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On halving-edges graphs

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On halving-edges graphs

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In this paper we study halving-edges graphs corresponding to a set of halving lines. Particularly, we study the vertex degrees, path, cycles and cliques of such graphs. In doing so, we study a vertex-partition of said graph called chains which are equipped with interesting properties.

1. Introduction

Halving lines have been an interesting object of study for a long time. Let n points be in general position in $\mathbb{R}^2$, where n is even. A *halving line* is a line through two of the points that splits the remaining $n-2$ points into two sets of equal size. The minimum number of halving lines is $\frac{1}{2}n$. The maximum number of halving lines is unknown. The first bounds were found by Lovász [1971] and by Erdős et al. [1973]. The current asymptotic upper bound of $O(n^{4/3})$ was proven by Dey [1998].

We approach the subject of halving lines by studying the properties of the underlying graph. From our set of n points, we define a *halving-edges graph* of n vertices, where each point is a vertex and each pair of vertices is connected by an edge if and only if there is a halving line through the corresponding two points; see [Matoušek 2002].

In Section 2 we discuss some basic properties of halving-edges graphs including degrees and the number of connected components. We also prove that any graph can be an induced subgraph of a halving-edges graph. In Section 3 we show that a halving-edges graph with n vertices can contain an $(n-1)$-path, and an $(n-3)$-cycle at most and provide a construction to show that the bound is exact. We give an example of a halving-edges graph containing a clique of size of at least $\sqrt{\frac{1}{2}n}$. We continue by studying chains, introduced by Dey [1998], in Section 4. The chain methods allow us to prove more properties of halving-edges graphs. In particular, we show that the largest clique cannot exceed a size of $\sqrt{2n}+1$.

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2. Basic properties of the halving-edges graph

The following properties of halving edges graphs are well known.

Lemma 2.1. *A halving-edges graph does not have isolated vertices. It has at least three leaves.*

Theorem 2.2. *Each vertex of a halving-edges graph has an odd degree.*

We will use another related result in the future.

Lemma 2.3. *Given two halving lines VP and VQ sharing a vertex V, there exists another halving line VR such that R lies in the opposite angle of $\angle PVQ$. Equivalently, the vectors $\vec{VP}$, $\vec{VQ}$, $\vec{VR}$ do not all lie on a single half-plane.*

As each vertex has at least one halving line passing through it, the minimum number of halving lines is $\frac{1}{2}n$. This number is achieved when points form a convex n-gon. Any number of halving lines between the lower bound and the upper bound is achievable as the following theorem states [Khovanova and Yang 2012].

Theorem 2.4. *For a fixed n, if there exist two configurations with k_1 and k_2 halving lines respectively, then for all k such that $k_1 \le k \le k_2$, there exists a configuration with k halving lines.*

Segmenterizing. We will use this construction a lot through this paper. Suppose we have a set of points. Any affine transformation does not change its halving-edges graph. Sometimes it is useful to picture that our points are squeezed into a long narrow rectangle. This way our points are almost on a segment. We call this procedure *segmenterizing*. Figure 1 shows three pictures. The first picture has six points, that we would squeeze towards the line $y = 0$. The second picture shows the configuration squeezed by a factor of 10, and if we make the factor arbitrarily large the points all lie very close to a segment, as shown in the last picture.

This procedure makes all the points very close to a single line segment and all the halving lines very close to this line. If we add a point not too close to this line, then it lies on the same side of all the halving lines. Moreover, it lies on the same side of all the lines connecting any two original points. Note that by the nature of affine transformations, we do not necessarily have to squeeze along the direction that is perpendicular to the segment.

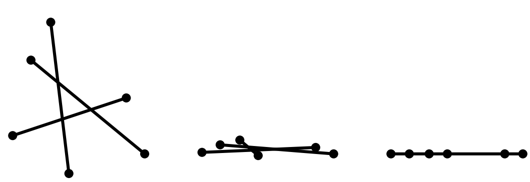

Figure 1. Segmenterizing.

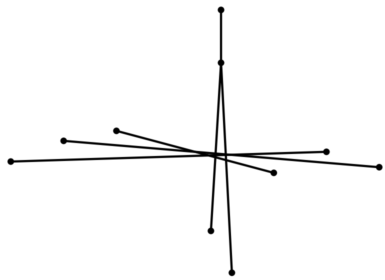

Figure 2. The cross construction.

