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2018 vol. 11, no. 1

INVOLVE 11:1 (2018)
dx.doi.org/10.2140/involve.2018.11.67

Labeling crossed prisms with a condition at distance two

Matthew Beaudouin-Lafon, Serena Chen, Nathaniel Karst, Jessica Oehrlein and Denise Sakai Troxell

(Communicated by Jerrold Griggs)

An *L(2,1)-labeling* of a graph is an assignment of nonnegative integers to its vertices such that adjacent vertices are assigned labels at least two apart, and vertices at distance two are assigned labels at least one apart. The *λ-number* of a graph is the minimum span of labels over all its L(2,1)-labelings. A *generalized Petersen graph* (GPG) of order n consists of two disjoint cycles on n vertices, called the *inner* and *outer cycles*, respectively, together with a perfect matching in which each matching edge connects a vertex in the inner cycle to a vertex in the outer cycle. A *prism* of order $n \geq 3$ is a GPG that is isomorphic to the Cartesian product of a path on two vertices and a cycle on n vertices. A *crossed prism* is a GPG obtained from a prism by crossing two of its matching edges; that is, swapping the two inner cycle vertices on these edges. We show that the λ-number of a crossed prism is 5, 6, or 7 and provide complete characterizations of crossed prisms attaining each one of these λ-numbers.

1. Introduction

The labelings of graphs with a condition at distance two, also known as L(2,1)-labelings, have provided a fertile area of research for about a quarter of a century since their introduction in [Griggs and Yeh 1992]. These labelings were first used to model simplified instances of the channel assignment problem [Hale 1980] where geographically close transmitters in a communications network must receive frequency channels that are sufficiently far apart to avoid signal interference. The scholarly works on L(2,1)-labelings and their variations are numerous and touch upon a wide range of applied as well as purely theoretical aspects of such labelings. Notably, optimization questions concerning the minimum span of labels required by different types of graphs have consistently attracted a great deal of interest.

MSC2010: primary 68R10, 94C15; secondary 05C15, 05C78.

Keywords: L(2,1)-labeling, L(2,1)-coloring, distance two labeling, channel assignment, generalized Petersen graph.

An L(2,1)-labeling of a graph G, or *k-labeling* for short, is a function $f : \mathrm{V}(G) \to \{0, 1, \ldots, k\}$ such that $|f(u) - f(v)| \geq 2$ if u and v are adjacent vertices, and $|f(u) - f(v)| \geq 1$ if u and v are at distance 2. The minimum k so that G has a k-labeling is called the *λ-number* of G and is denoted by $\lambda(G)$. Arguably, the appeal of this number has its roots in the long-standing conjecture stating that $\lambda(G) \leq \Delta^2(G)$ for $\Delta(G) \geq 2$, where $\Delta(G)$ denotes the maximum degree of G [Griggs and Yeh 1992]. This conjecture, which is sometimes referred to as the *Δ^2-conjecture*, holds for very large graphs (with $\Delta(G)$ larger than approximately 10^{69} [Havet et al. 2012]), for sufficiently small graphs (with at most $(\lfloor \Delta(G)/2 \rfloor + 1)(\Delta^2(G) - \Delta(G) + 1) - 1$ vertices [Franks 2015]), and for several particular classes of graphs. In addition, it has been possible to determine tighter bounds and even exact λ-numbers within some of these classes through interesting, nontrivial techniques, contributing to the incremental progress toward settling the Δ^2-conjecture. An extensive annotated bibliography of related articles can be found in [Calamoneri 2011] and in its 2014 updated online version.

Determining exact λ-numbers can be a complex task even when considering seemingly basic graphs, such as the following generalizations of the classic Petersen graph (shown on the right of Figure 1 together with a 9-labeling).

Definition 1.1. A *generalized Petersen graph* (GPG) of order $n \geq 3$ consists of two disjoint cycles, called *outer* and *inner cycles*, so that each vertex on the outer (resp., inner) cycle is adjacent to exactly one vertex on the inner (resp., outer) cycle. More formally, a GPG has vertices $\{v_0, v_1, \ldots, v_{n-1}\} \cup \{w_0, w_1, \ldots, w_{n-1}\}$ with edges $\{v_i, v_{i+1}\}$ and $\{w_i, w_{i+1}\}$ for all $i = 0, 1, \ldots, n-1$, where subscript addition is taken modulo n, and each v_i (resp., w_i), $i = 0, 1, \ldots, n-1$ is adjacent to exactly one w_j (resp., v_j) for some $0 \leq j \leq n-1$. The cycle on vertices $\{v_0, v_1, \ldots, v_{n-1}\}$ (resp., $\{w_0, w_1, \ldots, w_{n-1}\}$) is the outer (resp., inner) cycle.

