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Continuous dependence and differentiating solutions of a second order boundary value problem with average value condition

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Using a few conditions, continuous dependence, and a result regarding smoothness of initial conditions, we show that derivatives of solutions to the second order boundary value problem $y'' = f(x, y, y')$, $a < x < b$, satisfying $y(x_1) = y_1$, $1/(d-c)\int_c^d y(x)\,\mathrm{d}x = y_2$, where $a < x_1 < c < d < b$ and $y_1, y_2 \in \mathbb{R}$ with respect to each of the boundary data x_1, y_1, y_2, c, d solve the associated variational equation with interesting boundary conditions. Of note is the second boundary condition, which is an average value condition.

1. Introduction

Our concern is characterizing derivatives of solutions to the second order boundary value problem

$$y'' = f(x, y, y'), \quad a < x < b, \tag{1-1}$$

satisfying

$$y(x_1) = y_1, \quad \frac{1}{d-c}\int_c^d y(x)\,\mathrm{d}x = y_2, \tag{1-2}$$

where $a < x_1 < c < d < b$, and $y_1, y_2 \in \mathbb{R}$ with respect to the boundary data. We make note of the average value condition.

The history and breadth of work on the subject of smoothness of conditions for various problems is quite rich and stretches back to the time of Peano as attributed by Hartman [1964]. Peano's result characterized the smoothness of initial conditions for initial value problems (IVPs). Subsequently, many researchers expanded the result to smoothness of boundary conditions for boundary value problems. The key to making the jump was utilizing a continuous dependence result for boundary

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conditions. Once invoked, there were many articles published in the realm of boundary value problems for differential equations [Ehme 1993; Ehrke et al. 2007; Henderson 1987; Lyons 2011; Lyons and Miller 2015; Spencer 1975], difference equations [Benchohra et al. 2007; Datta 1998; Henderson and Jiang 2015; Hopkins et al. 2009; Lyons 2014a], and dynamic equations on time scales [Baxter et al. 2016; Lyons 2014b] with a host of interesting of boundary conditions.

Our main motivation for this paper is a recent result [Janson et al. 2014] in which the authors sought an analogue of Peano's theorem for a second order boundary value problem with an integral boundary condition. The novelty we contribute to the literature is employing an average value boundary condition, which, although similar, is very fascinating in its own right.

At first, the average value condition might seem unusual. However, the idea of an average value condition is quite useful when one is not concerned with what occurs at a specific point but instead the average over a range of points. For example, one may not need to specify the temperature at a certain time as long as the average temperature is fixed over a range of time. We point the reader to [Chua 2010] and the references therein for more discussion on average value conditions and more general functional conditions.

The remainder of the paper is organized as follows. In Section 2, we introduce the definition of a variational equation and place conditions upon the boundary value problem. Section 3 is comprised of interesting and crucial results for our research. We prove our main result and a corollary in Section 4.

2. Preliminaries

Throughout our work and previous research on the topic, a very important equation emerges which we now define.

Definition 2.1. Given a solution $y(x)$ of (1-1), we define the *variational equation along* $y(x)$ by

$$z'' = \frac{\partial f}{\partial u_1}(x, y(x), y'(x))z + \frac{\partial f}{\partial u_2}(x, y(x), y'(x))z', \tag{2-1}$$

where u_1 and u_2 are the second and third components of f, respectively.

Next, we place five hypotheses upon the boundary value problem:

(i) $f(x, u_1, u_2) : (a, b) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

(ii) For $i = 1, 2$, the map $\partial f/\partial u_i(x, u_1, u_2) : (a, b) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

(iii) Solutions of IVPs for (1-1) extend to (a, b).

(iv) Given $a < x_1 < c < d < b$, if $y(x_1) = z(x_1)$ and $1/(d-c)\int_c^d y(x)\,\mathrm{d}x = 1/(d-c)\int_c^d z(x)\,\mathrm{d}x$, where $y(x)$ and $z(x)$ are solutions of (1-1), then, on (a, b), we have $y(x) \equiv z(x)$.

