P_k$ with k even, then (A, B) either has all edges crossing the partition or exactly one edge with both endpoints in B (and all other edges crossing the partition). Thus, in either case, $a(P_k) = k - 1 + \lceil \frac{1}{2}k \rceil - 2 = \lceil \frac{1}{2}(3k-6) \rceil$. □

When $H = C_k$, a graph G being H-linked is equivalent to G being k-ordered. A graph G is *k-ordered* if for every ordered set of vertices S such that $|S| = k$, the graph G contains a cycle C encountering the vertices S in the given order. When $H = P_k$, a graph G being H-linked is equivalent to G being k-ordered connected. A graph G is *k-ordered connected* if for every ordered set of vertices $S = \{v_1, v_2, \dots, v_k\}$, the graph G contains a path P from v_1 to v_k encountering S in the given order. Note that by forcing the cycle (or path) that encounters the vertices of S in order to be a hamiltonian cycle (or a hamiltonian path), we get the property k-ordered hamiltonian (k-ordered hamiltonian connected). The concept of k-ordered was introduced by Ng and Schutz [1997].

Using Theorem 1.2 and Theorem 4.3, we have the following corollary.

Corollary 4.4. *Let G be a graph on n vertices and let $k \geq 3$.*

(i) *If $n > 20k$ and $\sigma_2(G) \geq n + \lceil \frac{1}{2}(3k-9) \rceil$, then G is k-ordered.*

(ii) *If $n > 20(k-1)$ and $\sigma_2(G) \geq n + \lceil \frac{1}{2}(3k-10) \rceil$, then G is k-ordered connected.*

The bounds on σ_2 in both cases are best possible.

To the best of our knowledge, the above σ_2 conditions for k-ordered and k-ordered connected are not explicitly stated in the literature, although it is implied in several sources that the σ_2 conditions for k-ordered and k-ordered connected should be the same as the σ_2 conditions for k-ordered hamiltonian and k-ordered hamiltonian connected.

There are a number of degree-sum results for k-ordered hamiltonian and k-ordered hamiltonian connected graphs. Ng and Schultz [1997] proved a sharp σ_2 condition for any graph on $n \geq 3$ vertices to be k-ordered hamiltonian. J. Faudree et al. [2000] proved the bound for σ_2 could be reduced for graphs on n vertices with $n \geq 53k^2$. The same σ_2 condition in [Faudree et al. 2000] was shown to work for $n \geq 2k$ by R. Faudree et al. [2003b]. Note that the sharpness example they construct in [Faudree et al. 2003b] is neither k-ordered hamiltonian nor k-ordered.

Theorem 4.5 [Faudree et al. 2003b]. *Let k be an integer with $3 \leq k \leq \frac{1}{2}n$, and let G be a graph of order n. If $\sigma_2(G) \geq n + \frac{1}{2}(3k-9)$, then G is k-ordered hamiltonian. The bound on $\sigma_2(G)$ is sharp.*

For k-ordered hamiltonian connected, a σ_2 condition for large n is mentioned (without proof) in [Faudree et al. 2003a]. A stronger and sharp σ_2 condition for k-ordered hamiltonian connected was proven by Nicholson and Wei [2015].

Theorem 4.6 [Nicholson and Wei 2015]. *If G is a graph on n vertices with $\sigma_2(G) \geq n + \frac{1}{2}(3k-10)$, where $4 \leq k \leq \frac{1}{2}(n+1)$, then G is k-ordered hamiltonian connected.*

Overall, while the concepts of k-ordered and k-ordered hamiltonian are distinct, the σ_2 condition is the same for both when n is large as shown in Corollary 4.4. Similarly, the σ_2 conditions for k-ordered connected and k-ordered hamiltonian connected are the same for large n. Corollary 4.4 has higher bounds on n than the optimal known results, but by utilizing Theorem 1.2, the proofs are much less technical.

