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Normal forms of endomorphism-valued power series

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We show for $n, k \geq 1$, and an n-dimensional complex vector space V that if an element $A \in \mathrm{End}(V)[[z]]$ has constant term similar to a Jordan block, then there exists a polynomial gauge transformation g such that the first k coefficients of gAg^{-1} have a controlled normal form. Furthermore, we show that this normal form is unique by demonstrating explicit relationships between the first nk coefficients of the Puiseux series expansion of the eigenvalues of A and the entries of the first k coefficients of gAg^{-1}.

Introduction

From Galois theory, we know that polynomials of degree greater than 4 are not solvable by radicals. So finding the eigenvalues of a companion matrix of the form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \beta_{n-1} & \beta_{n-2} & \beta_{n-3} & \cdots & \beta_0 \end{pmatrix}$$

algebraically in terms of the β_i is not possible. If, however, the β_i have expansions $\beta_i(z)$ in terms of some other variable z with $\beta_i(0) = 0$, we may then ask to find the coefficients in the series expansions of these eigenvalues in terms of these $\beta_i(z)$.

In this paper, we work with a formal power series $A \in \mathrm{End}(V)[[z]]$ whose constant term is a regular nilpotent endomorphism. We want to compute the coefficients of the Puiseux expansion of the eigenvalues of A, but since this is not possible algebraically we search for some normal form obtained via conjugating by an invertible transformation. Clearly, conjugating does not modify the eigenvalues

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of A, and our aim is to conjugate $A(z)$ to a simple shape that allows us to compute explicit relationships between coefficients of the series expansion of the eigenvalues and the coefficients of the conjugate.

In [Ivanics et al. 2016], this problem arose in taking an endomorphism of a vector bundle with some fixed local behavior and searching for the base locus of its corresponding spectral curves. They work with the special case of rank-2 vector bundles E and irregular Higgs fields $\theta(z)$, i.e., meromorphic sections of the endomorphism bundle of E tensored by the canonical bundle. Specifically, the endomorphism θ is assumed to have a single pole of order 4 at $z = 0$ with leading-order term having nontrivial nilpotent part, and the authors show that its polar part may be brought to a simple form up to applying some holomorphic gauge transformations. The authors also note that the case of endomorphisms having two distinct eigenvalues is much simpler. Let us point out that the rank-2 cases can be tackled algebraically due to the existence of the quadratic formula, but that method breaks down in higher-rank cases for the Galois-theoretic reason alluded to above. Another observation is that up to a shift of the index of summation, it is equivalent to consider power series or Laurent series with a fixed finite pole order. Therefore, in this paper we content ourselves with working with power series, however the role of the pole order (the number of terms in the normal form to be controlled) is played by our parameter k.

We cover the general rank-n case for endomorphism-valued power series where the leading-order term is a regular nilpotent endomorphism. That is, we maintain the assumptions of [Ivanics et al. 2016], aside from the pole of order 4 and the rank being equal to 2, extending their results to vector bundles of arbitrary rank and an arbitrary number of terms in the expansion of the endomorphism by presenting existence and uniqueness statements for the normal form of endomorphism-valued power series. This has the same consequence as in [Ivanics et al. 2016] concerning the base locus of generic irregular Higgs bundles with a regular nilpotent leading-order term.

This question is significantly more involved if the constant coefficient of A is a regular matrix with more than one eigenvalue, and even more so if the constant coefficient of A is not regular. The next step we would take to obtain future results would be to examine the case of the constant term of A being regular with more than one eigenvalue.

1. Preliminaries: endomorphisms, gauge transformations, Puiseux series

In this section we describe what kinds of endomorphisms and gauge transformations we plan to examine.

1A. ***Constraints on endomorphisms.*** We begin by putting constraints on the endomorphisms we want to examine. We remark that the results in this paper hold over any algebraically closed field of characteristic zero, but we will only be considering vector spaces over $\mathbb{C}$. Let V be a vector space over $\mathbb{C}$ of dimension n. Suppose that z is a complex variable, and let $A \in \mathrm{End}(V)[\![z]\!]$, that is, A has the form

$$A(z) = \sum_{m=0}^{\infty} A_m z^m, \quad \text{with } A_m \in M_{n,n}(\mathbb{C}).$$

We observe $A_0 = A(0)$. We also place the following condition of regularity on A_0.

Definition 1.1. For a vector space V over an algebraically closed field, an $n \times n$ matrix A_0 is *regular* if and only if its Jordan normal form is of the form

$$J_{d_1}(\lambda_1) \oplus \cdots \oplus J_{d_s}(\lambda_s),$$

with $i \neq j \implies \lambda_i \neq \lambda_j$, and where each $J_{d_i}(\lambda_i)$ is a Jordan block of size d_i with corresponding eigenvalue λ_i.