Observe that if G is a GPG of order $n \geq 3$, then G is 3-regular, so the Δ^2-conjecture states that $\lambda(G) \leq 9$. This upper bound is tight if G is the Petersen graph since it has diameter 2 and a 9-labeling [Griggs and Yeh 1992]. In contrast, if G is anything other than the Petersen graph, then $\lambda(G) \geq 5$, $\lambda(G) \leq 7$ if $n \leq 6$, and $\lambda(G) \leq 8$ if $n \geq 7$ [Georges and Mauro 2002]. Therefore, the GPGs satisfy the Δ^2-conjecture. As no GPG with λ-number exactly 8 is known, it has been conjectured that if G is a GPG of order $n \geq 7$, then $\lambda(G) \leq 7$. This *GPG conjecture* has remained open since 2002 but has been verified for all GPGs of orders between 7 and 12 for which exact λ-numbers were completely determined [Adams et al. 2006; 2007; 2012; Huang et al. 2012]. In an attempt to expand the list of graphs satisfying the GPG conjecture, some articles have focused on the exact λ-numbers of infinite subclasses of GPGs that exhibit certain symmetric features. For instance, a *prism* (resp., an *n-star* for odd n) is a GPG of order $n \geq 3$ wherein the edges between

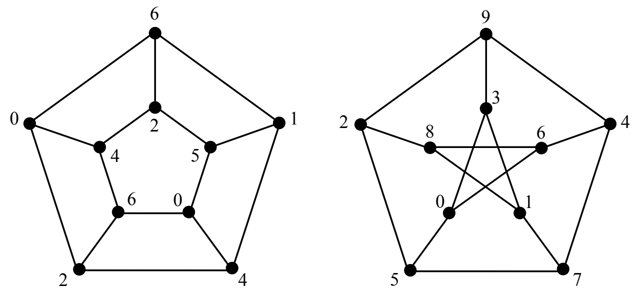

Figure 1. The prism of order 5 and the 5-star (Petersen graph) with respective L(2, 1)-labelings.

vertices on the outer and inner cycles are precisely $\{v_i, w_i\}$, $i = 0, 1, \ldots, n-1$ (resp., $\{v_{(n-1)i/2}, w_i\}$ for $i = 0, 1, \ldots, n-1$ where subscripts are taken modulo n), and the notation is as introduced in Definition 1.1. The prism of order 5 and the 5-star with respective L(2,1)-labelings are shown in Figure 1.

The λ-numbers of prisms have been completely determined in [Georges and Mauro 2002; Jha et al. 2000; Klavžar and Vesel 2003; Kuo and Yan 2004], and of n-stars in [Adams et al. 2007] using nontrivial techniques. Key to some of these were ingenious connections between the regularity and symmetry of these graphs used in [Georges and Mauro 2003; Adams et al. 2007] that would force impossible configurations of labels within 5-labelings for certain values of n. We were curious to see if the same strategies could be extended to other subclasses of GPGs where this symmetry would be slightly disturbed. This motivated our focus on GPGs obtained from prisms by "crossing" two edges connecting the outer cycle to the inner cycle:

Definition 1.2. Let n and d be integers so that $n \geq 3$ and $1 \leq d \leq n/2$. The *crossed prism* $XPr(n, d)$ is a prism of order n where the edges $\{v_0, w_0\}$ and $\{v_d, w_d\}$ are replaced by the crossed edges $\{v_0, w_d\}$ and $\{v_d, w_0\}$, with the notation as introduced in the definition of prisms. The *cross* $X(d)$ is the graph isomorphic to the subgraph of $XPr(n, d)$ induced by the vertices $\{v_0, v_1, \ldots, v_d\} \cup \{w_0, w_1, \ldots, w_d\}$.

Figure 2 shows the crossed prism $XPr(5, i)$ with the cross $X(i)$ within the dashed oval for $i = 1, 2$, respectively.

It will be helpful to visualize the crossed prism $XPr(n, d)$ as copies of the two crosses $X(d)$ and $X(n-d)$ sharing the same crossed edges but otherwise disjoint. To illustrate, Figure 3 shows a 3-dimensional cylindrical representation of $XPr(9, 4)$ on the left and the crosses $X(4)$ and $X(5)$ on the top and bottom right, respectively (crossed edges in bold to facilitate their visualization within the graphs).

Let f be an L(2,1)-labeling of $XPr(n, d)$. It will be often convenient to provide f as a 2-by-n matrix $A(n, d)$ where the entry on the i-th row, j-th column will be

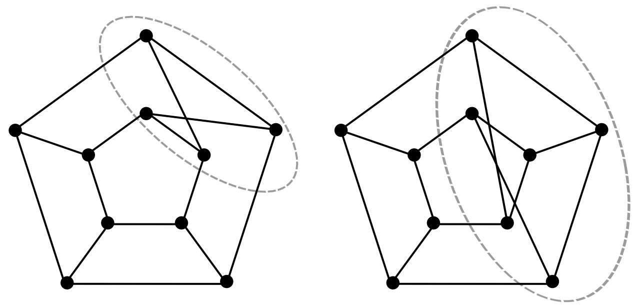

Figure 2. The crossed prism $XPr(5, i)$ with the cross $X(i)$ within the dashed oval for $i = 1, 2$, respectively.

