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What is odd about binary Parseval frames?

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This paper examines the construction and properties of binary Parseval frames. We address two questions: When does a binary Parseval frame have a complementary Parseval frame? Which binary symmetric idempotent matrices are Gram matrices of binary Parseval frames? In contrast to the case of real or complex Parseval frames, the answer to these questions is not always affirmative. The key to our understanding comes from an algorithm that constructs binary orthonormal sequences that span a given subspace, whenever possible. Special regard is given to binary frames whose Gram matrices are circulants.

1. Introduction

Much of the literature on frames, from its beginnings in nonharmonic Fourier analysis [Duffin and Schaeffer 1952] to comprehensive overviews of theory and applications [Christensen 2003; Kovačević and Chebira 2007a; 2007b], assumes an underlying structure of a real or complex Hilbert space to study approximate expansions of vectors. Indeed, the correspondence between vectors in Hilbert spaces and linear functionals given by the Riesz representation theorem provides a convenient way to characterize Parseval frames, sequences of vectors that behave in a way that is similar to orthonormal bases without requiring the vectors to be linearly independent [Christensen 2003]. Incorporating linear dependence relations is useful to permit more flexibility for expansions and to suppress errors that may model faulty signal transmissions in applications [Marshall 1984; 1989; Rath and Guillemot 2003; 2004; Holmes and Paulsen 2004; Puschel and Kovačević 2005; Bodmann and Paulsen 2005].

The concept of frames has also been established even in vector spaces without a positive definite inner product [Bodmann et al. 2009; Han et al. 2007]. In fact, the

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well-known theory of binary codes can be seen as a form of frame theory in which linear dependence relations among binary vectors are examined [MacWilliams and Sloane 1977; Haemers et al. 1999; Betten et al. 2006]. Here, binary vector spaces are defined over the finite field with two elements; a frame for a finite-dimensional binary vector space is simply a spanning sequence [Bodmann et al. 2009]. In a preceding paper [Bodmann et al. 2014], the study of binary codes from a frame-theoretic perspective has led to additional combinatorial insights in the design of error-correcting codes.

The present paper is concerned with binary Parseval frames. These binary frames provide explicit expansions of binary vectors using a bilinear form that resembles the dot product in Euclidean spaces. In contrast to the inner product on real or complex Hilbert spaces, there are many nonzero vectors whose dot product with themselves vanishes. Such vectors have special significance in our results. Due to the number of nonzero entries they contain, we call them *even* vectors, and if a vector is not even, we call it *odd*. As a consequence of the degeneracy of the bilinear form, there are some striking differences with frame theory over real or complex Hilbert spaces. In this paper, we explore the construction and properties of binary Parseval frames, and compare them with real and complex ones. Our main results are as follows.

In the real or complex case, it is known that each Parseval frame has a Naimark complement [Christensen 2003; Han and Larson 2000]. The complementarity is most easily formulated by stating that the Gram matrices of two complementary Parseval frames sum to the identity. We show that in the binary case, not every Parseval frame has a Naimark complement. We also show that a necessary and sufficient condition for its existence is that the Parseval frame contains at least one even vector.

Moreover, we study the structure of Gram matrices. The Gram matrices of real or complex Parseval frames are characterized as symmetric or hermitian idempotent matrices. The binary case requires the additional condition that at least one column vector of the matrix is odd.

The general results we obtain are illustrated with examples. Special regard is given to cyclic binary Parseval frames, whose Gram matrices are circulants.

2. Preliminaries

We define the notions of a binary frame and a binary Parseval frame as in a previous paper [Bodmann et al. 2009]. The vector space that these sequences of vectors span is the direct sum $\mathbb{Z}_2^n = \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$ of n copies of $\mathbb{Z}_2$ for some $n \in \mathbb{N}$. Here, $\mathbb{Z}_2$ is the field of binary numbers with the two elements 0 and 1, the neutral element with respect to addition and the multiplicative identity, respectively.

Definition 2.1. A *binary frame* is a sequence $\mathcal{F} = \{f_1, \dots, f_k\}$ in a binary vector space $\mathbb{Z}_2^n$ such that $\operatorname{span} \mathcal{F} = \mathbb{Z}_2^n$.