(v) Given $a < x_1 < c < d < b$ and a solution $y(x)$ of (1-1), if $u(x_1) = 0$ and $1/(d-c)\int_c^d u(x)\,\mathrm{d}x = 0$, where $u(x)$ is a solution of (2-1) along $y(x)$, then, on (a, b), we have $u(x) \equiv 0$.

Note that even though (i) and (ii) may seem to be very strict conditions, we remind the reader that since our aim is to compute derivatives of solutions to (1-1) and (1-2), they are not unusual. Condition (iii) is not necessary but instead allows us to suppress verbiage of finding an interval inside (a, b) where the solution of the boundary value problem converges. Finally, conditions (iv) and (v) are required to ensure the uniqueness of the solution and variational equation.

3. Background theorems

We now introduce two theorems that play a key role in the proof of the main result. The first result is attributed to Peano and is in essence the type of result we seek for (1-1) and (1-2). We direct the reader to Hartman's book [1964] for more details.

Theorem 3.1 (Peano's theorem). *Assume that, with respect to* (1-1), *conditions (i)–(iii) are satisfied. Let $x_0 \in (a, b)$ and $y(x) := y(x, x_0, c_1, c_2)$ denote the solution of* (1-1) *satisfying the initial conditions $y(x_0) = c_1$ and $y'(x_0) = c_2$. Then*:

(a) *For $i = 1, 2$, $\partial y/\partial c_i(x)$ exists on (a, b), and $\alpha_i(x) := \partial y/\partial c_i(x)$ is the solution of the variational equation* (2-1) *along $y(x)$ satisfying the respective initial conditions*

$$\alpha_1(x_0) = 1, \quad \alpha_1'(x_0) = 0, \quad \alpha_2(x_0) = 0, \quad \alpha_2'(x_0) = 1.$$

(b) *$\partial y/\partial x_0(x)$ exists on (a, b), and $\beta(x) := \partial y/\partial x_0(x)$ is the solution of the variational equation* (2-1) *along $y(x)$ satisfying the initial conditions*

$$\beta(x_0) = -y'(x_0), \quad \beta'(x_0) = -y''(x_0).$$

(c) $\dfrac{\partial y}{\partial x_0}(x) = -y'(x_0)\dfrac{\partial y}{\partial c_1}(x) - y''(x_0)\dfrac{\partial y}{\partial c_2}(x).$

The next result permits the leap from IVPs to boundary value problems. The proof requires mapping initial data to boundary data and an application of the Brouwer invariance of domain theorem. For a typical proof, we refer the reader to [Henderson et al. 2005].

Theorem 3.2 (continuous dependence for boundary value problems). *Assume (i)–(iv) are satisfied with respect to* (1-1). *Let $y(x)$ be a solution of* (1-1) *on (a, b), and let $a < \alpha < x_1 < c < d < \beta < b$ and $y_1, y_2 \in \mathbb{R}$ be given. Then, there exists a $\delta > 0$ such that, for $|x_1 - t_1| < \delta$, $|c - \xi| < \delta$, $|d - \Delta| < \delta$, $|y(x_1) - y_1| < \delta$, and $\left|1/(d-c)\int_c^d y(x)\,\mathrm{d}x - y_2\right| < \delta$, there exists a unique solution $y_\delta(x)$ of* (1-1) *such that $y_\delta(t_1) = y_1$ and $1/(\Delta - \xi)\int_\xi^\Delta y_\delta(x)\,\mathrm{d}x = y_2$ and, for $i = 1, 2$, $\{y_\delta^{(i)}(x)\}$ converges uniformly to $y^{(i)}(x)$ as $\delta \to 0$ on $[\alpha, \beta]$.*

4. Main result

In light of the information in the previous sections, we now present the main result. A reminder that the novel portion of our result is differentiation with respect to the terms in the average value condition, namely c and d. We will only show the proof of part (d) as (c) is similar. In fact, each part (a)–(d) employs the same idea for a proof.