More abstractly, this is equivalent to considering the space of complex $n \times n$ matrices as a Lie algebra and requiring that the centralizer of A_0 has minimal dimension. The importance of this will become clearer later with the discussion of the transformation applied to A.

1B. ***Constraints on gauge transformations.*** Consider $g \in \mathrm{Aut}(V)[\![z]\!]$, supposing that g has a power series expansion

$$g(z) = \sum_{m=0}^{\infty} g_m z^m, \quad \text{with } g_m \in M_{n,n}(\mathbb{C}),\ g_0 \in \mathrm{GL}_n(\mathbb{C}).$$

We call g an "analytic/formal gauge transformation" (according to whether the radius of convergence of the power-series is 0 or positive), and require that g_0 be invertible because we intend to conjugate A by g. It is a well-known fact about rings of formal power series that an element is invertible if and only if its constant term is invertible. Since g is a power series of matrices, this means we must have $g_0 \in \mathrm{GL}_n(\mathbb{C})$ for g to be invertible.

We turn our attention to the conjugation of A by g, and rename it B:

$$g(z)A(z)g^{-1}(z) = B(z) = \sum_{m=0}^{\infty} B_m z^m. \tag{1-1}$$

Our first goal is to design g such that we may control any finite number of the matrix coefficients in the conjugation. Because eigenvalues are invariant under conjugation, transforming A into B will make computation of the eigenvalues of A simpler. We obtain the following theorem, which will be restated later as Theorem 2.1.

Theorem 1.2. *Suppose $k, n \geq 1$, V is an n-dimensional vector space over $\mathbb{C}$, and $A \in \mathrm{End}(V)[[z]]$. If A is such that A_0 is similar to a Jordan block with eigenvalue 0, then we may construct a polynomial gauge transformation g such that B_0 is an upper triangular Jordan block of dimension n and the first k coefficients $B_1, \ldots, B_k$ of $gAg^{-1} = B$ are matrices with nonzero coefficients only in their n-th row.*

The series B will be referred to as "the normal form" from now on. With the existence of this established, we move towards our second goal of determining explicit relationships between the eigenvalues of A and the entries of the coefficients of B. Let us enumerate the possibly nonzero entries of B_m from left to right as $b_{mn-n+1}, \ldots, b_{mn}$. We obtain the following result, which will be restated later as Theorem 3.2.

Theorem 1.3. *Let B be the normal form of A as described in Theorem 1.2, and suppose that the bottom left coefficient b_1 of B_1 determined by the normal form is nonzero. The eigenvalues of A have a Puiseux expansion*

$$\zeta(z) = \sum_{m=1}^{\infty} a_m z^{m/n},$$

and for fixed $s \geq 1$, the first s coefficients $a_1, \ldots, a_s$ of the Puiseux expansion explicitly determine and are determined by the first s entries $b_1, \ldots, b_s$ of the matrices making up the normal form B.

In particular, this theorem tells us that for fixed k the normal form B of A is uniquely determined. In all cases we assume $A(z) = \sum_{m=0}^{\infty} A_m z^m$ is such that A_0 is similar to a Jordan block. Thus we may define $g_0 \in \mathrm{GL}_n(\mathbb{C})$ such that

$$B_0 = g_0 A_0 g_0^{-1}$$

has the desired Jordan block form. This is a constant transformation, which is notable since the final g will be a finite product of polynomials. Specifically, we will build g as a product of g_0 introduced above and nonconstant factors h_ℓ of the form

$$h_\ell(z) = I_n + g_\ell z^\ell,$$

where I_n is the $n \times n$ identity matrix and $1 \leq \ell \leq k \in \mathbb{Z}^+$. This is an important point, because it means that g will be a polynomial, hence everywhere convergent, so applying them to A will not affect the convergence radius of A. This means that the portion of our results concerning gauge transformations will apply to rings of power series where convergence is a relevant concern. Furthermore, since we only consider the terms of A up to the k-th degree we will be applying k of these h_ℓ transformations, so instead of computing an explicit form for g^{-1}, we will only need that $h_\ell^{-1}(z) =$

$I_n - g_\ell z^\ell + O(z^{\ell+1})$. Then conjugation of A by one of the factors h_ℓ looks like

$$\begin{aligned} h_\ell(z)A(z)h_\ell^{-1}(z) &= (I_n + g_\ell z^\ell)\biggl(\sum_{m=0}^{\infty} A_m z^m\biggr)(I_n - g_\ell z^\ell) + O(z^{\ell+1}) \\ &= \biggl(\sum_{m=0}^{\ell-1} A_m z^m\biggr) + (A_\ell - [A_0, g_\ell])z^\ell + O(z^{\ell+1}), \end{aligned}$$

where $[A_0, g_\ell] = A_0 g_\ell - g_\ell A_0$ represents the commutator. In this manipulation we see that g affects the ℓ-th term of A without changing the first $\ell - 1$ terms. This is important because we apply the transformations $I_n - g_\ell z^\ell$ iteratively for $1 \le \ell \le k$ for ℓ *increasing*, ultimately obtaining a polynomial transformation of the form

$$\begin{aligned} g(z) &= h_k(z)h_{k-1}(z)\dots h_1(z)g_0 \\ &= (I_n + g_k z^k)(I_n + g_{k-1}z^{k-1})\dots(I_n + g_1 z)g_0. \end{aligned} \tag{1-2}$$