A simple example of a frame is the canonical basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbb{Z}_2^n$. The i-th vector has components $(e_i)_j = \delta_{i,j},\ j \in \{1, 2, \ldots, n\}$, and thus $(e_i)_i = 1$ is the only nonzero entry for e_i. Consequently, a vector $x = (x_i)_{i=1}^n$ is expanded in terms of the canonical basis as $x = \sum_{i=1}^n x_i e_i$.

Frames provide similar expansions of vectors in linear combinations of the frame vectors. Parseval frames are especially convenient for this purpose because the linear combination can be determined with little effort. In the real or complex case, this only requires computing values of inner products between the vector to be expanded and the frame vectors. Although we cannot introduce a nondegenerate inner product in the binary case, we define Parseval frames using a bilinear form that resembles the dot product on $\mathbb{R}^n$. Other choices of bilinear forms and a more general theory of binary frames have been investigated elsewhere; see [Hotovy et al. 2015].

Definition 2.2. The *dot product* on $\mathbb{Z}_2^n$ is the bilinear map $(\cdot, \cdot) : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{Z}_2$ given by

$$\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right) := \sum_{i=1}^{n} x_i y_i .$$

With the help of this dot product, we define a Parseval frame for $\mathbb{Z}_2^n$.

Definition 2.3. A *binary Parseval frame* is a sequence of vectors $\mathcal{F} = \{f_1, \ldots, f_k\}$ in $\mathbb{Z}_2^n$ such that, for all $x \in \mathbb{Z}_2^n$, the sequence satisfies the reconstruction identity

$$x = \sum_{j=1}^{k} (x, f_j) f_j . \tag{2-1}$$

To keep track of the specifics of such a Parseval frame, we then also say that $\mathcal{F}$ is a *binary* (k, n)*-frame*.

In the following, we use matrix algebra whenever it is convenient for establishing properties of frames. We write $A \in M_{m,n}(\mathbb{Z}_2)$ when A is an $m \times n$ matrix with entries in $\mathbb{Z}_2$ and identify A with the linear map from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2^m$ induced by left multiplication of any (column) vector $x \in \mathbb{Z}_2^n$ with A. We let A^* denote the adjoint of $A \in M_{m,n}(\mathbb{Z}_2)$; that is, $(Ax, y) = (x, A^* y)$ for all $x \in \mathbb{Z}_2^n,\ y \in \mathbb{Z}_2^m$ and consequently, A^* is the transpose of A.

Definition 2.4. Each frame $\mathcal{F} = \{f_1, \ldots, f_k\}$ is associated with its *analysis matrix* $\Theta_{\mathcal{F}}$, whose i-th row is given by the i-th frame vector for $i \in \{1, 2, \ldots, k\}$. Its transpose $\Theta_{\mathcal{F}}^*$ is called the *synthesis matrix*.

With the help of matrix multiplication, the reconstruction formula (2-1) of a binary (k, n)-frame $\mathcal{F}$ with analysis matrix $\Theta_{\mathcal{F}}$ is simply expressed as

$$\Theta_{\mathcal{F}}^* \Theta_{\mathcal{F}} = I_n, \tag{2-2}$$

where I_n is the $n \times n$ identity matrix. We also note that for any $x, y \in \mathbb{Z}_2^n$, their dot product is unchanged by applying $\Theta_{\mathcal{F}}$,

$$(\Theta_{\mathcal{F}} x, \Theta_{\mathcal{F}} y) = (x, y),$$

which motivates speaking of $\Theta_{\mathcal{F}}$ as an *isometry*, as in the case of real or complex inner product spaces.

Another way to interpret identity (2-2) is in terms of the *column* vectors of $\Theta_{\mathcal{F}}$. Again borrowing a concept from Euclidean spaces, we introduce orthonormality.

Definition 2.5. We say that a sequence of vectors $\{v_1, v_2, \ldots, v_r\}$ in $\mathbb{Z}_2^n$ is *orthonormal* if $(v_i, v_j) = \delta_{i,j}$ for $i, j \in \{1, 2, \ldots, r\}$; that is, the dot product of the pair v_i and v_j vanishes unless $i = j$, in which case it is equal to 1.

