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A solution to a problem of Frechette and Locus

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In a recent paper, Frechette and Locus examined and found expressions for the infinite product $D_m(q) := \prod_{t=1}^{\infty}(1-q^{mt})/(1-q^t)$ in terms of products of q-series of the Rogers–Ramanujan type coming from Hall–Littlewood polynomials, when $m \equiv 0, 1, 2 \pmod 4$. These q-series were originally discovered in 2014 by Griffin, Ono, and Warnaar in their work on the framework of the Rogers–Ramanujan identities. Extending this framework, Rains and Warnaar also recently discovered more q-series and their corresponding infinite products. Frechette and Locus left open the case where $m \equiv 3 \pmod 4$. Here we find such an expression for the infinite products for $m \equiv 3 \pmod 4$ by making use of the new q-series obtained by Rains and Warnaar.

1. Introduction

The Rogers–Ramanujan identities [Andrews 1971]

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \quad (1\text{-}1)$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}, \quad (1\text{-}2)$$

have inspired research and discoveries in many areas of mathematics and physics, such as modular forms and elliptic curves, conformal field theory, knot theory, probability, and statistical mechanics. (See next citation for some discussion.) Given the importance of these identities, it had been an open problem for nearly a century to build a theory suggested by these two Rogers–Ramanujan identities. In 2014 Griffin, Ono, and Warnaar [Griffin et al. 2016] discovered[1] a more general framework for identities similar to that of Rogers–Ramanujan, where an infinite sum, defined using Hall–Littlewood polynomials $P_\lambda(x; q)$, is equal to an infinite product with periodic exponents.

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[1]Their work was named the 15th top story in science in 2014 by *Discover* magazine.

In order to define the Hall–Littlewood polynomials, we recall the definition of an integer partition and the following notation. A *partition* is a nonincreasing sequence of nonnegative integers with finitely many nonzero terms. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, we define the *weight* of the partition to be $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and the *length* of the partition λ to be n. In addition, we let $2\lambda := (2\lambda_1, 2\lambda_2, \ldots, 2\lambda_n)$. Let m_i denote the multiplicity of size i parts. Also, let $(q)_k = (q; q)_k$ denote the q-Pochhammer symbol, which is defined as follows:

$$(a)_k := (a; q)_k = \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1}) & \text{if } k \geq 0, \\ \prod_{n=0}^{\infty}(1-aq^n) & \text{if } k = \infty. \end{cases}$$

If λ has length n, the Hall–Littlewood polynomial is a symmetric function in n variables, namely $x_1, x_2, \ldots, x_n$, defined as

$$P_\lambda(x; q) = \frac{1}{v_\lambda(q)} \sum_{w \in S_n} w\left(x^\lambda \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j}\right), \tag{1-3}$$

where $x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} \ldots x_n^{\lambda_n}$, $v_\lambda(q) := \prod_{i=0}^{n} (q)_{m_i}/(1-q)^{m_i}$, and the symmetric group S_n acts on x by permuting all the x_i.

For ordered pairs $\nu = (c, d) \in \{(1, -1), (2, -1), (1, 0), (2, -2)\}$ and arbitrary $a, b \geq 1$, Griffin, Ono, and Warnaar [Griffin et al. 2016], and more recently Rains and Warnaar [2015], defined the q-series

$$R_\nu(a, b; q) = \sum_{\lambda, \lambda_1 \leq a} q^{c|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^{2b+d}), \tag{1-4}$$

$$S(a, b; q) = \sum_{\lambda, \lambda_1 \leq a} q^{|\lambda|/2} P_\lambda(1, q, q^2, \ldots; q^{2b}), \tag{1-5}$$

and

$$T(a, b; q) = \sum_{\lambda, \lambda_1 \leq a} q^{|\lambda|} \left(\prod_{i=1}^{a-1} (-q^{b-1/2}; q^{b-1/2})_{m_i}(\lambda)\right) P_\lambda(1, q, q^2, \ldots; q^{2b-1}). \tag{1-6}$$