the label $f(v_j)$ if $i = 0$, and $f(w_j)$ if $i = 1$, for $j = 0, 1, \dots, n-1$. Notice that the matrix $A(d)$ given by the first $d+1$ columns of $A(n, d)$ is an L(2,1)-labeling of the cross $X(d)$. Similarly, the matrix $A(n-d)$ given by the last $n-d$ columns followed by the first column of $A(n, d)$ is an L(2,1)-labeling of the cross $X(n-d)$. These conventions are illustrated in Figure 4 with 6-labelings of $XPr(9, 4)$, $X(4)$, and $X(5)$ of Figure 3 given by the matrices $A(9, 4)$, $A(4)$, and $A(5)$, respectively.

The strategies used to find the λ-numbers of prisms leveraged the symmetries of these graphs, and even the minor breaks in symmetry introduced in crossed prisms prohibit the simple extension of these proof techniques into this new context. Nevertheless, we were able to use certain properties of crosses to determine the λ-numbers of all crossed prisms. In Section 2, we find the exact λ-number of

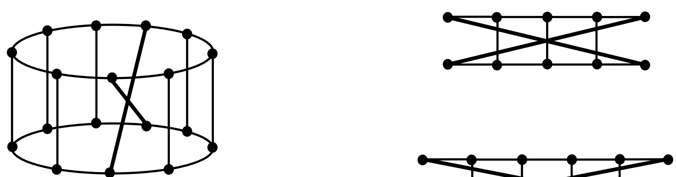

Figure 3. The crossed prism $XPr(9, 4)$, left, and the crosses $X(4)$, top right, and $X(5)$, bottom right.

$$A(9,4) = \boxed{\begin{matrix} 3 & 6 & 0 & 3 & 6 & 2 & 0 & 4 & 1 \\ 0 & 4 & 2 & 5 & 0 & 4 & 6 & 2 & 5 \end{matrix}}$$

$$A(4) = \boxed{\begin{matrix} 3 & 6 & 0 & 3 & 6 \\ 0 & 4 & 2 & 5 & 0 \end{matrix}} \qquad A(5) = \boxed{\begin{matrix} 6 & 2 & 0 & 4 & 1 & 3 \\ 0 & 4 & 6 & 2 & 5 & 0 \end{matrix}}$$

Figure 4. The 6-labelings $A(9, 4)$, $A(4)$, and $A(5)$ of $XPr(9, 4)$, $X(4)$, and $X(5)$, respectively.

$X(d)$ for all $d \geq 1$, as well as exhibit all possible 5-labelings when $d \geq 2$ using an auxiliary directed graph where the vertices are particular 2-by-2 matrices with entries in $\{0, 1, \dots, 5\}$. These results allow us to raise the general lower bound for the λ-number of a GPG from 5 to 6 if it contains a subgraph isomorphic to certain crosses, ultimately enabling us to verify the following result in Section 3.

Theorem 1.3. *Let n and d be integers so that $n \geq 3$ and $1 \leq d \leq n/2$. If G is the crossed prism $XPr(n, d)$, then $\lambda(G) = 5$ when*

(a) $d = 1$ *and* $n = 3$; *or*

(b) $d \equiv 0 \pmod 3$ *and* $(n - d) \equiv 0 \pmod 3$; *or*

(c) $d \equiv 1 \pmod 3$ *and* $(n - d) \equiv 1 \pmod 3$ *with* $d \geq 7$.

Furthermore, $\lambda(G) = 7$ *when* $d = 1$ *and* $n = 4$; *otherwise* $\lambda(G) = 6$.

2. The λ-number of crosses

Now, we will completely determine the λ-number of crosses $X(d)$ with $d \geq 1$ in Theorem 2.4, the main result in this section. The following definitions will simplify the description of an auxiliary directed graph that will be helpful in the preliminary discussion. A sequence of nonnegative integers $x_1, x_2, \dots, x_m$ *induces* a k-labeling of the path P_m with vertices $u_1, u_2, \dots, u_m$ and edges $\{u_i, u_{i+1}\}$ for $i = 1, 2, \dots, m - 1$, if the assignment of x_i to u_i for $i = 1, 2, \dots, m$ produces a k-labeling of P_m.

Let D be the directed graph with vertex set containing the 2-by-2 matrices M with entries in $\{0, 1, \dots, 5\}$ such that:

- the sequence $M_{0,0}, M_{0,1}, M_{1,1}, M_{1,0}$ induces a 5-labeling of P_4, in which case M is called a *left-vertex*; or
- the sequence $M_{0,1}, M_{0,0}, M_{1,0}, M_{1,1}$ induces a 5-labeling of P_4, in which case M is called a *right-vertex*.