Inspecting the matrix identity (2-2), we see that a binary $k \times n$ matrix Θ is the analysis matrix of a binary Parseval frame if and only if the columns of Θ form an orthonormal sequence in $\mathbb{Z}_2^k$.

The orthogonality relations between the frame vectors are recorded in the Gram matrix, whose entries consist of the dot products of all pairs of vectors.

Definition 2.6. The *Gram matrix* of a binary frame $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ for $\mathbb{Z}_2^n$ is the $k \times k$ matrix G with entries $G_{i,j} = (f_j, f_i)$.

It is straightforward to verify that the Gram matrix of $\mathcal{F}$ is expressed as the composition of the analysis and synthesis matrices,

$$G = \Theta_{\mathcal{F}} \Theta_{\mathcal{F}}^*.$$

The identity (2-2) implies that the Gram matrix of a Parseval frame satisfies

$$G = G^* = G^2.$$

For frames over the real or complex numbers, these equations characterize the set of all Gram matrices of Parseval frames as orthogonal projection matrices. However, in the binary case, this is only a necessary condition, as shown in the following proposition and the subsequent example.

Proposition 2.7. *If M is binary matrix that satisfies $M = M^2 = M^*$ and it has only even column vectors, then M is not the Gram matrix of a binary Parseval frame.*

Proof. If G is the Gram matrix of a Parseval frame with analysis operator Θ, then $G\Theta = \Theta\Theta^*\Theta = \Theta$, and thus for each column ω of Θ, we obtain the eigenvector equation $G\omega = \omega$. By the orthonormality of the columns of Θ, each ω is odd.

On the other hand, if M has only even columns, then any eigenvector corresponding to eigenvalue 1 is even, because it is a linear combination of the column vectors of M. This means M cannot be the Gram matrix of a binary Parseval frame. $\square$

The following example shows that idempotent symmetric matrices that are not Gram matrices of binary Parseval frames exist for any odd dimension $k \geq 3$.

Example 2.8. Let $k \geq 3$ be odd and let M be the $k \times k$ matrix whose entries are all equal to 1 except for vanishing entries on the diagonal, $M_{i,j} = 1 - \delta_{i,j}$, $i, j \in \{1, 2, \ldots, k\}$. This matrix satisfies $M = M^2 = M^*$, but only has even columns, and by the preceding proposition, it is not the Gram matrix of a binary Parseval frame.

As shown in Section 4, having only even column vectors is the only way a binary symmetric idempotent matrix can fail to be the Gram matrix of a Parseval frame. The construction of Example 2.8 is intriguing because the alternative choice where k is odd and all entries of M are equal to 1 *is* the Gram matrix of a binary Parseval frame. The relation between these two alternatives can be interpreted as complementarity, which will be explored in more detail in the next section.

3. Complementarity for binary Parseval frames

Over the real or complex numbers, each Parseval frame has a so-called Naimark complement [Christensen 2003], also called a strong complement [Han and Larson 2000]; if G is the Gram matrix of a real or complex Parseval frame, then it is an orthogonal projection matrix, and so is $I - G$, which makes it the Gram matrix of a complementary Parseval frame.

We adopt the same definition for the binary case.

Definition 3.1. Two binary Parseval frames $\mathcal{F}$ and $\mathcal{G}$ having analysis operators $\Theta_{\mathcal{F}} \in M_{k,n}(\mathbb{Z}_2)$ and $\Theta_{\mathcal{G}} \in M_{k,k-n}(\mathbb{Z}_2)$ are *complementary* if

$$\Theta_{\mathcal{F}} \Theta_{\mathcal{F}}^* + \Theta_{\mathcal{G}} \Theta_{\mathcal{G}}^* = I_k.$$

We also say that $\mathcal{F}$ and $\mathcal{G}$ are *Naimark complements* of each other.

There is an equivalent statement of complementarity in terms of the block matrix $U = (\Theta_{\mathcal{F}}\, \Theta_{\mathcal{G}})$, formed by adjoining $\Theta_{\mathcal{F}}$ and $\Theta_{\mathcal{G}}$, being *orthogonal*, meaning $UU^* = U^*U = I$, just as in the real case (or as U being unitary in the complex case).