5. Complete graphs and complete bipartite graphs

Our last results provide $a(H)$ when H is the complete graph K_k or the complete bipartite graph $K_{r,s}$. First, we consider the complete graph K_k.

Theorem 5.1. *For any integer $k \geq 3$, we have $a(K_k) = \lfloor \frac{1}{4}k^2 \rfloor$.*

Proof. Suppose $|A| = t$ and $|B| = k-t$. Then, $a(A, B) = (k-t)(t) + (k-t) - (k-t) = kt - t^2$. Let $f(t) = kt - t^2$. Then, $f'(t) = -2t + k$. So, $f'(t) = 0$ implies that $t = \frac{1}{2}k$. Since $f''(t) < 0$, $f(t)$ has a global maximum at $t = \frac{1}{2}k$. If k is even, then $a(K_k) = \frac{1}{4}k^2$. If k is odd, then either

$$a(K_k) = \left[k - \left(\tfrac{1}{2}(k-1)\right)\right]\left(\tfrac{1}{2}(k-1)\right) \quad \text{or} \quad a(K_k) = \left[k - \tfrac{1}{2}(k+1)\right]\left(\tfrac{1}{2}(k+1)\right).$$

In both cases, $a(K_k) = \frac{1}{4}(k^2-1)$. Therefore, $a(K_k) = \lfloor \frac{1}{4}k^2 \rfloor$ for any integer $k \geq 3$. □

Now, we prove the value of $a(H)$ when is the complete bipartite graph $K_{r,s}$. Note that $K_{1,1}$ and $K_{1,2}$ are covered by previous results in this article.

Theorem 5.2. *For $r \geq s \geq 2$, we have $a(K_{r,s}) = rs$.*

Proof. Let $(A, B) \in \mathbb{P}(K_{r,s})$ such that $a(A, B) = a(K_{r,s})$. Using the canonical bipartition of $K_{r,s}$, let X and Y be the partite sets. Let $X_A = X \cap A$, $Y_B = Y \cap B$, $Y_A = Y \cap A$, and $X_B = X \cap B$. Additionally, let $|X_A| = x_A$, $|X_B| = x_B$, $|Y_A| = y_A$, and $|Y_B| = y_B$.

Suppose that exactly one of the sets X_A, X_B, Y_A, and Y_B is empty. Assume without loss of generality that the partite sets of $K_{r,s}$ are labeled X and Y so that either X_A or X_B is empty. Assume first that only $X_A = \varnothing$. Let $v \in Y_B$. Consider the partition (A', B') starting with (A, B) and moving v from B to A. That is, we have $X_{B'} = X_B = X$, $Y_{A'} = Y_A \cup \{v\}$, and $Y_{B'} = Y_B - \{v\}$. Then,

$$a(A', B') = a(A, B) + |X_B| - 1 = a(A, B) + |X| - 1.$$

As $|X| \geq 2$, we have $a(A', B') > a(A, B)$, which contradicts our choice of (A, B).

Assume now that X_B is the only empty set among X_A, X_B, Y_A, and Y_B. Let $w \in Y_A$. Consider the partition (A', B') starting with (A, B) and moving w from B to A as in the first case. Then,

$$a(A', B') = a(A, B) + |X_A| + 1 - 1 = a(A, B) + |X|.$$

Since $|X| \geq 2$, we have $a(A', B') > a(A, B)$, which contradicts our choice of (A, B).

Assume now that each of X_A, X_B, Y_A, and Y_B is nonempty. Then,

$$a(A, B) = (x_A)(y_B) + (y_A)(x_B) + x_B + y_B - \max\{x_B, y_B\}.$$

Note that $x_B + y_B - \max\{x_B, y_B\} = \min\{x_B, y_B\}$.