Notice that a vertex can be both a left- and right-vertex. Given a left-vertex M and a right-vertex N different from M, the directed edge set of D contains:

- the *solid edge* (M, N), if $M_{i,1} = N_{i,0}$ for $i = 0, 1$ (i.e., the last column of M is equal to the first column of N), $M_{0,0} \neq N_{0,1}$, and $M_{1,0} \neq N_{1,1}$; and
- the *dashed edge* (N, M), if there exists a directed path of solid edges from M to N of length at least 1 so that the two sequences $(N_{0,0}, N_{0,1}, M_{1,0}, M_{1,1})$ and $(N_{1,0}, N_{1,1}, M_{0,0}, M_{0,1})$ each induces a 5-labeling of P_4.

Observe that there is a natural one-to-one relationship between the set of 5-labelings of crosses $X(d)$ with $d \geq 2$ and the set of the *X-cycles* defined as directed cycles in D containing exactly one dashed edge. More specifically, for a 5-labeling

$$A(7) = \boxed{\begin{matrix} 0 & 2 & 5 & 0 & 2 & 4 & 1 & 5 \\ 0 & 4 & 1 & 3 & 5 & 0 & 3 & 5 \end{matrix}}$$

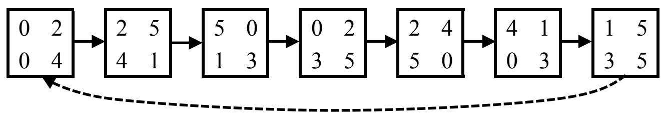

Figure 5. A 5-labeling $A(7)$ of $X(7)$ and corresponding X-cycle.

of the cross $X(d)$ represented by a 2-by-$(d+1)$ matrix $A(d)$, consider for each $i = 0, 1, \ldots, d-1$, the 2-by-2 submatrix $M(i)$ with the i-th and $(i+1)$-th columns of $A(d)$. From the definition of D, it is straightforward to verify that the vertices $M(i)$ for $i = 0, 1, \ldots, d-1$ induce an X-cycle and, moreover, this correspondence is one-to-one. We illustrate this correspondence in Figure 5 with a 5-labeling $A(7)$ of $X(7)$ and its associated X-cycle. In particular, solid and dashed edges are represented by solid and dashed arrows, respectively. The start and end vertices of the maximal directed path with solid edges within the X-cycle are left- and right-vertices, respectively. For the sake of simplicity, we will sometimes abuse the notation and use a 5-labeling in matrix form to refer to the corresponding X-cycle and vice-versa.

We define three operations on subgraphs D^* of D that will simplify the description of some of its properties:

- *dual* of D^*: replace entry j of every vertex of D^* with its *dual* $5-j$.
- *flip* of D^*: swap the two rows of every vertex of D^*.
- *reverse* of D^*: swap the two columns of every vertex of D^*, and reverse the direction of every edge of D^*.

Notice that each of these operations coincides with its inverse and preserves the structure of X-cycles.

To generate the directed graph D, a computer program classified each of the 6^4 matrices with entries in $\{0, 1, \ldots, 5\}$ as a left- and/or right-vertex of D if possible, and discarded it otherwise. The algorithm then considered each pair of a left-vertex M and right-vertex N different from M, and added a solid edge (M, N) and/or dashed edge (N, M) if the pair satisfied the associated definition stated above. This algorithm relies on brute force — every vertex pair is considered individually — and could certainly be improved by cleverly integrating results about duals, flips, and reverses. Still, this algorithm is sound in the sense that every edge added satisfies either the solid or dashed edge definition, and complete, in the sense that every 2-by-2 matrix was considered from the outset and every pair of left- and right-vertices was tested for both solid and dashed connections.

$$A(7+3q) = \boxed{\begin{array}{ccccccccccc} 0 & 2 & 5 & 0 & 2 & 4 & 0 & 2 & 4 & 1 & 5 \\ 0 & 4 & 1 & 3 & 5 & 1 & 3 & 5 & 0 & 3 & 5 \end{array}}$$

$$A(3+3q) = \boxed{\begin{array}{ccccccc} 2 & 4 & 0 & 2 & 4 & 0 & 2 \\ 5 & 1 & 3 & 5 & 1 & 3 & 5 \end{array}} \text{ or } \boxed{\begin{array}{ccccccc} 4 & 0 & 2 & 4 & 0 & 2 & 4 \\ 1 & 3 & 5 & 1 & 3 & 5 & 1 \end{array}}$$

$$\text{or } \boxed{\begin{array}{ccccccc} 0 & 2 & 4 & 0 & 2 & 4 & 0 \\ 3 & 5 & 1 & 3 & 5 & 1 & 3 \end{array}}$$

$$A(3) = \boxed{\begin{array}{cccc} 2 & 4 & 1 & 5 \\ 2 & 0 & 3 & 5 \end{array}} \text{ or } \boxed{\begin{array}{cccc} 0 & 2 & 5 & 3 \\ 0 & 4 & 1 & 3 \end{array}}$$