Proposition 3.2. *Two binary Parseval frames $\mathcal{F}$ and $\mathcal{G}$ having analysis operators $\Theta_{\mathcal{F}} \in M_{k,n}(\mathbb{Z}_2)$ and $\Theta_{\mathcal{G}} \in M_{k,k-n}(\mathbb{Z}_2)$ are complementary if and only if the block matrix $(\Theta_{\mathcal{F}}\, \Theta_{\mathcal{G}})$ is an orthogonal $k \times k$ matrix.*

Proof. In terms of the block matrix $(\Theta_{\mathcal{F}}\, \Theta_{\mathcal{G}})$, the complementarity is expressed as

$$(\Theta_{\mathcal{F}}\, \Theta_{\mathcal{G}})(\Theta_{\mathcal{F}}\, \Theta_{\mathcal{G}})^* = I_k\,.$$

Since $U = (\Theta_{\mathcal{F}}\, \Theta_{\mathcal{G}})$ is a square matrix, $UU^* = I$ is equivalent to U^* also being a left inverse of U, meaning $UU^* = U^*U = I_k$, and thus U is orthogonal. □

In the binary case, not every Parseval frame has a Naimark complement. For example, if $k \geq 3$ is odd and $n = 1$, the frame consisting of k vectors $\{1, 1, \dots, 1\}$ in $\mathbb{Z}_2$ is Parseval, and the Gram matrix G is the $k \times k$ matrix whose entries are all equal to 1. However, $I - G \equiv I + G$ is the matrix M appearing in Example 2.8, which is not the Gram matrix of a binary Parseval frame. This motivates the search for a condition that characterizes the existence of complementary Parseval frames.

A simple condition for the existence of complementary Parseval frames. We observe that if $\mathcal{F}$ is a Parseval frame with analysis operator $\Theta_{\mathcal{F}}$ that extends to an orthogonal matrix, then the column vectors of $\Theta_{\mathcal{F}}$ are a subset of a set of n orthonormal vectors. This is true in the binary as well as the real or complex case. Thus, one could try to relate the construction of a complementary Parseval frame to a Gram–Schmidt orthogonalization strategy. Indeed, this idea allows us to formulate a concrete condition that characterizes when $\mathcal{F}$ has a complementary Parseval frame. We prepare this result with a lemma about extending orthonormal sequences.

Lemma 3.3. *A binary orthonormal sequence $\mathcal{Y} = \{v_1, v_2, \dots, v_r\}$ in $\mathbb{Z}_2^k$ with $r \leq k-1$ extends to an orthonormal sequence $\{v_1, v_2, \dots, v_k\}$ if and only if $\sum_{i=1}^r v_i \neq \iota_k$, where ι_k is the vector in $\mathbb{Z}_2^k$ whose entries are all equal to* 1.

Proof. If the sequence extends, then $\{v_1, v_2, \dots, v_k\}$ forms a Parseval frame for $\mathbb{Z}_2^k$, and by the orthonormality, $\sum_{i=1}^k v_i = \sum_{i=1}^k (\iota_k, v_i) v_i = \iota_k$. On the other hand, the orthonormality forces the set $\{v_1, v_2, \dots, v_k\}$ to be linearly independent, so ι_k cannot be expressed as a linear combination of a proper subset.

To show the converse, we use an inductive proof. Let V be the analysis operator associated with an orthonormal sequence $\{v_1, v_2, \dots, v_s\}$, $r \leq s \leq k-1$, satisfying $\sum_{i=1}^s v_i \neq \iota_k$. To extend the sequence by one vector, we need to find v_{s+1} with $(v_{s+1}, v_{s+1}) = (v_{s+1}, \iota_k) = 1$ and with $(v_j, v_{s+1}) = 0$ for all $1 \leq j \leq s$. Using block matrices this is summarized in the equation

$$\begin{pmatrix} V \\ \iota_k^* \end{pmatrix} v_{s+1} = \begin{pmatrix} 0_s \\ 1 \end{pmatrix}, \tag{3-1}$$

where 0_s is the zero vector in $\mathbb{Z}_2^s$.