Degrees and connected components. In the proof of the following lemma we need a construction we call a *cross*. Given two sets of points with n_1 and n_2 points respectively whose halving-edges graphs are G_1 and G_2, the cross is the construction of $n_1 + n_2$ points on the plane whose halving-edges graph has two isolated components G_1 and G_2. We form the cross as follows. Segmenterize graphs G_1 and G_2 and intersect the resulting segments at middle lines, so that half of the points of each segment lie on one side of all halving lines that pass through the points of the other segment (see Figure 2).

Lemma 2.5. *Any odd degree between* 1 *and* $n-1$ *can appear in a halving-edges graph of* n *vertices. Any number of connected components between* 1 *and* $\frac{1}{2}n$ *inclusive can appear in a halving-edges graph of* n *vertices.*

Proof. Consider a configuration with $2k$ vertices, where all but one of them are on a convex hull. The resulting halving graph is a star. We build a cross of this star graph and of a convex polygon with $n-2k$ vertices. The cross has $\frac{1}{2}n-k+1$ connected components. It has $n-1$ leaves and one vertex of degree $2k-1$. □

Degree sequence. The *degree sequence* of a graph is the nonincreasing sequence of its vertex degrees. The Erdős–Gallai [1960] theorem describes which sequences could be degree sequences of graphs.

Theorem 2.6 (Erdős–Gallai theorem). *A nondecreasing sequence of* n *numbers* d_i *is the degree sequence of a simple graph if and only if the total sum of degrees is even and*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k) \quad \textit{for } k \in \{1, \ldots, n\}.$$

The following lemma is about vertices of large degrees in a halving graph.

Lemma 2.7. *At most one vertex can have degree* $n-1$, *at most three vertices can have degree* $n-3$. *If the halving-edges graph has a vertex of degree* $n-1$, *then it is a star graph.*

The proof is straightforward [Khovanova and Yang 2012].

Lemma 2.8. *Any degree sequence consisting of only ones and threes, with at least* 3 *ones, is achievable by the halving-edges graph of some configuration.*

Proof. The degree sequence with 3 ones and everything else threes corresponds to the configuration in the path construction in Lemma 3.1. This configuration crossed with a matching graph can produce any odd number of ones with the rest being threes.

To achieve an even number of ones, we can use the following modified version of the path construction: replace the two vertices lying on the y-axis by two vertices that form a horizontal segment which makes the bottom side of the convex hull. Under this configuration, the four vertices on the convex hull have degree 1, and the remaining vertices have degree 3. □

3. Paths cycles, and cliques

Paths. Here we consider the size of non-self-intersecting paths in halving graphs. A path cannot have more than two leaves, so an easy upper bound for the largest path is $n-1$ vertices. It turns out that this bound is exact.

Lemma 3.1. *For every n, there exists a halving-edges graph of size n having a path through $n-1$ vertices.*

Proof. Figure 3 shows the path construction for a configuration with eight points. To avoid clutter, only relevant halving lines are shown by thin lines and thick lines show the path in the halving-edges graph. We generalize this construction to any n.

Consider $\frac{1}{2}(n-2)$ points that lie on a concave function. We segmenterize these points onto a segment lying on the x-axis. Now we place one such segment onto a line $y = x$, to the right of the origin, and another segment on the line $y = -x$ to the left of the origin. We keep the segments oriented in such a way that a line that passes through any two neighboring points of a segment has the remaining $\frac{1}{2}(n-2)-2$ points of the segment below it. Now add two more points: $(0, -1)$ and $(0, -2)$. Thus, every line that passes through two neighboring points of a segment

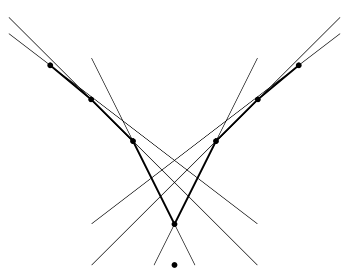

Figure 3. Path.

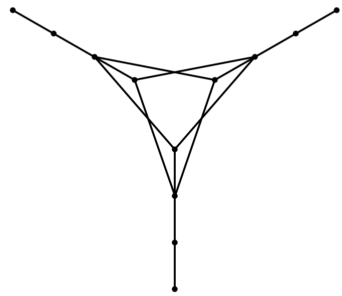

Figure 4. The Y-shape construction.

becomes a halving line. In addition, the point $(0, -1)$ forms halving lines with the rightmost point of the left segment and the leftmost point of the right segment.