Theorem 4.1. *Assume conditions (i)–(v) are satisfied. Let $y(x)$ be a solution of* (1-1) *on (a,b). Let $a < x_1 < c < d < b$ and $y_1, y_2 \in \mathbb{R}$ be given so that $y(x) = y(x, x_1, y_1, y_2, c, d)$, where*

$$y(x_1) = y_1, \qquad \frac{1}{d-c}\int_c^d y(x)\,\mathrm{d}x = y_2.$$

Then:

(a) *For $i = 1, 2$, $u_i(x) := \partial y/\partial y_i(x)$ exists on (a,b) and is the solution of the variational equation* (2-1) *along $y(x)$ satisfying the respective boundary conditions*

$$u_1(x_1) = 1 \quad \textit{and} \quad \frac{1}{d-c}\int_c^d u_1(x)\,\mathrm{d}x = 0,$$
$$u_2(x_1) = 0 \quad \textit{and} \quad \frac{1}{d-c}\int_c^d u_2(x)\,\mathrm{d}x = 1.$$

(b) *$z_1(x) := \partial y/\partial x_1(x)$ exists on (a,b) and is the solution of the variational equation* (2-1) *along $y(x)$ satisfying the respective boundary conditions*

$$z_1(x_1) = -y'(x_1) \quad \textit{and} \quad \frac{1}{d-c}\int_c^d z_1(x)\,\mathrm{d}x = 0.$$

(c) *$C(x) := \partial y/\partial c(x)$ exists on (a,b) and is the solution of the variational equation* (2-1) *along $y(x)$ satisfying the boundary conditions*

$$C(x_1) = 0 \quad \textit{and} \quad \frac{1}{d-c}\int_c^d C(x)\,\mathrm{d}x = \frac{y(c) - y_2}{d-c}.$$

(d) *$D(x) := \partial y/\partial d(x)$ exists on (a,b) and is the solution of the variational equation* (2-1) *along $y(x)$ satisfying the boundary conditions*

$$D(x_1) = 0 \quad \textit{and} \quad \frac{1}{d-c}\int_c^d D(x)\,\mathrm{d}x = \frac{y_2 - y(d)}{d-c}.$$

Proof. Since only x and d are not fixed, we denote $y(x, x_1, y_1, y_2, c, d)$ by $y(x,d)$.

Let $\delta > 0$ be as in Theorem 3.2, $0 < |h| < \delta$ be given, and define the difference quotient

$$D_h(x) = \frac{1}{h}[y(x, d+h) - y(x,d)].$$

Our goal is to show that the limit of D_h exists, solves the variational equation, and satisfies the correct boundary conditions. First, we investigate the boundary conditions.

For every $h \neq 0$,

$$D_h(x_1) = \frac{1}{h}[y(x_1, d+h) - y(x_1, d)] = \frac{1}{h}[y_1 - y_1] = 0,$$

and by using the mean value theorem for integrals,

$$\begin{aligned}
\frac{1}{d-c}\int_c^d D_h(x)\,\mathrm{d}x &= \frac{1}{d-c}\int_c^d \frac{y(x,d+h)-y(x,d)}{h}\,\mathrm{d}x \\
&= \frac{1}{d-c}\int_c^d \frac{y(x,d+h)}{h}\,\mathrm{d}x - \frac{1}{d-c}\int_c^d \frac{y(x,d)}{h}\,\mathrm{d}x \\
&= \frac{1}{d-c}\left[\int_c^{d+h} \frac{y(x,d+h)}{h}\,\mathrm{d}x + \int_{d+h}^d \frac{y(x,d+h)}{h}\,\mathrm{d}x\right] - \frac{y_2}{h} \\
&= \frac{1}{d-c}\,\frac{(d+h)-c}{(d+h)-c}\int_c^{d+h} \frac{y(x,d+h)}{h}\,\mathrm{d}x + \frac{y(e)(d-(d+h))}{h(d-c)} - \frac{y_2}{h} \\
&= \frac{((d+h)-c)y_2}{h(d-c)} - \frac{y(e)}{d-c} - \frac{y_2(d-c)}{h(d-c)} = \frac{y_2 - y(e)}{d-c}
\end{aligned}$$

for some e between d and $d+h$.