Here the $P_\lambda(1, q, q^2, \ldots; q^T)$ are Hall–Littlewood q-series in infinitely many parameters. To define these Hall–Littlewood q-series, we must first express the Hall–Littlewood polynomial $P_\lambda(x_1, x_2, \ldots, x_n; q^T)$ in terms of the r-th power sum symmetric function, $x_1^r + x_2^r + \cdots + x_n^r$. This is possible due to a well-known fact in abstract algebra which states that every symmetric polynomial can be written as a sum of products of r-th power sum symmetric functions with rational coefficients [Macdonald 1995]. Now we obtain the Hall–Littlewood polynomial $P_\lambda(1, q, q^2, \ldots; q^T)$ by replacing $(x_1^r + x_2^r + \cdots + x_n^r)$ with $1^r + q^r + q^{2r} + \cdots = 1/(1-q^r)$.

To motivate our work, consider another interesting property of the Roger–Ramanujan identities. When we take the product of both Rogers–Ramanujan identities, we can see that

$$\begin{aligned} G(q)\cdot H(q) &= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \cdot \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} \\ &= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})(1-q^{5n+2})(1-q^{5n+3})} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^{5n})}{(1-q^{n})}. \end{aligned}$$

Inspired by this emergence of this infinite product, Frechette and Locus [2016] explored the natural question of more generalized products of q-series of the Rogers–Ramanujan type that result in an infinite product of the form

$$D_m(q) := \prod_{t=1}^{\infty} \frac{(1-q^{mt})}{(1-q^{t})}. \tag{1-7}$$

Making use of the Rogers–Ramanujan framework of [Griffin et al. 2016], Frechette and Locus obtained explicit formulas for $D_m(q)$ when $m \equiv 0, 1, 2 \pmod 4$. When $m \geq 8$ and is even, they found that

$$D_m(q) = \frac{R_{(1,0)}\big(2, \frac{m}{2}-3; q\big) \cdot R_{(2,-2)}\big(\frac{m}{2}, 2; q\big)}{R_{(2,-2)}\big(\frac{m}{2}-3, 3; q\big)}. \tag{1-8}$$

They also found for $m \equiv 1 \pmod 4$ and $m > 1$,

$$D_m(q) = \frac{R_{(1,-1)}\big(\frac{m-1}{2}-1, 1; q\big) \cdot R_{(2,-1)}\big(\frac{m-1}{4}, \frac{m-1}{4}; q\big)}{R_{(2,-1)}\big(\frac{m-1}{4}+1, \frac{m-1}{4}-1; q\big)}. \tag{1-9}$$

However, they were unable to construct $D_m(q)$ for positive integers $m \equiv 3 \pmod 4$. Their difficulty arose from the fact that the Rogers–Ramanujan framework of [Griffin et al. 2016]. Rains and Warnaar [2015] recently found this extension. In this paper, we address this case and provide such a formula.

Theorem 1.1. *If $m \equiv 3 \pmod 4$ and $m \geq 7$, we have*

$$D_m(q) = \frac{T\big(\frac{m+1}{2}, \frac{m+1}{4}; q\big) S\big(\frac{m+1}{2}, \frac{m+1}{4}-1; q\big) R_{(1,-1)}\big(\frac{m-1}{2}-1, 1; q\big)}{T\big(\frac{m+1}{2}+2, \frac{m+1}{4}-1; q\big) S\big(\frac{m+1}{2}-2, \frac{m+1}{4}; q\big)}.$$

In Section 2, we cover preliminaries on q-series and state the results of [Griffin et al. 2016; Rains and Warnaar 2015; Frechette and Locus 2016]. In Section 3, we use these results to prove Theorem 1.1.