$$A(2) = \boxed{\begin{array}{ccc} 0 & 2 & 4 \\ 1 & 5 & 3 \end{array}} \text{ or } \boxed{\begin{array}{ccc} 2 & 0 & 4 \\ 1 & 3 & 5 \end{array}} \text{ or } \boxed{\begin{array}{ccc} 0 & 2 & 4 \\ 0 & 5 & 3 \end{array}}$$

$$\text{or } \boxed{\begin{array}{ccc} 2 & 0 & 5 \\ 1 & 3 & 5 \end{array}} \text{ or } \boxed{\begin{array}{ccc} 2 & 4 & 1 \\ 3 & 0 & 5 \end{array}} \text{ or } \boxed{\begin{array}{ccc} 0 & 5 & 2 \\ 4 & 1 & 3 \end{array}}$$

Figure 6. X-cycles in component D_1 as 5-labelings $A(d)$ of crosses $X(d)$ with $d \geq 2$ (respective flips are not shown).

Excluding isolated vertices, the directed graph D consists of four connected components D_i for $i = 1, 2, 3, 4$, and their respective duals, where D_1, D_2, and D_3 are provided in the online supplement, and D_4 is the flip of D_3. The symmetry of crosses implies the following relationships among these components that can be verified by inspection:

(a) the flip of D_1 is D_1;

(b) the reverse of D_1 is the dual of D_1;

(c) the flip of D_2 is the dual of D_2;

(d) the reverse of D_2 is D_2;

(e) the reverse of D_3 (resp., D_4) is the dual of D_4 (resp., D_3).

In Lemmas 2.1 through 2.3, we exhibit all the X-cycles in components D_i for $i = 1, 2, 3, 4$. These cycles together with their duals (which are the X-cycles in the dual of D_i for $i = 1, 2, 3, 4$) are all possible 5-labelings $A(d)$ of crosses $X(d)$ with $d \geq 2$.

Lemma 2.1. *The X-cycles in component D_1 are given by the 5-labelings $A(d)$ of crosses $X(d)$ with $d \geq 2$ in Figure 6 and their respective flips (i.e., the flip of a matrix with two rows is obtained by swapping its rows). Each shaded block of three consecutive columns within a matrix can be replaced with $q \geq 0$ copies of itself, arranged consecutively as needed to reach the desired value of d (this convention will be used from this point forward).*

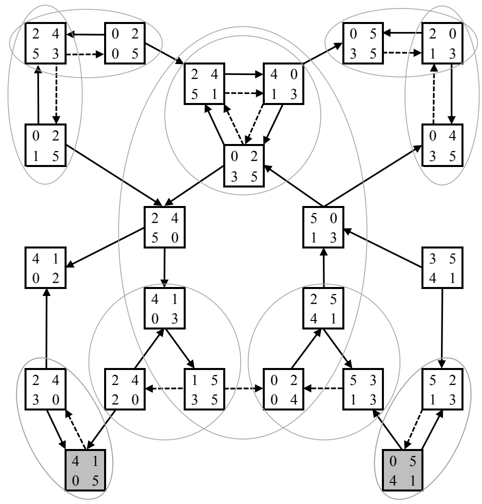

Figure 7. The directed subgraph H of D_1 and its X-cycles (circled).

Proof. Let H be the directed subgraph of D_1 in the online supplement induced by the two shaded vertices and all the vertices above them; H is shown in Figure 7. By inspection, one can verify that the flip of H is exactly the directed subgraph of D_1 induced by the two shaded vertices and all the vertices below them. Moreover, an X-cycle in component D_1 must be either completely within H or completely within the flip of H. Therefore, the lemma follows by exhibiting all the X-cycles within H. They are circled in Figure 7 and their corresponding 5-labelings of crosses are given in Figure 6. □

Lemma 2.2. *The X-cycles in component D_2 are given by the 5-labelings $A(2)$ of crosses $X(2)$ in Figure 8.*

Proof. Since all solid edges in D_2 in the online supplement are directed from left to right and the only four dashed edges are directed from right to left, it is straightforward to verify that there are only four X-cycles of length 2 with corresponding 5-labelings given in Figure 8. □

Lemma 2.3. *The only X-cycle in component D_3 (resp., D_4) is given by the 5-labeling $A(3)$ (resp., flip of $A(3)$) of cross $X(3)$ in Figure 9.*

$$A(2) = \boxed{\begin{matrix} 3 & 1 & 5 \\ 2 & 4 & 0 \end{matrix}} \quad \text{or} \quad \boxed{\begin{matrix} 1 & 5 & 2 \\ 4 & 0 & 3 \end{matrix}} \quad \text{or} \quad \boxed{\begin{matrix} 2 & 5 & 1 \\ 3 & 0 & 4 \end{matrix}} \quad \text{or} \quad \boxed{\begin{matrix} 5 & 1 & 3 \\ 0 & 4 & 2 \end{matrix}}$$