Specifically, considering the map

$$\mathrm{ad}_{A_0} : M_{n,n}(\mathbb{C}) \to M_{n,n}(\mathbb{C}), \quad g_\ell \mapsto [A_0, g_\ell] = A_0 g_\ell - g_\ell A_0, \tag{1-3}$$

will tell us how to construct g to generate a normal form for the conjugated series.

1C. *Factorization of the characteristic polynomial of A.* We consider the eigenvalues of endomorphisms in the variable ζ. Let $A(z) = \sum_{m=0}^{\infty} A_m z^m$ be an element of $\mathrm{End}(V)[\![z]\!]$. We have that the characteristic polynomial of $A(z)$ has the form

$$\chi_{A(z)}(\zeta) = \chi_A(z, \zeta) = \det(\zeta I - A(z)) = \zeta^n + a_1(z)\zeta^{n-1} + \dots + a_n(z), \tag{1-4}$$

with $a_1, \dots, a_n \in \mathbb{C}[\![z]\!]$. We then recall the following particular case of a result attributed to Puiseux and Newton.

Theorem 1.4 (Newton–Puiseux). *The characteristic polynomial* (1-4) *factors as*

$$\chi_A(w^n, \zeta) = \prod_{i=1}^{n}(\zeta - \zeta_i(w)), \quad \textit{with} \quad \zeta_i \in \mathbb{C}[\![w]\!].$$

This version of the theorem is taken from [Abhyankar 1990, Lecture 12], except for identifying the ramification index as n instead of some unspecified divisor of $n!$; this latter identification in turn follows from [Serre 1979, Chapter I, Proposition 17]. Indeed, according to the assumption $b_1 \neq 0$ the z-adic valuation of a_n is 1, on the other hand the coefficients $a_1(0), \dots, a_{n-1}(0)$ clearly vanish as A_0 is a nilpotent endomorphism. These conditions mean that $\chi_{A(z)}$ is an Eisenstein polynomial in ζ, thus it is totally ramified, i.e., of ramification index n.

For us, the above theorem means that we may decompose the characteristic polynomial of A into linear factors, with the roots being represented by Puiseux series. Furthermore, we will be able to obtain each root of the polynomial by considering all of the conjugates (in the Galois-theory sense) of a single root by multiplying $w = z^{1/n}$ by some power of a primitive n-th root of unity ω. Specifically, after a branch cut we may fix a choice $z^{1/n}$ of n-th root of z, and then all the roots of the characteristic polynomial are expressible in the form

$$\zeta_i(z) = \sum_{m=1}^{\infty} a_m (\omega^i z^{1/n})^m \tag{1-5}$$

for $i = 0, \dots, n-1$. Different choices of $z^{1/n}$ only amount to a permutation of the n roots ζ_i.

2. Existence of the normal form

In this section we present the construction of a normal form for A where the dimension of the ambient vector space V is an arbitrary integer $n \geq 2$. Furthermore, we fix an arbitrary $k \in \mathbb{Z}^+$.

Theorem 2.1. *Take V to be a vector space over $\mathbb{C}$ of dimension n, and suppose that $A(z) = \sum_{m=0}^{\infty} A_m z^m$ is an endomorphism of V such that A_0 is similar to a Jordan matrix with a single eigenvalue. Then for fixed $k \geq 1$ we may construct a gauge transformation g of the form* (1-2) *such that the coefficient B_0 of $gAg^{-1}(z) = B(z) = \sum_{m=0}^{\infty} B_m z^m$ has the form*

$$B_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and the subsequent coefficients have the form

$$B_\ell = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ b_{n(\ell-1)+1} & b_{n(\ell-1)+2} & \cdots & b_{n\ell-1} & b_{n\ell} \end{pmatrix}$$

for $1 \leq \ell \leq k$.