Next, we view $y(x)$ in terms of the solution of an IVP at x_1 so that we may employ Theorem 3.1.

To that end, let

$$\mu = y'(x_1, d) \quad \text{and} \quad \nu = \nu(h) = y'(x_1, d+h) - \mu.$$

Then, in terms of an IVP,

$$y(x) = u(x, x_1, y_1, \mu),$$

and we have

$$D_h(x) = \frac{1}{h}[u(x, x_1, y_1, \mu+\nu) - u(x, x_1, y_1, \mu)].$$

By Theorem 3.1 and the mean value theorem, we obtain

$$D_h(x) = \frac{1}{h}[\alpha_2(x, u(x, x_1, y_1, \mu+\bar{\nu}))(\mu+\nu-\mu)],$$

where $\alpha_2(x, u(\,\cdot\,))$ is the solution of (1-1) along $u(\,\cdot\,)$ satisfying

$$\alpha_2(x_1) = 0, \quad \alpha_2'(x_1) = 1.$$

Furthermore, $\mu+\bar{\nu}$ is between μ and $\mu+\nu$. Simplifying,

$$D_h(x) = \frac{\nu}{h}\alpha_2(x, u(x, x_1, y_1, \mu+\bar{\nu})).$$

Thus, to show $\lim_{h\to 0} D_h(x)$ exists, it suffices to show $\lim_{h\to 0} \nu/h$ exists. By condition (v), the fact that $\alpha_2(x, u(\cdot))$ is a nontrivial solution of (2-1) along $u(\cdot)$ and $\alpha_2(x_1, u(\cdot)) = 0$, we have

$$\frac{1}{d-c}\int_c^d \alpha_2(x, u(\cdot))\,\mathrm{d}x \neq 0 \quad \Longrightarrow \quad \int_c^d \alpha_2(x, u(\cdot))\,\mathrm{d}x \neq 0.$$

Recall,

$$\frac{1}{d-c}\int_c^d D_h(x)\,\mathrm{d}x = \frac{y_2 - y(e)}{d-c},$$

and so,

$$\frac{1}{d-c}\int_c^d \frac{\nu}{h}\alpha_2(x, u(x, x_1, y_1, \mu+\bar{\nu}))\,\mathrm{d}x = \frac{y_2 - y(e)}{d-c}.$$

Hence, we obtain

$$\lim_{h\to 0}\frac{\nu}{h} = \frac{(y_2 - y(e))}{(d-c)}\,\frac{1}{1/(d-c)\int_c^d \alpha_2(x, u(\cdot))\,\mathrm{d}x} = \frac{y_2 - y(e)}{\int_c^d \alpha_2(x, u(\cdot))\,\mathrm{d}x} := U.$$

Now let

$$D(x) = \lim_{h\to 0} D_h(x),$$

and note by construction of $D_h(x)$,

$$D(x) = \frac{\partial y}{\partial d}(x).$$

Furthermore,

$$D(x) = \lim_{h\to 0} D_h(x) = U\alpha_2(x, y(x)),$$

which is a solution of the variational equation (2-1) along $y(x)$. In addition,

$$D(x_1) = \lim_{h\to 0} D_h(x_1) = \lim_{h\to 0} 0 = 0,$$

and

$$\frac{1}{d-c}\int_c^d D(x)\,\mathrm{d}x = \lim_{h\to 0}\left[\frac{1}{d-c}\int_c^d D_h(x)\,\mathrm{d}x\right] = \lim_{h\to 0}\frac{y_2 - y(e)}{d-c} = \frac{y_2 - y(d)}{d-c}. \quad \square$$

Finally, we present an analogue to (c) of Theorem 3.1 (Peano's theorem).

Corollary 4.2. *Under the assumptions of the previous theorem, we have*

(a) $z_1(x) = -y'(x_1)u_1(x)$,

(b) $C(x) = -\dfrac{y_2 - y(c)}{y_2 - y(d)}D(x)$,

(c) $C(x) = \dfrac{y(c) - y_2}{d-c}u_2(x)$.

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