Figure 8. X-cycles in component D_2 as 5-labelings $A(2)$ of crosses $X(2)$.

$$A(3) = \begin{array}{|cccc|} \hline 1 & 5 & 2 & 4 \\ 1 & 3 & 0 & 4 \\ \hline \end{array}$$

Figure 9. The only X-cycle in D_3 as a 5-labeling $A(3)$ of cross $X(3)$.

$$A(4) = \begin{array}{|ccccc|} \hline 3 & 6 & 0 & 3 & 6 \\ 0 & 4 & 2 & 5 & 0 \\ \hline \end{array} \qquad A(5+3q) = \begin{array}{|ccccccccc|} \hline 3 & 1 & 6 & 4 & 0 & 6 & 4 & 2 & 6 \\ 0 & 5 & 3 & 1 & 5 & 3 & 1 & 5 & 0 \\ \hline \end{array}$$

Figure 10. 6-labelings of crosses $X(d)$ for $d = 4$ and for [$d \equiv 2 \pmod 3$ and $d \geq 5$], respectively.

Proof. It is straightforward to verify that there is only one X-cycle in D_3 in the online supplement with corresponding 5-labeling given in Figure 9. Since D_4 is the flip of D_3, the flip of $A(3)$ corresponds to the only X-cycle in D_4. □

We can finally state the main result of this section.

Theorem 2.4. *If G is the cross $X(d)$ with $d \geq 1$, then $\lambda(G) = 4$ when $d = 1$, $\lambda(G) = 6$ when $d = 4$ or when [$d \equiv 2 \pmod 3$ with $d \geq 5$], otherwise $\lambda(G) = 5$. In addition, the only possible 5-labelings of G when $d \geq 2$ are the ones in Figure 6 and 9 with their respective flips, the ones in Figure 8, and all the respective duals (i.e., the dual of a matrix is obtained by replacing each entry j with $5 - j$).*

Proof. The cross $X(1)$ is a cycle on four vertices which has λ-number 4 [Griggs and Yeh 1992]. The second sentence in the theorem's statement follows from the construction of D and Lemmas 2.1 through 2.3. Hence, if G has a 5-labeling, then $d = 2$, $d \equiv 0 \pmod 3$, or [$d \equiv 1 \pmod 3$ and $d \geq 7$] (refer to Figures 6, 8, and 9), thus $\lambda(G) = 5$ (recall from Section 1 that GPGs have λ-number at least 5). On the other hand, if G does not have a 5-labeling, then $\lambda(G) \geq 6$ and either $d = 4$ or [$d \equiv 2 \pmod 3$ and $d \geq 5$], thus $\lambda(G) = 6$ follows from the 6-labelings of G in Figure 10. □

We close this section by mentioning that the directed subgraph of D induced by its vertices that are simultaneously left- and right-vertices (vertices within double-lined squares in the online supplement and their respective duals) was used in [Klavžar and Vesel 2003] to exhibit 5-labelings of prisms.

3. The λ-number of crossed prisms

Let n and d be integers so that $n \geq 3$ and $1 \leq d \leq n/2$. The main goal of this section is to find the λ-number of the crossed prism $XPr(n, d)$; that is, prove Theorem 1.3 of Section 1. In Lemma 3.1 we will discuss the case $d \geq 2$, and the case $d = 1$ will be examined in Lemma 3.2.

The following construction of k-labelings of $XPr(n, d)$ for $d \geq 2$ using k-labelings of the crosses $X(d)$ and $X(n-d)$ will be useful in the proof of Lemma 3.1. Consider a k-labeling of the cross $X(d)$ given as a 2-by-$(d+1)$ matrix M. In addition, consider a k-labeling of the cross $X(n-d)$ given as a 2-by-$(n-d+1)$ matrix N. We will say that M and N *mesh* if the following two conditions are satisfied:

(i) $M_{i,d} = N_{i,0}$ for $i = 0, 1$ (i.e., the last column of M is equal to the first column of N), $M_{0,d-1} \neq N_{0,1}$, and $M_{1,d-1} \neq N_{1,1}$;

(ii) $N_{i,n-d} = M_{i,0}$ for $i = 0, 1$ (i.e., the last column of N is equal to the first column of M), $N_{0,n-d-1} \neq M_{0,1}$, and $N_{1,n-d-1} \neq M_{1,1}$.

Observe that if M and N mesh, then the matrix $mesh(M, N)$ obtained by combining the first d columns of M immediately followed by the first $n-d$ columns of N provides a k-labeling of the crossed prism $XPr(n, d)$. For example, $A(9, 4) = mesh(A(4), A(5))$ as seen in Figure 4 is a 6-labeling of $XPr(9, 4)$.