In order to verify that this equation is consistent, we note that by the orthonormality of the sequence $\{v_1, v_2, \dots, v_s\}$, the vector ι_k is a linear combination if and only if $\sum_{i=1}^s v_i = \iota_k$. Thus, there exists v_{s+1} which extends the orthonormal sequence. This is all that is needed if $s = k-1$.

Next, we need to show that if $s \leq k-2$, then a solution v_{s+1} can be chosen so that $\sum_{i=1}^{s+1} v_i \neq \iota_k$, so that the iterative extension procedure can be continued. The solution set of (3-1) forms an affine subspace of $\mathbb{Z}_2^k$ having dimension $k - (s+1)$, and thus contains 2^{k-s-1} elements. If $s \leq k-2$, there are at least two elements in this affine subspace. Consequently, there is one choice of v_{s+1} such that $\sum_{i=1}^{s+1} v_i \neq \iota_k$. $\square$

We are ready to characterize the complementarity property for binary Parseval frames. The condition that determines the existence of a Naimark complement is whether at least one frame vector is even, that is, its entries sum to zero.

Theorem 3.4. *A binary (k,n)-frame $\mathcal{F}$ with $n<k$ has a complementary Parseval frame if and only if at least one frame vector is even.*

Proof. The existence of a complementary Parseval frame is by Proposition 3.2 equivalent to the sequence of column vectors $\{\omega_1,\omega_2,\dots,\omega_n\}$ of $\Theta_{\mathcal{F}}$ having an extension to an orthonormal sequence of k elements.

The condition that at least one frame vector is even can be stated as $\Theta_{\mathcal{F}}\iota_n \neq \iota_k$ or, expressed in terms of the column vectors, as $\sum_{i=1}^{n}\omega_n \neq \iota_k$.

The preceding lemma thus provides the existence of a complementary Parseval frame via the extension of $\{\omega_1,\omega_2,\dots,\omega_n\}$ if and only if $\sum_{i=1}^{n}\omega_n \neq \iota_k$. $\square$

A catalog of binary Parseval frames with the complementarity property. A previous work contained a catalog of binary Parseval frames for $\mathbb{Z}_2^n$ when n was small [Bodmann et al. 2009]. Here, we wish to compile a list of the binary Parseval frames that have a complementary Parseval frame. For notational convenience, we consider $\Theta_{\mathcal{F}}$ instead of the sequence of frame vectors. By Proposition 3.2, every such $\Theta_{\mathcal{F}}$ is obtained by a selection of columns from a binary orthogonal matrix, so we could simply list the set of all orthogonal matrices for small k. However, such a list quickly becomes extensive as k increases. To reduce the number of orthogonal matrices, we note that although the frame depends on the order in which the columns are selected to form $\Theta_{\mathcal{F}}$, the Gram matrix does not. Identifying frames whose Gram matrices coincide has already been used to avoid repeating information when examining real or complex frames [Balan 1999] and binary frames [Bodmann et al. 2009]. We consider an even coarser underlying equivalence relation [Goyal et al. 2001; Holmes and Paulsen 2004; Bodmann and Paulsen 2005] that has also appeared in the context of binary frames [Bodmann et al. 2009].

Definition 3.5. Two families $\mathcal{F}=\{f_1,f_2,\dots,f_k\}$ and $\mathcal{G}=\{g_1,g_2,\dots,g_k\}$ in $\mathbb{Z}_2^n$ are called *switching equivalent* if there is an orthogonal $n\times n$ matrix U and a permutation π of the set $\{1,2,\dots,k\}$ such that

$$f_j = U g_{\pi(j)} \quad \text{for all } j\in\{1,2,\dots\}.$$

Representing the permutation π by the associated permutation matrix P with entries $P_{i,j}=\delta_{i,\pi(j)}$ gives that if $\mathcal{F}$ and $\mathcal{G}$ are switching equivalent, then $\Theta_{\mathcal{F}} = P\Theta_{\mathcal{G}}U$, where U is an orthogonal $n\times n$ matrix and P is a $k\times k$ permutation matrix. Alternatively, switching equivalence is stated in the form of an identity for the corresponding Gram matrices.