2. Preliminaries

In order to simplify the products involved in writing $R_\nu(a,b;q)$, $S(a,b;q)$, and $T(a,b;q)$, we use a modified theta function

$$\theta(a;q) := (a;q)_\infty (q/a;q)_\infty, \tag{2-1}$$

where $(a;q)_\infty$ denotes the q-Pochhammer symbol we previously defined. We then define

$$\theta(a_1,a_2,\ldots,a_n;q) = \theta(a_1;q)\cdot\theta(a_2;q)\cdots\theta(a_n;q). \tag{2-2}$$

Theorem 2.1 [Griffin et al. 2016, Theorem 1.1]. *If a and b are positive integers and $\kappa := 2a+2b+1$, we have*

$$\begin{aligned} R_{(1,-1)}(a,b;q) &:= \sum_{\lambda,\lambda_1\le a} q^{|\lambda|} P_{2\lambda}(1,q,q^2,\ldots;q^{2b-1}) \\ &= \frac{(q^\kappa;q^\kappa)_\infty^b}{(q)_\infty^b}\cdot \prod_{i=1}^{b}\theta(q^{i+a};q^\kappa)\prod_{1\le i<j\le b}\theta(q^{j-i},q^{i+j-1};q^\kappa) \\ &= \frac{(q^\kappa;q^\kappa)_\infty^a}{(q)_\infty^a}\cdot \prod_{i=1}^{a}\theta(q^{i+1};q^\kappa)\prod_{1\le i<j\le a}\theta(q^{j-i},q^{i+j+1};q^\kappa). \end{aligned}$$

This result has generalizations to the functions S and T:

Theorem 2.2 [Rains and Warnaar 2015, Theorem 5.10]. *If a and b are positive integers and $\kappa := a+2b+1$, we have*

$$\begin{aligned} S(a,b;q) &:= \sum_{\lambda,\lambda_1\le a} q^{|\lambda|/2} P_{\lambda}(1,q,q^2,\ldots;q^{2b}) \\ &= \frac{(q^\kappa;q^\kappa)_\infty^{b-1}(q^{\kappa/2};q^{\kappa/2})_\infty}{(q)_\infty^{b-1}(q^{1/2};q^{1/2})_\infty} \prod_{i=1}^{b}\theta(q^{i};q^{\kappa/2})\prod_{1\le i<j\le b}\theta(q^{j-i},q^{i+j};q^\kappa). \end{aligned}$$

Theorem 2.3 [Rains and Warnaar 2015, Remark 5.13]. *If a and b are positive integers and $\kappa := a+2b-1$, we have*

$$\begin{aligned} T(a,b;q) &:= \sum_{\lambda,\lambda_1\le a} q^{|\lambda|}\left(\prod_{i=1}^{a-1}(-q^{b-1/2};q^{b-1/2})_{m_i(\lambda)}\right) P_{\lambda}(1,q,q^2,\ldots;q^{2b-1}) \\ &= \frac{(q^\kappa;q^\kappa)_\infty^{b}}{(q)_\infty^{b-1}(q^{1/2};q^{1/2})_\infty} \prod_{i=1}^{b}\theta(q^{i-1/2};q^{\kappa})\prod_{1\le i<j\le b}\theta(q^{j-i},q^{i+j-1};q^\kappa). \end{aligned}$$

To prove our result, we combine Theorem 2.3 with the following proposition.

Proposition 2.4. *If $i \in \mathbb{Z}^+$, we have*

$$\theta(q^i; q^{m/2}) = \theta(q^i; q^m)\theta(q^{m/2-i}; q^m).$$

Proof. Using the definition of the modified theta function, we have

$$\begin{aligned}\theta(q^i; q^{m/2}) &= (q^i; q^{m/2})_\infty (q^{m/2-i}; q^{m/2})_\infty \\ &= \prod_{n=0}^{\infty}(1 - q^i \cdot q^{mn/2}) \prod_{n=0}^{\infty}(1 - q^{m/2-i} \cdot q^{mn/2}).\end{aligned}$$