Lemma 3.1. *Let n and d be integers so that $n \geq 3$ and $2 \leq d \leq n/2$. If G is the crossed prism $XPr(n, d)$, then $\lambda(G) = 5$ when*

(a) $d \equiv 0 \pmod 3$ *and* $(n-d) \equiv 0 \pmod 3$; *or*

(b) $d \equiv 1 \pmod 3$ *and* $(n-d) \equiv 1 \pmod 3$ *with* $d \geq 7$.

Otherwise $\lambda(G) = 6$.

Proof. Suppose (a) holds. Select the first of the corresponding three choices for $A(3+3q)$ in Figure 6 (we could also select the second or the third choice instead). Let q_1 and q_2 be integers so that $d = 3+3q_1$ and $n-d = 3+3q_2$. Hence $A(3+3q_1)$ and $A(3+3q_2)$ are 5-labelings of the crosses $X(d)$ and $X(n-d)$, respectively, and these matrices mesh. From the observation right before the lemma, the matrix $mesh(A(3+3q_1), A(3+3q_2))$ is a 5-labeling of G, hence $\lambda(G) \leq 5$. Recall from Section 1 that GPGs have λ-number at least 5, therefore $\lambda(G) = 5$.

Suppose (b) holds. Select the $A(7+3q)$ in Figure 6. Let q_1 and q_2 be integers so that $d = 7+3q_1$ and $n-d = 7+3q_2$ (note that $n-d \geq d \geq 7$). Hence $A(7+3q_1)$and the dual of $A(7+3q_2)$ are 5-labelings of the crosses $X(d)$ and $X(n-d)$, respectively, and these matrices mesh. Similarly to the previous paragraph, we conclude that $\lambda(G) = 5$.

Suppose for the remainder of the proof that neither (a) nor (b) is satisfied. We will first show that $\lambda(G) \geq 6$. If $d = 4$ or [$d \equiv 2 \pmod 3$ with $d \geq 5$], then the λ-number of $X(d)$ is 6 by Theorem 2.4 and therefore $\lambda(G) \geq 6$ since $X(d)$ is a subgraph of G. Likewise, we can replace d with $n-d$ in the previous sentence and reach the same conclusion. To verify the remaining cases, we suppose for contradiction that G has a 5-labeling given by a 2-by-n matrix $A(n, d)$. The matrix M given by the first $d+1$ columns of $A(n, d)$ is a 5-labeling of the cross $X(d)$, and the matrix N given by the last $n-d$ columns followed by the first column of $A(n, d)$ is a 5-labeling of the cross $X(n-d)$. Thus M and N mesh and are instances of the set of matrices described in Theorem 2.4. We will examine the following remaining cases and reach a contradiction in all of them, which implies $\lambda(G) \geq 6$.

Case 1: [$d \equiv 0 \pmod 3$ and $(n-d) \equiv 1 \pmod 3$ with $n-d \geq 7$] or [$(n-d) \equiv 0 \pmod 3$ and $d \equiv 1 \pmod 3$ with $d \geq 7$]. Suppose [$d \equiv 0 \pmod 3$ and $(n-d) \equiv 1 \pmod 3$ with $n-d \geq 7$]. Note that N or its dual must be an instance of $A(7+3q)$ in Figure 6 with their respective flips, so the first and last columns of N are different and have entries in {0, 5}. Since M and N mesh, the first and last columns of M must also be different and have entries in {0, 5}. Unfortunately, the same does not hold for any instance of $A(3+3q)$ and $A(3)$ in Figure 6 and 9 with their respective flips and all their respective duals, a contradiction. Similarly, we also reach a contradiction in the case [$(n-d) \equiv 0 \pmod 3$ and $d \equiv 1 \pmod 3$ with $d \geq 7$] by switching the roles of M and N in the discussion above.

Case 2: [$d = 2$ and $(n-d) \equiv 0 \pmod 3$] or [$d = 2$ and $(n-d) \equiv 1 \pmod 3$ with $(n-d) \geq 7$]. Note that each instance $A(2)$ in Figure 6 with their respective flips, or in Figure 8, and all their respective duals, uses at least three different labels in the first and last columns combined. In contrast, each instance of $A(7+3q)$, $A(3+3q)$, and $A(3)$ in Figures 6 and 9 with their respective flips and all their respective duals, uses only two different labels in the first and last columns. So M and N cannot mesh, a contradiction.

Case 3: $d = 2$ and $(n-d) = 2$. We can verify by inspection that all pairs of instances of $A(2)$ in Figure 6 with their respective flips, or in Figure 8, and all their respective duals do not mesh (note that no component has directed cycles of length 4 containing only solid edges). So M and N cannot mesh, a final contradiction.