Proof. We want to find a way to conjugate A into B such that $A_0 = B_0$ and the subsequent B_ℓ for $1 \leq \ell \leq k$ have the indicated form. So we consider the map $\mathrm{ad}_{A_0} : V \to V$ for an arbitrary matrix G given by $G \mapsto [A_0, G]$, with the bracket representing the commutator of A_0 and G. To examine the image of this map, label the entries of G in the usual way and expand:

$$\left[\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}\right]$$
$$= \begin{pmatrix} g_{21} & g_{22}-g_{11} & g_{23}-g_{12} & \cdots & g_{2,n}-g_{1,n-1} \\ g_{31} & g_{32}-g_{21} & g_{33}-g_{22} & \cdots & g_{3,n}-g_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{n,1} & g_{n,2}-g_{n-1,1} & g_{n,3}-g_{n-1,2} & \cdots & g_{n,n}-g_{n-1,n-1} \\ 0 & -g_{n,1} & -g_{n,2} & \cdots & -g_{n,n-1} \end{pmatrix}.$$

Name the above matrix C, and name the entries in the usual way. Then see that we may write each entry in the last row as

$$c_{n,t} = -\sum_{j=1}^{t-1} c_{n-j,t-j},$$

as t ranges from 1 to n. That is, each entry in the last row is the negative of the sum of entries along the diagonal up and to the left of $c_{n,t}$. We set $c_{n,1} = 0$ by convention. Now although we considered the matrix G to be arbitrary, we may pick the entries of G so that we can make $A_\ell - [A_0, G]$ have a desired form. Specifically, the dependence of the last row of C on the first $n-1$ rows ensures that we can eliminate the first $n-1$ rows of A_ℓ. This almost certainly affects the last row of A_ℓ, but this does not matter to us. Thus from the iterative process described at the end of Section 1B, we may find a polynomial of the form (1-2) that we may conjugate A by to turn the ℓth coefficient of $B(z)$ into the form

$$B_\ell = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ b_{n(\ell-1)+1} & b_{n(\ell-1)+2} & \cdots & b_{n\ell-1} & b_{n\ell} \end{pmatrix},$$

for $1 \le \ell \le k$. Turning A_0 into B_0 is much easier, since it is achieved by a constant transformation, and we are assuming that A_0 is similar to a matrix of the form B_0. This is the desired normal form for the first k coefficients of B. □

3. Uniqueness of the normal form

In this section we again fix an arbitrary $k \in \mathbb{Z}^+$ and show that the coefficients b_i for $1 \le i \le kn$ are uniquely determined by the shape of the normal form B and the coefficients $a_1, \dots, a_{kn}$ of the Puiseux expansion of the eigenvalues of A. We begin the search for relationships between the series of eigenvalues and the entries of the B_ℓ with a lemma. For the remainder of this section we now suppose that $A_0 = B_0$ is as in Theorem 2.1 and that the first k coefficients of $B(z)$ may have nonzero entries only in the n-th row.

Lemma 3.1. *Let t be an integer with $n > t \ge 1$, and $w_1, \dots, w_{t+1} \in \mathbb{Z}$ be such that*

$$-n < w_1 < 0, \quad 0 < w_2, \dots, w_{t+1} < n, \quad \sum_{\ell=1}^{t+1} w_\ell = 0.$$

Define ω to be a primitive n-th root of unity. Then we have that

$$\frac{1}{t!} \sum_{\substack{1 \le s_1, \dots, s_{t+1} \le n \\ s_j \ne s_\ell, \forall \ell \ne j}} \omega^{w_1 s_1 + \cdots + w_{t+1} s_{t+1}} = (-1)^t n.$$

Proof. First, note the following basic identity regarding sums of powers of primitive n-th roots of unity: for any $w \in \mathbb{Z}$ such that $n \nmid w$ we have

$$\sum_{j=0}^{n-1} \omega^{jw} = \frac{\omega^{wn} - 1}{\omega^w - 1} = 0. \tag{3-1}$$

For our application below, let us point out that in the sum of the left-hand side the summation index j may equally be chosen to range from 1 to n without changing the value of the sum, because $\omega^{0w} = \omega^{nw}$. Then we proceed by induction on t. Starting with $t = 1$, we see that we must have $w_2 = -w_1$, since $w_1 < 0$, and $w_1 + w_2 = 0$. Then see that

$$\frac{1}{1!} \sum_{\substack{1 \le s_1,\, s_2 \le n \\ s_1 \ne s_2}} \omega^{w_1(s_1 - s_2)},$$

and relabeling $u = (s_1 - s_2) \bmod n$ gives

$$n \cdot \frac{1}{1!} \sum_{u=1}^{n-1} \omega^{w_1 u} = n \cdot \frac{1}{1!} \cdot (-1) = (-1)^1 \cdot n,$$

using (3-1) and observing each u is obtained in n possible ways. So the base case is proven.