Separating terms in the infinite product on the right-hand side based on the parity of n, we have

$$\begin{aligned}\theta(q^i; q^{m/2}) &= \prod_{n=0}^{\infty}(1-q^i\cdot q^{mn})(1-q^{i+m/2}\cdot q^{mn}) \prod_{n=0}^{\infty}(1-q^{m/2-i}\cdot q^{mn})(1-q^{m-i}\cdot q^{mn}) \\ &= \prod_{n=0}^{\infty}(1-q^i\cdot q^{mn})(1-q^{m-i}\cdot q^{mn})(1-q^{m/2-i}\cdot q^{mn})(1-q^{i+m/2}\cdot q^{mn}) \\ &= (q^i; q^m)_\infty (q^{m-i}; q^m)_\infty (q^{m/2-i}; q^m)_\infty (q^{m/2+i}; q^m)_\infty \\ &= \theta(q^i; q^m)\theta(q^{m/2-i}; q^m). \qquad \square\end{aligned}$$

3. Proof of Theorem 1.1

We shall now prove Theorem 1.1. By Theorem 2.3, when $m = a + 2b - 1$, we have

$$\begin{aligned}T\Big(\frac{m+1}{2}, \frac{m+1}{4}; q\Big) &\\ = \frac{(q^m; q^m)_\infty^{(m+1)/4}}{(q)_\infty^{(m+1)/4-1}(q^{1/2}; q^{1/2})_\infty} &\prod_{i=1}^{(m+1)/4} \theta(q^{i-1/2}; q^m) \\ &\prod_{1\le i<j\le (m+1)/4} \theta(q^{j-i}, q^{i+j-1}; q^m) \qquad (3\text{-}1)\end{aligned}$$

and

$$\begin{aligned}T\Big(\frac{m+1}{2} + 2, \frac{m+1}{4} - 1; q\Big) &\\ = \frac{(q^m; q^m)_\infty^{(m-3)/4}}{(q)_\infty^{(m-7)/4}(q^{1/2}; q^{1/2})_\infty} &\prod_{i=1}^{(m-3)/4} \theta(q^{i-1/2}; q^m) \\ &\prod_{1\le i<j\le (m-3)/4} \theta(q^{j-i}, q^{i+j-1}; q^m), \qquad (3\text{-}2)\end{aligned}$$

which gives

$$
\begin{aligned}
&\frac{T\left(\frac{m+1}{2}, \frac{m+1}{4}; q\right)}{T\left(\frac{m+1}{2}+2, \frac{m+1}{4}-1; q\right)} \\
&\qquad = \frac{(q^m;q^m)_\infty}{(q)_\infty}\theta(q^{(m-1)/4};q^m)\prod_{i=1}^{(m+1)/4-1}\theta(q^{(m+1)/4-i},q^{(m+1)/4-1+i};q^m) \\
&\qquad = \frac{(q^m;q^m)_\infty}{(q)_\infty}\theta(q^{(m-1)/4};q^m)\prod_{i=1}^{(m+1)/4-1}\theta(q^i;q^m)\prod_{i=(m+1)/4}^{(m+1)/2-2}\theta(q^i;q^m) \\
&\qquad = \frac{(q^m;q^m)_\infty}{(q)_\infty}\theta(q^{(m-1)/4};q^m)\prod_{i=1}^{(m-1)/2}\theta(q^i;q^m). \qquad (3\text{-}3)
\end{aligned}
$$

Using Theorem 2.2 and Proposition 2.4, we have

$$
\begin{aligned}
&S\Big(\frac{m+1}{2}-2, \frac{m+1}{4}; q\Big) \\
&\quad = \frac{(q^m;q^m)_\infty^{(m+1)/4-1}(q^{m/2};q^{m/2})_\infty}{(q)_\infty^{(m+1)/4-1}(q^{1/2};q^{1/2})_\infty}\prod_{i=1}^{(m+1)/4}\theta(q^i;q^m)\theta(q^{m/2-i};q^m) \\
&\qquad\qquad \prod_{1\le i<j\le(m+1)/4}\theta(q^{j-i},q^{i+j};q^m), \qquad (3\text{-}4)
\end{aligned}
$$

and

$$
\begin{aligned}
&S\Big(\frac{m+1}{2}, \frac{m+1}{4}-1; q\Big) \\
&\quad = \frac{(q^m;q^m)_\infty^{(m+1)/4-2}(q^{m/2};q^{m/2})_\infty}{(q)_\infty^{(m+1)/4-2}(q^{1/2};q^{1/2})_\infty}\prod_{i=1}^{(m+1)/4-1}\theta(q^i;q^m)\theta(q^{m/2-i};q^{m/2}) \\
&\qquad\qquad \prod_{1\le i<j\le(m+1)/4-1}\theta(q^{j-i},q^{i+j};q^m). \qquad (3\text{-}5)
\end{aligned}
$$