Theorem 3.6 [Bodmann et al. 2009]. *Two binary (k, n)-frames $\mathcal{F}$ and $\mathcal{G}$ are switching equivalent if and only if their Gram matrices are related by conjugation with a $k \times k$ permutation matrix P,*

$$G_{\mathcal{F}} = P G_{\mathcal{G}} P^*.$$

We deduce a consequence for switching equivalence and Naimark complements, which is inferred from the role of the Gram matrices in the definition of complementarity.

Corollary 3.7. *If $\mathcal{F}$ and $\mathcal{G}$ are switching-equivalent binary (k, n)-frames, then $\mathcal{F}$ has a Naimark complement if and only if $\mathcal{G}$ does.*

Thus, to provide an exhaustive list, we only need to ensure that at least one representative of each switching equivalence class appears as a selection of columns in the orthogonal matrices we include. To reduce the number of representatives, we identify matrices up to row and column permutations.

Definition 3.8. Two matrices $A, B \in M_{k,k}(\mathbb{Z}_2)$ are called *permutation equivalent* if there are two permutation matrices $P_1, P_2 \in M_{k,k}(\mathbb{Z}_2)$ such that $A = P_1 B P_2^*$.

Proposition 3.9. *If U_1 and U_2 are permutation-equivalent binary orthogonal matrices, then each (k, n)-frame $\mathcal{F}$ formed by a sequence of n columns of U_1 is switching equivalent to a (k, n)-frame $\mathcal{G}$ formed with columns of U_2.*

Proof. Without loss of generality, we can assume that the analysis matrix $\Theta_{\mathcal{F}}$ is formed by the first n columns of U_1. By the equivalence of U_1 and U_2, we have $U_1 P_2 = P_1 U_2$ with permutation matrices P_1 and P_2. The right multiplication of U_1 with P_2 gives a column permutation, which identifies a sequence of columns in $P_1 U_2$ that is identical to the first n columns of U_1. If $\mathcal{G}$ is obtained with the corresponding columns in U_2, then the Gram matrices of $\mathcal{F}$ and $\mathcal{G}$ are related by $G_{\mathcal{F}} = P_1 G_{\mathcal{G}} P_1^*$, which proves the switching equivalence. □

A list of permutation-inequivalent orthogonal $k \times k$ matrices allows us to obtain the Gram matrix of each binary (k, n)-frame with a Naimark complement by selecting an appropriate choice of n columns from an orthogonal $k \times k$ matrix to form Θ and then by applying a permutation matrix P to obtain $G_{\mathcal{F}} = P\Theta\Theta^* P^*$.

Each representative of an equivalence class of orthogonal matrices can be chosen so that the columns are in lexicographical order. Table 1 contains a complete list of representatives of binary orthogonal matrices for $k \in \{3, 4, 5, 6\}$ from each permutation equivalence class. Each column vector in our list is recorded by the integer obtained from the binary expansion with the entries of the vector. For example, if a frame vector in $\mathbb{Z}_2^4$ is $f_1 = (1, 0, 1, 1)$, then it is represented by the integer $2^0 + 2^2 + 2^3 = 13$. Accordingly, in $\mathbb{Z}_2^4$, the standard basis is recorded as the sequence of numbers $1, 2, 4, 8$.

k	nonequivalent $k \times k$ orthogonal matrices
3	(1, 2, 4)
4	(1, 2, 4, 8) (7, 11, 13, 14)
5	(1, 2, 4, 8, 16) (4, 11, 19, 25, 26) (7, 8, 19, 21, 22) (7, 11, 13, 14, 16)
6	(1, 2, 4, 8, 16, 32) (4, 8, 19, 35, 49, 50) (4, 11, 16, 35, 41, 42) (4, 11, 19, 25, 26, 32) (7, 8, 16, 35, 37, 38) (7, 8, 19, 21, 22, 32) (7, 11, 13, 14, 16, 32) (13, 14, 28, 44, 55, 59) (21, 22, 28, 47, 52, 59) (25, 26, 28, 47, 55, 56) (31, 37, 38, 44, 52, 59) (31, 41, 42, 44, 55, 56) (31, 47, 49, 50, 52, 56) (31, 47, 55, 59, 61, 62)

Table 1. All permutation-inequivalent binary orthogonal $k \times k$ matrices, $3 \leq k \leq 6$. Up to switching equivalence, the Gram matrix of each binary (k, n)-frame with a Naimark complement is obtained by selecting appropriate columns in one of the listed $k \times k$ orthogonal matrices.