Now suppose that the claim holds for $t-1 \geq 1$. For t, we then have

$$\frac{1}{t!}\sum_{\substack{1\leq s_1,\dots,s_{t+1}\leq n\\ s_j\neq s_\ell,\forall \ell\neq j}} \omega^{w_1s_1+\cdots+w_{t+1}s_{t+1}} = \frac{1}{t!}\sum_{\substack{1\leq s_2,\dots,s_{t+1}\leq n\\ s_j\neq s_\ell,\forall \ell\neq j}} \omega^{w_2s_2+\cdots+w_{t+1}s_{t+1}}\Bigg(\sum_{\substack{s_1=1\\ s_1\notin\{s_2,\dots,s_{t+1}\}}}^{n} \omega^{w_1s_1}\Bigg)$$

and since $w_1 \not\equiv 0 \bmod n$, we may rewrite the inner sum using (3-1):

$$\begin{aligned}
\frac{1}{t!}&\sum_{\substack{1\leq s_2,\dots,s_{t+1}\leq n\\ s_j\neq s_\ell,\forall \ell\neq j}} \omega^{w_2s_2+\cdots+w_{t+1}s_{t+1}}\Bigg(\sum_{\substack{s_1=1\\ s_1\notin\{s_2,\dots,s_{t+1}\}}}^{n} \omega^{w_1s_1}\Bigg)\\
&= \frac{1}{t!}\sum_{\substack{1\leq s_2,\dots,s_{t+1}\leq n\\ s_j\neq s_\ell,\forall \ell\neq j}} \omega^{w_2s_2+\cdots+w_{t+1}s_{t+1}}(-\omega^{w_1s_2}-\cdots-\omega^{w_1s_{t+1}})\\
&= -\frac{1}{t!}\sum_{\substack{1\leq s_1,\dots,s_{t+1}\leq n\\ s_j\neq s_\ell,\forall \ell\neq j}} \omega^{(w_2+w_1)s_2+w_3s_3+\cdots+w_{t+1}s_{t+1}} - \cdots\\
&\qquad -\frac{1}{t!}\sum_{\substack{1\leq s_1,\dots,s_{t+1}\leq n\\ s_j\neq s_\ell,\forall \ell\neq j}} \omega^{w_2s_2+(w_1+w_3)s_3+w_4s_4+\cdots+w_{t+1}s_{t+1}} - \cdots\\
&\qquad\qquad -\frac{1}{t!}\sum_{\substack{1\leq s_1,\dots,s_{t+1}\leq n\\ s_j\neq s_\ell,\forall \ell\neq j}} \omega^{w_2s_2+w_3s_3+\cdots+w_ts_t+(w_1+w_{t+1})s_{t+1}}.
\end{aligned}$$

In each of the t terms in the final sum, we may relabel the indices $w'_1, w'_2, \dots, w'_{t+1}$ such that $w'_1 = w_1 + w_\ell$ for $\ell = 1, \dots, t+1$. The remaining w'_j are assigned lexicographically according to what is left; that is, if w'_1 takes the ℓ-th spot in the list, then

$$w'_2 = w_2,\quad w'_3 = w_3, \dots, w'_{\ell-1} = w_{\ell-1},\quad w'_\ell = w_{\ell+1}, \dots, w'_{t-1} = w_t,\quad w'_t = w_{t+1}.$$

These relabeled terms still satisfy $\sum_{u=1}^{t} w_s = 0$ since the original w terms satisfy this relation. They also satisfy $-n < w'_1 < 0$ and $0 < w'_2, \dots, w'_t < n$. This is clear for w'_j with $j > 1$, and also holds for w'_1 since we have

$$w'_1 = w_1 + w_\ell < \sum_{j=1}^{t+1} w_j = 0.$$

So we may apply the induction assumption to each of these sums to turn the last expression in the above manipulation to

$$-\frac{1}{t}\Bigg(\frac{1}{(t-1)!}(-1)^{t-1}(t-1)!\cdot n + \frac{1}{(t-1)!}(-1)^{t-1}(t-1)!\cdot n \\ \cdots + \frac{1}{(t-1)!}(-1)^{t-1}(t-1)!\cdot n\Bigg) = (-1)^t\cdot n. \quad \square$$

This lemma is crucial in determining the coefficients we're ultimately looking for. We now present the argument for the coefficient relationships of the rank-n case.

Let $k \geq 1$, $A \in \operatorname{End}(V)[\![z]\!]$ have A_0 similar to a Jordan block and have normal form B as in Theorem 2.1 with $b_1 \neq 0$. Letting

$$\zeta(z) = \sum_{m=1}^{\infty} a_m z^{m/n}$$

denote the Puiseux expansion of the eigenvalues of A, our aim is to show that the coefficients $\{a_1, \ldots, a_s\}$ determine and are determined by $\{b_1, \ldots, b_s\}$ for arbitrary $1 \leq s \leq kn$. More precisely, writing

$$s = n\ell - t \tag{3-2}$$

for a unique $1 \leq \ell \leq k$ and $0 \leq t \leq n-1$, we have the following.