The path goes through every point except $(0, -2)$, forming a V-shape. □

Cycles. Here we consider the size of cycles in halving-edges graphs. Vertices on the convex hull cannot be part of a cycle, so an easy upper bound for the length of the largest cycle is $n - 3$. It turns out that this bound is asymptotically exact. But first we need to introduce the Y-shape construction.

Suppose we have three configurations G_1, G_2, and G_3 with n points each and k_1, k_2, and k_3 halving lines correspondingly. The Y-shape construction allows us to build a new configuration with $3n$ points which has each of the three initial configurations as a subgraph and has a total of $k_1 + k_2 + k_3 + \frac{3}{2}n$ halving lines.

The construction works as follows. We segmenterize each set of points G_i. Then we draw three rays emanating from the origin, forming an angle of 120 degrees between each other, and place each segmenterized set of points along one of the rays; see Figure 4. This makes a Y-shape of $3n$ points, with n points on each branch.

On an individual branch, all halving lines prior to segmenterization remain halving lines. In addition, we can find halving lines that go through two points on different branches of the Y-shape. There are a total of $\frac{3}{2}n$ such lines, so we have produced a configuration with $3n$ points and $k_1 + k_2 + k_3 + \frac{3}{2}n$ halving lines.

Lemma 3.2. *Suppose a configuration of points with two neighboring points on the convex hull, denoted by A and B, is given. We can segmenterize in such a way and choose a direction on the segment so that A becomes the first point of the segment and B the k-th point, for $1 < k \leq n$, where n is the total number of points.*

See the proof in [Khovanova and Yang 2012]. The *halving difference* of a line is the difference of the number of points on each side of the line. Sometimes we will produce a construction that does not disturb the halving difference of certain lines. That is to say, we add the same number of points on both sides of the line and the difference is preserved.

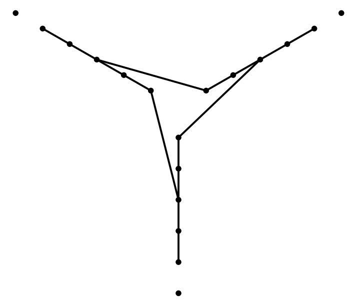

Figure 5. Cycle.

Theorem 3.3. *If n is a multiple of* 6, *the maximum length of a cycle is exactly $n-3$.*

Proof. We can write $n = 3b$, where b is even. Using Lemma 3.1, we can create a configuration of b points with a path of length $b-1$. Note that the endpoints of the path (of the V-shape) are neighboring points on a convex hull. This allows us to use Lemma 3.2 to segmenterize this configuration so that the endpoints of the path occupy the positions 1 and $\frac{1}{2}b$.

Now we use three copies of this segment in the Y-shape construction. We orient segments in such a way that the point 1 is oriented closer to the center of the construction. The edges of the $(b-1)$-path inside each branch all remain edges, and we also have edges between the first points of the branches and $\frac{1}{2}b$ points connecting all these paths together. This creates a cycle of length $n-3$ as desired. In Figure 5 we demonstrate this cycle for 18 vertices. Note that each branch has six vertices and the outermost vertex of each branch does not belong to the cycle. □

Induced subgraphs. We just showed that a halving-edges graph can contain a large path/cycle as a subgraph. If we restrict the graph to the vertices of the path/cycle that we constructed above, we can see that the graph has extra edges in addition to the path/cycle. To differentiate any subgraph from a subgraph that retains all the edges, the notion of induced subgraph is used.

A subgraph H of graph G is said to be an *induced* subgraph if any pair of vertices in H is connected by an edge if and only if it is connected by an edge in G.