Finally, to prove that $\lambda(G) = 6$, it suffices to show that $\lambda(G) \leq 6$. Observe that $\{d, n-d\} = \{d_1, d_2\}$ for a combination of values d_1 and d_2 described by one of the rows of the table in the online supplement. This row exhibits two 6-labelings of the crosses $X(d_1)$ and $X(d_2)$, respectively, as two matrices that mesh. From the observation right before Lemma 3.1, we can conclude that $\lambda(G) \leq 6$. □

Lemma 3.2. *If G is the crossed prism $XPr(n, 1)$ with $n \geq 3$, then $\lambda(G) = 5$ if $n = 3$; $\lambda(G) = 7$ if $n = 4$; otherwise $\lambda(G) = 6$.*

Proof. Recall from Section 1 that an L(2,1)-labeling f of $XPr(n, d)$ is given by a 2-by-n matrix $A(n, d)$ where the entry on the i-th row, j-th column will be the label $f(v_j)$ if $i = 0$, and $f(w_j)$ if $i = 1$, for $j = 0, 1, \ldots, n-1$, and the notation is as introduced in Definition 1.2 (the ends of crossed edges are in the 0-th and d-th columns). If $n = 3$, then the 5-labeling $A(3, 1)$ of G in Figure 11 implies $\lambda(G) = 5$ (recall from Section 1 that GPGs have λ-number at least 5). If $n = 4$, then G has diameter 2 and therefore $\lambda(G) \geq |V(G)| - 1 = 2n - 1 = 7$ [Griggs and Yeh 1992]; the 7-labeling $A(4, 1)$ of G in Figure 11 implies $\lambda(G) = 7$.

Assume $n \geq 5$. We will show first that $\lambda(G) \geq 6$. Suppose for contradiction that G has a 5-labeling given by a 2-by-n matrix $A(n, 1)$. The matrix M given by the first two columns of $A(n, 1)$ is a 5-labeling of the cross $X(1)$, and the matrix N given

$A(3,1) = \boxed{\begin{matrix} 0 & 2 & 4 \\ 5 & 3 & 1 \end{matrix}}$ $\qquad$ $A(4,1) = \boxed{\begin{matrix} 0 & 2 & 5 & 3 \\ 4 & 6 & 1 & 7 \end{matrix}}$

Figure 11. A 5-labeling of $XPr(3, 1)$ and a 7-labeling of $XPr(4, 1)$, respectively.

$$A(5+3q,1) = \boxed{\begin{matrix} 0 & 4 & 6 & 1 & 3 & 6 & 1 & 3 \\ 1 & 5 & 2 & 4 & 0 & 2 & 4 & 6 \end{matrix}}$$

$$A(6+3q,1) = \boxed{\begin{matrix} 0 & 4 & 6 & 3 & 1 & 6 & 1 & 5 & 2 \\ 1 & 5 & 2 & 4 & 0 & 2 & 4 & 0 & 6 \end{matrix}}$$

$$A(7+3q,1) = \boxed{\begin{matrix} 0 & 4 & 6 & 3 & 1 & 6 & 0 & 2 & 4 & 6 \\ 1 & 5 & 2 & 0 & 5 & 2 & 4 & 6 & 0 & 3 \end{matrix}}$$

Figure 12. The 6-labelings of $XPr(n, 1)$ for $n \geq 5$.

by the last $n-1$ columns followed by the first column of $A(n, 1)$ is a 5-labeling of the cross $X(n-1)$. Since N has $n \geq 5$ columns, the X-cycle corresponding to N must be in component D_1 or the dual of D_1 of the directed graph D constructed in Section 2. We may assume without loss of generality that this X-cycle is in D_1. The cross $X(1)$ has diameter 2 so its four vertices must be assigned different labels, thus the first and last columns of N must contain four different labels. Unfortunately, this is not the case for $A(7+3q)$ or $A(3+3q)$ in Figure 6 and their respective flips, implying that $n < 5$, a contradiction, and so $\lambda(G) \geq 6$ holds. The desired equality follows from the 6-labelings of G provided in Figure 12. □

Finally, Theorem 1.3 in Section 1 is a straightforward consequence of Lemmas 3.1 and 3.2.

4. Closing remarks

In this work, we made progress towards closing the GPG conjecture by showing that any crossed prism G satisfies $\lambda(G) \leq 7$ and that, in fact, all but one G satisfy $\lambda(G) \leq 6$. These crossed prisms were of particular interest, as they allowed us to examine how the controlled introduction of asymmetries to prisms would impact both the λ-number and overall proof strategies. The complications these breaks in symmetry introduced were nontrivial, and we ultimately determined $\lambda(G)$ by constructing and inspecting an auxiliary directed graph motivated by previous studies. We hope that these ideas help the community examine other families of graphs — for instance, prisms with more than one pair of crossed edges, perturbations of the n-stars — as we move closer to putting the general GPG conjecture to rest.

Acknowledgements

The authors would like to thank Sarah Spence Adams for handling administrative requirements regarding student research credits. The authors are also in debt to the referee for his/her helpful comments and suggestions. Denise Sakai Troxell would like to thank Babson College for its support through the Babson Research Scholar award.

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Received: 2016-07-09 Revised: 2016-08-18 Accepted: 2016-09-07

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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2018 vol. 11 no. 1

1944-4176(2018)11:1;1-8