Theorem 3.2. *With the above assumptions, there exist polynomials $P_{s,n} \in \mathbb{C}[x_1, \ldots, x_{s-1}]$ only depending on s, n such that we have*

$$b_s = (-1)^t n a_1^t a_s + P_{s,n}(a_1, \ldots, a_{s-1}).$$

Conversely, there exist rational functions of the form $Q_{s,n} \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_{s-1}]$ such that

$$a_s = \frac{(-1)^s}{n} b_1^{-s/n} b_s + Q_{s,n}(b_1^{1/n}, \ldots, b_{s-1}).$$

In particular, for any given $A \in \operatorname{End}(V)[\![z]\!]$ and fixed k, the parameters $\{b_1, \ldots, b_{kn}\}$ appearing in Theorem 2.1 are uniquely determined.

Proof. Let ω be a primitive n-th root of unity and recall our notation (1-5) for the eigenvalues of A. The key idea is to compare two different representations for the characteristic polynomial

$$\chi_{B(z)}(\zeta) = \chi_{A(z)}(\zeta).$$

Namely, up to order k with respect to the variable z, the polynomial $\chi_{B(z)}$ can be read off directly from the form of the matrices $B_0, B_1, \ldots, B_k$ given in Theorem 2.1. On the other hand, as we have seen in Theorem 1.4 we may expand $\chi_{A(z)}$ into linear factors $(\zeta - \zeta_i(z))$. This provides us the identity

$$\zeta^n + \zeta^{n-1}\biggl(\sum_{\ell=1}^{k} b_{n\ell} z^\ell + O(z^{k+1})\biggr) + \cdots + \biggl(\sum_{\ell=1}^{k} b_{n\ell-(n-1)} z^\ell + O(z^{k+1})\biggr) \tag{3-3}$$
$$= \prod_{i=0}^{n-1}\Bigl(\zeta - \zeta_i(z)\Bigr)$$

$$= \Big(\zeta - \sum_{m=1}^{\infty} a_m z^{m/n}\Big)\Big(\zeta - \sum_{m=1}^{\infty} a_m (\omega z^{1/n})^m\Big)\cdots$$
$$\Big(\zeta - \sum_{m=1}^{\infty} a_m (\omega^{n-1} z^{1/n})^m\Big). \quad (3\text{-}4)$$

The generic term of (3-3) is

$$\zeta^{n-1-t}\Big(\sum_{\ell=1}^{k} b_{n\ell-t} z^{\ell} + O(z^{k+1})\Big).$$

We proceed now by comparing coefficients of (3-3) and (3-4), and to do this we apply induction on s.

Before starting the induction, we do some preliminary work in computing the coefficient in (3-4) of $\zeta^{n-1-t} z^{\ell}$, that is, the coefficient that corresponds to $b_{n\ell-t}$ in (3-3). We exclude the case where $\ell = 1$ and $t = n-1$ (i.e., b_1), since this first nonzero term has simpler combinatorial structure than subsequent ones. We would like to have a general form for the subsequent terms.

To this end, we know that the coefficient of $\zeta^{n-1-t} z^{\ell}$ in (3-4) will be a complex linear combination of the products $a_{m_1} \dots a_{m_{t+1}}$ such that $\sum_{i=1}^{t+1} m_i = n\ell$, with constants given in terms of a sum of powers of ω. This is equivalent to noticing that the indices m_i partition $n\ell$ into $t+1$ nonempty parts. To explain why there are $t+1$ parts, we first see that $n-1-t = n-(t+1)$, and in the expansion (3-4), each term will have n components. These components are formed by picking one term from each of the n factors in (3-4), and are thus split into those that are just ζ and those that come from the a_i. In the particular case of ζ^{n-1-t} we can imagine that we use $n-1-t$ choices on ζ, and the remaining $t+1$ choices on various a_{m_i}. The correct coefficient in (3-4) to compare to $b_{n\ell-t}$ will be then those combinations of a_{m_i} such that the indices m_i sum to n times the exponent of z multiplying $b_{n\ell-t}$, that is, the m_i sum to $n\ell$. We see that the parts must be nonempty since any $m_i = 0$ would give us a factor of $a_0 = 0$ in the product of all a_{m_i}, thus annihilating the product.

So we need to consider the set of all partitions of the integer $n\ell$ as a sum of $t+1$ positive integers, say in decreasing order:

$$\mathcal{P}_{\ell,t} = \{m_1 \geq \dots \geq m_{t+1} \geq 1 \mid m_1 + \cdots + m_{t+1} = n\ell\}.$$

With this notation, we can produce an initial expression for the general coefficient:

$$b_{n\ell-t} = \sum_{\mathcal{P}_{\ell,t}} a_{m_1} \dots a_{m_{t+1}} \mu_{m_1,\dots,m_{t+1}}, \quad (3\text{-}5)$$

where $\mu_{m_1,\dots,m_{t+1}}$ denotes a yet undetermined linear combination of powers of ω with rational coefficients that depends on the partition $(m_1, \dots, m_{t+1})$.