Theorem 3.4. *Any graph with $2k$ vertices and e edges can be an induced subgraph of a halving-edges graph with at most $2k+2ek-4e+2\binom{2k}{2}$ vertices.*

Proof. Notice that if the number of vertices is even, then every line has an even halving difference. We process the configuration line by line. Take a line. Suppose we want to make it a halving line. For this we need to add an even number of points on one side of the line without disturbing the halving difference of other lines. If it is a halving line and we want to make it a nonhalving line we can add 2 points on

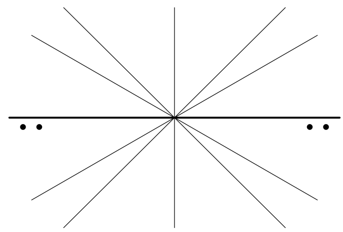

Figure 6. Zooming out and adding points.

one side. Let us draw all possible lines connecting the points and zoom out. From a big distance the point configuration will look like a bunch of lines intersecting at one point; see Figure 6. In Figure 6 the thick line is the line we are processing. Suppose the line needs an addition of four points below it. We add half of the points (two in our example) below the line far away on each side.

Each line that should be an edge in the new halving-edges graph requires an addition of at most $2k-2$ vertices. All of the future edges require at most $2ek-2e$ extra points. Other lines require at most 2 points each for a total of $2\left(\binom{2k}{2}-e\right)$. □

Cliques. Halving-edges graphs with 4 vertices have cliques of size 2. We can have a clique of size 3 in a graph with six vertices. By Theorem 3.4 we can have a clique of size n as an induced subgraph in a halving-edges graph of size $O(n^3)$. In this subsection we would like to improve the bound by using a construction similar to the construction of Theorem 3.4, where we process several lines at a time. Clustering lines together allows us to reduce the total number of extra points that we need.

Theorem 3.5. *The largest possible clique in a halving-edges graph of n vertices is at least* $\Omega(\sqrt{n})$.

Proof. Let k be even. To produce a clique of size k, take a regular k-gon, and distort it a little bit using a projective transformation that makes one end of the k-gon slightly wider than the other end. This perturbs all the diagonals (and sides) of the k-gon that were once parallel, making them intersect somewhere far away from the polygon, but still remain nearly parallel. You can imagine the k-gon as drawn on the floor in a painting that respects the perspective properly. This way the lines that are parallel in the k-gon intersect on a point on the horizon line in the painting. We assume that the k-gon is in a general position, that is, no two vertices are connected by a line parallel to the horizon. Note that there are now k sets of nearly parallel diagonals and sides, each set having either $\frac{1}{2}k$ or $\frac{1}{2}k-1$ lines.

We will now add $O(k^2)$ points to turn this k-gon into a k-clique. Consider a set of $\frac{1}{2}k$ or $\frac{1}{2}k-1$ nearly parallel lines. We will process each cluster of lines separately. In Figure 7 we depict one cluster of near parallel sides and diagonals. We rotated the picture so that it fits better in the page, and now the imaginary horizon line is a

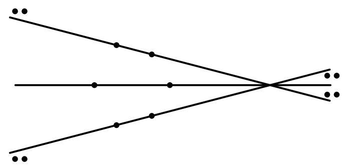

Figure 7. The cluster of nearly parallel lines.

vertical line through the intersection points on the right. On the half-plane beyond the horizon line add two points between every pair of consecutive lines. This way each line in the cluster becomes a halving line. In addition, we want every cluster to be independent. That means we want to add more points so that the halving difference of every line that is not in the cluster does not change.

We just added to the right of all other lines that are not in the cluster either $k-2$ or $k-4$ points. We need to add the same number of points to the left of all other lines as not to disturb the halving difference we just created in this cluster. The extra points you can see on the picture are put into two equal groups on the left above and below the current cluster.

This process requires a total of $2k-4$ or $2k-8$ new points, but turns all of our nearly parallel lines into halving lines without disturbing the other diagonals and sides. We do this a total of k times for each set of nearly parallel lines, and we have constructed a halving-edges graph with a k-clique by adding $2k^2-6k$ points.

Given n, we have shown how to construct a halving-edges graph with a clique of size at least $\sqrt{\frac{1}{2}n}$ with no more than n vertices. We can pad this graph to any number of vertices by crossing it with 2-paths. □

We will discuss the upper bound on the size of the clique later.

Any graph can be a subgraph of a clique, so an arbitrary graph with k vertices can always be found as a subgraph, not necessarily induced, of a halving-edges graph with no more than $O(k^2)$ vertices.

4. Chains

We define the following algorithm to group the halving lines into sets that are called *chains*, introduced by Dey [1998].