Now, we evaluate the following quotient:

$$
\begin{aligned}
&\frac{S\left(\frac{m+1}{2}-2, \frac{(m+1)}{4}; q\right)}{S\left(\frac{m+1}{2}, \frac{m+1}{4}-1; q\right)} \\
&\qquad\qquad = \frac{(q^m;q^m)_\infty}{(q)_\infty}\theta(q^{(m+1)/4};q^m)\theta(q^{(m-1)/4};q^m) \\
&\qquad\qquad\qquad \prod_{i=1}^{(m+1)/4-1}\theta(q^{(m+1)/4-i},q^{(m+1)/4+i};q^m)
\end{aligned}
$$

$$= \frac{(q^m;q^m)_\infty}{(q)_\infty}\theta(q^{(m+1)/4};q^m)\theta(q^{(m-1)/4};q^m) \prod_{i=1}^{(m+1)/4-1}\theta(q^i;q^m)\prod_{i=(m+1)/4+1}^{(m+1)/2-1}\theta(q^i;q^m)$$

$$= \frac{(q^m;q^m)_\infty}{(q)_\infty}\theta(q^{(m-1)/4;q^m})\prod_{i=1}^{(m-1)/2}\theta(q^i;q^m). \tag{3-6}$$

Dividing (3-3) by (3-6), we obtain

$$\begin{aligned}\frac{T\big(\frac{m+1}{2},\frac{m+1}{4};q\big)S\big(\frac{m+1}{2},\frac{m+1}{4}-1;q\big)}{T\big(\frac{m+1}{2}+2,\frac{m+1}{4}-1;q\big)S\big(\frac{m+1}{2}-2,\frac{m+1}{4};q\big)} &= \frac{(q^m;q^m)_\infty/(q)_\infty\,\theta(q^{(m-1)/4};q^m)\prod_{i=1}^{(m-3)/2}\theta(q^i;q^m)}{(q^m;q^m)_\infty/(q)_\infty\,\theta(q^{(m-1)/4};q^m)\prod_{i=1}^{(m-1)/2}\theta(q^i;q^m)} \\ &= \frac{1}{\theta(q^{(m-1)/2};q^m)}.\end{aligned} \tag{3-7}$$

By Theorem 2.1, we have the identity

$$R_{(1,-1)}\Big(\frac{m-1}{2}-1,1;q\Big) = \frac{(q^m;q^m)_\infty}{(q)_\infty}\theta(q^{(m-1)/2};q^m). \tag{3-8}$$

Multiplying (3-8) and (3-7) gives us our desired result:

$$\begin{aligned}\frac{T\big(\frac{m+1}{2}+1,\frac{m+1}{4};q\big)S\big(\frac{m+1}{2},\frac{m+1}{4}-1;q\big)R_{(1,-1)}\big(\frac{m-1}{2}-1,1;q\big)}{T\big(\frac{m+1}{2}+3,\frac{m+1}{4}-1;q\big)S\big(\frac{m+1}{2}-2,\frac{m+1}{4};q\big)} &= \frac{(q^m;q^m)_\infty}{(q)_\infty}\theta(q^{(m-1)/2};q^m)\cdot\frac{1}{\theta(q^{(m-1)/2};q^m)} \\ &= \frac{(q^m;q^m)_\infty}{(q)_\infty} = \prod_{t=1}^{\infty}\frac{(1-q^{mt})}{(1-q^t)} = D_m(q). \qquad \square\end{aligned}$$

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