The expression in (3-5) can be refined by noticing that we care only about the partitions with $m_1 = n\ell - t = s$, since this will be the highest possible index for a given s and given t, and the products coming from all partitions with $m_1 < n\ell - t$ will be absorbed in the polynomial $P_{s,n}(a_1^{1/n}, \dots, a_{s-1})$. This assignment of m_1 then necessarily forces $m_2 = \cdots = m_{t+1} = 1$, since we still require that the partition contains $t+1$ nonempty parts and that the m_i sum to $n\ell$. Let us now introduce

$$\mathcal{P}^0_{\ell,t} = \{(m_1, \dots, m_{t+1}) \in \mathcal{P}_{\ell,t} \mid n\ell - t > m_1\}.$$

This $\mathcal{P}^0_{\ell,t}$ captures all of the partitions whose m_1 index we do not need to keep track of, allowing us to rewrite (3-5). In rewriting, we suppress the m_i in the first term, instead presenting their actual values which we know to be $m_1 = n\ell - t$, $m_2 = \cdots = m_{t+1} = 1$:

$$a_1^t a_{n\ell-t} \mu_{n\ell-t,1,\dots,1} + \sum_{\mathcal{P}^0_{\ell,t}} a_{m_1} \dots a_{m_{t+1}} \mu_{m_1,\dots,m_{t+1}}. \tag{3-6}$$

Again, as the indices m_i of each term in the sum are all strictly less than $n\ell - t$, the second term in this formula only contributes to $P_{s,n}$, hence we only need to specify the constants $\mu_{n\ell-t,1,\dots,1}$.

To gain a better understanding of the structure of the constant $\mu_{n\ell-t,1,\dots,1}$ appearing in the above expression, we describe a way of visualizing each partition that will give more structure to the enumeration of the constant's summands. Consider the partition of $n\ell$ into parts $n\ell - t, 1, \dots, 1$ with 1 appearing t times. We align this partition with the combinatorial choice of picking a term out of each of the n factors of (3-4) by considering the m_i to be distributed among n boxes, not necessarily in increasing order. We label the positions of these m_i amongst the n boxes by the labels s_i for $i = 1, \dots, t+1$, such that $s_i \neq s_j$ for $i \neq j$. Observe however that since $m_2 = \cdots = m_{t+1}$, any fixed set $\{s_2, \dots, s_{t+1}\}$ of t distinct positions in $\{1, \dots, n\}$ and any further position $s_1 \notin \{s_2, \dots, s_{t+1}\}$ give rise to a *single* term in (3-6) of the form $\omega^s a_1^t a_{n\ell-t}$ for some integer s (to be specified below), independently of the order of $\{s_2, \dots, s_{t+1}\}$. So we may (and from now on, will) assume that the positions $\{s_2, \dots, s_{t+1}\}$ are in increasing order:

$$s_2 < \cdots < s_{t+1};$$

however, we have no restriction about the position of s_1 relative to the above increasing sequence. This gives us a way of picturing all possible configurations of the m_i. An example of one of these configurations is

ζ	ζ	x_2	$\dots$	ζ	x_j	ζ	$\dots$	ζ	x_1	ζ	$\dots$	x_t	x_{t+1}	ζ	$\dots$	ζ	ζ
1	2	s_2	$\dots$		s_j		$\dots$		s_1		$\dots$	s_t	s_{t+1}		$\dots$	$n-1$	n

,

with $x_j = -a_{m_j}(\omega^{s_j-1}z)^{m_j}$ for all $1 \le j \le t+1$. We note that the -1 attached to each s_i in the exponents occurs since the expansion in (3-4) is indexed from 0 to $n-1$, but we were considering the s_i as elements of $\{1, \dots, n\}$. This is a minor adjustment.

Computing $\mu_{m_1,\dots,m_{t+1}}$ involves writing an expression for μ that reflects the fixing of s_1, the position of m_1, outside of the strict ordering of the other labels. We express this now, adopting the standard notation $[n] = \{1, \dots, n\}$:

$$\mu_{n\ell-t,1,\dots,1} = \sum_{\substack{s_2,\dots,s_{t+1}\in\mathbb{Z}^+ \\ 1\le s_2<\cdots<s_{t+1}\le n}} \omega^{(s_2-1)} \cdots \omega^{(s_{t+1}-1)} \sum_{\substack{s_1\in[n]\setminus \\ \{s_2,\dots,s_{t+1}\}}} \omega^{(s_1-1)(n\ell-t)}. \tag{3-7}$$