Choose an orientation to define as "up". The $\frac{1}{2}n$ leftmost vertices are called the left half, and the rightmost vertices are called the right half. We assume that no two points are vertically aligned, so that leftmost and rightmost are well defined. Start with a vertex on the left half of the graph, and take a vertical line passing through this vertex. Rotate this line clockwise until it either aligns itself with an edge, or becomes vertical again. If it aligns itself with an edge in the halving-edges graph, define this edge to be part of the chain, and continue rotating the line about the rightmost vertex in the current chain. If the line becomes vertical, we terminate the process. The set of edges in our set is defined as the chain. Repeat on a different point on the left half of the halving-edges graph until every edge is part of a chain.

Note that the chains we get are determined by which direction we choose as “up”. The following properties of chains follow immediately. Later properties on the list follow from the previous ones:

- A vertex on the left half of the halving-edges graph is a left endpoint of a chain.
- The process is reversible. We could start each chain from the right half and rotate the line counterclockwise instead, and obtain the same chains.
- A vertex on the right half of the halving-edges graph is a right endpoint of a chain.
- Every vertex is the endpoint of exactly one chain.
- The number of chains is exactly $\frac{1}{2}n$.
- The degrees of the vertices are odd. Indeed, each vertex has one chain ending at it and several passing through it.
- Every halving line is part of exactly one chain.
- The length of each chain is bounded by $\frac{1}{2}n$.

The following property bounds the number of vertices with a large degree.

Lemma 4.1. *For every integer k, a halving graph has at most $2k$ vertices with degree $n-2k+1$.*

Proof. The i-th vertex from the left in the left half plane can have at most $i-1$ chains passing through it and is a start of exactly one chain. So its degree cannot be more than $2i-1$. Hence, only k rightmost vertices in the left plane and k leftmost vertices in the right plane can have degree $n-2k+1$. □

The sums of degrees of two vertices. Now we use our knowledge about chains to refine our knowledge about degrees of the vertices of the halving-edges graph.

Theorem 4.2. *The degrees of two distinct vertices sum to at most n, if they are connected by an edge, and at most $n-2$ otherwise.*

Proof. Denote the vertices in question as P and Q. Rotate the geometric graph until segment PQ is nearly vertical, so that there are no vertices between the horizontal projections of P and Q. If P and Q do not belong to the same chain, then each of the $\frac{1}{2}n$ chains contributes at most 2 to the sum of degrees of P and Q. We have to subtract 2 from this sum since P and Q are both endpoints of some chains. Thus the total sum does not exceed $n-2$.

If PQ is an edge, then it can add two more to the sum of degrees making it at most n. □

It immediately follows that the largest clique in the halving-edges graph cannot be bigger than $\frac{1}{2}n$. We can use chains to prove an upper bound on the size of the largest clique that is much closer to the lower bound. But before doing so we would like to introduce some definitions.

The straddling span and the largest clique. Given a line that does not pass through any vertex of a given graph, we call edges that intersect it *straddling* edges. The maximum number of straddling edges that can be produced by a line is called the *straddling span* of the halving-edges geometric graph. Naturally, this notion applies to subgraphs as well. Let us consider some examples (see [Khovanova and Yang 2012]).

- The straddling span of a k-clique is at least $\lfloor \frac{1}{4}k^2 \rfloor$.
- The straddling span of an (a, b)-complete bipartite graph is at least $\frac{1}{2}ab$.

Theorem 4.3. *If a halving-edges geometric graph has straddling span w, then it has at least $\frac{1}{2}w$ vertices.*

Proof. Choose the up direction along the line that produces the straddling span. We claim that no two straddling edges belong to the same chain. Indeed, if two edges are straddling, then their projections onto the x-axis must overlap at the point that is the projection of the line that produces the straddling span. But it is clear that the projections along the x-axis of the edges of any given chain must be mutually nonoverlapping. Therefore, our graph contains at least one chain for every straddling edge. Since there are at least w straddling edges, the number of chains must be at least the same and the number of vertices must be at least $\frac{1}{2}w$. □

Corollary 4.4. *If a halving-edges graph contains a k-clique, then it has at least $\lfloor \frac{1}{2}k^2 \rfloor$ vertices. Consequently, the largest clique in the halving-edges graph with n vertices cannot exceed $\sqrt{2n} + 1$ vertices.*

Corollary 4.5. *If a halving-edges graph contains an (a, b)-complete bipartite subgraph, then it has at least ab vertices.*

Note that now both the lower bound and the upper bound for the largest clique are on the order of $\sqrt{n}$.

5. Acknowledgements

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