Consider the partition (A', B') formed by starting with (A, B) and moving the vertices of X_B from B to A and moving the vertices of Y_A from A to B. Then,

$$a(A', B') = (x_A)(y_B) + (x_B)(y_A) + (x_A)(y_A) + (x_B)(y_B).$$

Note that $a(A', B') \geq a(A, B)$ whenever $(x_A)(y_A) + (x_B)(y_B) > \min\{x_B, y_B\}$. However, since x_B, x_A, y_A, and y_B are all at least one, $(x_A)(y_A) + (x_B)(y_B)$ is strictly larger than $\min\{x_B, y_B\}$. Thus, $a(A', B') \geq a(A, B)$, which contradicts our choice of (A, B).

Consequently, the only remaining possibility for (A, B) is either $A = X$ and $B = Y$, or $A = Y$ and $B = X$. In either case, $a(A, B) = rs$ and thus, $a(K_{r,s}) = rs$. $\square$

These results together with Theorem 1.2, we get the following corollary.

Corollary 5.3. *Let G be a graph on n vertices.*

(i) *Let $k \geq 3$. If $n > 20k$ and $\sigma_2(G) \geq n + \lfloor \frac{1}{4}k^2 \rfloor - 2$, then G is K_k-linked.*

(ii) *Let $r \geq s \geq 2$. If $n \geq 20rs$ and $\sigma_2(G) \geq n + rs - 2$, then G is $K_{r,s}$-linked.*

6. Final observations

For all of the classes of graphs examined above, an optimal partition (A, B) always exists where the vertex of maximum degree is in A. However, this is not always the case. Consider the graph J with $V(J) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and

$$E(J) = \{v_1v_2, v_1v_3, v_1v_5, v_2v_3, v_2v_4, v_2v_6, v_2v_7, v_3v_4, v_4v_6, v_4v_7, v_5v_7, v_6v_7\}.$$

Note that $\Delta(J) = 5$ and v_2 is the vertex of maximum degree. By checking all possible partitions of the vertex set (possibly with the aid of a computer), it can be shown that J has a unique optimal partition (A, B) given by $A = \{v_1, v_4, v_7\}$. From this partition, we have $a(J) = 10$. However, in this optimal partition, the vertex of maximum degree (i.e., v_2) is in B and $\Delta_B(A) = 3$. So, it is not always the case that a graph has an optimal partition (A, B) where the vertex of maximum degree is in A.

We conclude by making an observation about optimal partitions of the union of graphs. Consider two graphs M_1 and M_2 and assume an optimal partition of both graphs is known. We note that the union of these two optimal partitions is not necessarily an optimal partition for the union of M_1 and M_2. As an example, let M_1 be the five cycle $v_1v_2v_3v_4v_5$ with the additional edges v_2v_4 and v_3v_5. Let M_2 be the graph on the set $\{w_1, w_2, w_3, w_4, w_5\}$ where w_1, w_2, w_3, and w_4 form a K_4 and the only other edge is w_4w_5.

Now, the graph M_1 has exactly two optimal partitions which both give $a(M_1) = 6$. One of the optimal partitions of M_1, which we denote by (A_{M_1}, B_{M_1}), has $A_{M_1} = \{v_2, v_3\}$. The graph M_2 has nine different optimal partitions which give $a(M_2) = 5$. One of these optimal partitions of M_2, which we denote by (A_{M_2}, B_{M_2}), has $A_{M_2} = \{w_1, w_3\}$. Let M be the graph formed by the union of M_1 and M_2 and consider the partition

$$(A_M, B_M) = (A_{M_1} \cup A_{M_2}, B_{M_1} \cup B_{M_2}).$$

This partition gives $a(A_M, B_M) = 13$. However, $a(M) = 14$, which can be achieved using the optimal partition of M_1 given above and a different optimal partition of M_2 such as the partition (A'_{M_2}, B'_{M_2}), where $A'_{M_2} = \{w_1, w_4\}$. So, finding an optimal partition for a union of graphs is not simply a matter of taking any optimal partition of the graphs individually and forming the union of these partitions.

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