Now we manipulate (3-7) as follows, recognizing that since ω is an n-th root of unity, we may work with any of the sums in the exponents modulo n:

$$\begin{aligned}
&\sum_{\substack{s_2,\dots,s_{t+1}\in\mathbb{Z}^+ \\ 1\le s_2<\cdots<s_{t+1}\le n}} \sum_{\substack{s_1\in[n]\setminus \\ \{s_2,\dots,s_{t+1}\}}} \omega^{s_2+\cdots+s_{t+1}-t+s_1\ell n-s_1t-\ell n+t} \\
&\qquad = \sum_{\substack{s_2,\dots,s_{t+1}\in\mathbb{Z}^+ \\ 1\le s_2<\cdots<s_{t+1}\le n}} \sum_{s_1\in[n]\setminus\{s_2,\dots,s_{t+1}\}} \omega^{s_2+\cdots+s_{t+1}-s_1t} \\
&\qquad = \sum_{\substack{s_2,\dots,s_{t+1}\in\mathbb{Z}^+ \\ 1\le s_2<\cdots<s_{t+1}\le n}} \omega^{s_2+\cdots+s_{t+1}} \sum_{s_1\in[n]\setminus\{s_2,\dots,s_{t+1}\}} \omega^{-s_1t}.
\end{aligned} \tag{3-8}$$

We may recognize (3-8) as an ordered version of the sum examined by Lemma 3.1. Indeed, we have bounded weights that sum to zero and an exponent sum in $t+1$ terms, namely $w_1 = -t$, $w_2 = \cdots = w_{t+1} = 1$. In Lemma 3.1 we have $t+1$ unordered terms, but here we have t ordered terms and one independent term. Multiplying (3-8) by $t!$ allows us to rewrite it without the ordering and allows us to apply the lemma, since we obtain sums over $t+1$ unordered terms. But then the lemma gives that dividing by $t!$ again allows us to compute the sum, and so the sum from the lemma and the sum in (3-8) are equivalent. So we find

$$\sum_{\substack{s_2,\dots,s_{t+1}\in\mathbb{Z}^+ \\ 1\le s_2<\cdots<s_{t+1}\le n}} \omega^{s_2+\cdots+s_{t+1}} \sum_{\substack{s_1\in[n]\setminus \\ \{s_2,\dots,s_{t+1}\}}} \omega^{-s_1t} = (-1)^t n.$$

We conclude that the leading-index term for $b_{n\ell-t}$ is $(-1)^t n a_1^t a_{n\ell-t}$.

Now we can start the induction on s, which will actually be a double induction, first on $\ell \in \{1, \dots, k\}$ in increasing order then on $t \in \{0, \dots, n-1\}$ in decreasing order; see (3-2). We determine b_1 by inspection, and apply the above argument for $b_2, \dots, b_n$. So we have

$$b_1 = a_1^n, \quad b_2 = (-1)^{n-2} n a_1^{n-2} a_2, \quad \dots, \quad b_p = (-1)^{n-p} n a_1^{n-p} a_p, \quad \dots, \quad b_n = n a_n.$$

We note that each of these b_i relations matches that in the theorem statement, depending on a_1 and a_i. These relationships are certainly invertible in terms of the a_i:

$$a_1 = \sqrt[n]{b_1}, \quad a_2 = \frac{(-1)^{2-n} b_2}{n b_1^{1-2/n}}, \quad \dots, \quad a_p = \frac{(-1)^{p-n} b_p}{n b_1^{1-p/n}}, \quad \dots, \quad a_n = b_n/n.$$

We fix an n-th root of b_1 here so that everything is uniquely determined. Changing the choice of the root is equivalent to multiplying a_1 by a primitive n-th root of unity, which then affects all subsequent coefficients a_k in the same way, eventually leading to a permutation of the roots $\zeta_j(z)$ in (3-4); thus, fixing an n-th root of b_1 is not a restrictive choice. Furthermore, we note that in one direction we have the desired polynomial relations, and in the other direction we have the desired rational relations. Thus, the statement holds for $\ell = 1$ and all t.

Then supposing that the claim holds for $2, \dots, s-1$, we consider general s. From the earlier partition argument we also know that any terms a_i in the full expression for b_s that do not contain $a_{n\ell-t}$ will have indices at most $i \le n\ell - t - 1 = s - 1$, so applying the induction hypothesis gives

$$b_{n\ell-t} = (-1)^t n a_1^t a_{n\ell-t} + \mathrm{P}_{s,n}(a_1, \dots, a_{s-1}),$$

since we have invertible relationships for the expressions contained in $\mathrm{P}_{s,n}(a_1, \dots, a_{s-1})$. This new set of relationships will also be invertible since the only new term is $(-1)^t n a_1^t a_{n\ell-t}$, which is a nonzero multiple of $a_{n\ell-t}$ since we are working over a field of characteristic zero with $a_1 \neq 0$. So $b_{n\ell-t}$ is determined explicitly by this expression, and vice versa. Thus we have shown that the claim holds for general s. □

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