4. Gram matrices of binary Parseval frames

The preceding section on complementarity hinged on the problem that even if G is the Gram matrix of a binary Parseval frame, $I - G$ may not be, even though it is symmetric and idempotent. Again, there is a simple condition that needs to be added; Gram matrices of binary Parseval frames are symmetric and idempotent *and* have at least one odd column, that is, a column whose entries sum to 1. Because of the identity $G^2 = G$, having an odd column is equivalent to having a nonzero diagonal entry. Indeed, it has been shown that for *any* binary symmetric matrix G without vanishing diagonal, there is a factor Θ such that $G = \Theta\Theta^*$ and the rank of Θ is equal to that of G [Lempel 1975]. The assumptions needed for our proof are stronger, but our algorithm for producing Θ appears to be more straightforward than the factorization procedure for general symmetric binary matrices.

Theorem 4.1. *A binary symmetric idempotent matrix M is the Gram matrix of a Parseval frame if and only if it has at least one odd column.*

Proof. First, we re-express the condition on the columns of a symmetric $k \times k$ matrix M in the equivalent form of the matrix $I_k + M$ having at least one even column or row. This, in turn, is equivalent to the inequality $(I_k + M)\iota_k \neq \iota_k$.

Next, we recall that both M and $I_k + M$ are assumed to be idempotent. We observe that any vector $y \in \mathbb{Z}_2^k$ is in the range of an idempotent P if and only if $Py = y$ if and only if y is in the kernel of $I_k + P$.

Assuming M is the Gram matrix of a Parseval frame, we have $M = \Theta\Theta^*$ where Θ has orthonormal columns and $(I_k + M)\Theta = 0$. Combining the two properties gives

$$\begin{pmatrix} I_k + M \\ \iota_k^* \end{pmatrix} \Theta = \begin{pmatrix} 0_{k,n} \\ \iota_n^* \end{pmatrix}.$$

This is inconsistent if and only if ι_k is in the span of the columns of the idempotent $I_k + M$, which is equivalent to $(I_k + M)\iota_k = \iota_k$.

Conversely, assuming that M is symmetric and idempotent and has at least one odd column, we construct a matrix Θ with orthonormal columns such that $M = \Theta\Theta^*$.

We follow an inductive strategy similar to an earlier proof and construct an orthonormal sequence $\{\omega_1, \dots, \omega_n\}$ as described in the following paragraph such that n is the rank of M and $M\omega_i = \omega_i$ for all $i \in \{1, 2, \dots, n\}$. In that case, the range of M is the span of the sequence, and so is the range of M^*. Moreover, if Θ contains the column vectors $\{\omega_1, \dots, \omega_n\}$, then $M\omega_i = \omega_i = \Theta\Theta^*\omega_i$ implies $M = \Theta\Theta^*$ because the two matrices have rank at most n and provide the identity map on the span of n linearly independent vectors.

To begin with the induction, if M has an odd column, then the fact that M is idempotent gives that the equation $(I_k + M)\omega_1 = 0$ has this column vector as an odd solution. If this solution is unique, then $I_k + M$ has rank $k - 1$, M has rank 1 and $\{\omega_1\}$ is the desired sequence. If the solution is not unique, then we can choose $\omega_1 \neq \iota_k$ and proceed with the induction.

Next, we extend a given orthonormal sequence $\{\omega_1, \dots, \omega_s\}$ in the kernel of $I_k + M$ with $s \le n - 2$ by one vector. Let V be a matrix formed by a maximal set of linearly independent rows in $I_k + M$. Then if M has rank n, the rank-nullity theorem gives that V has $k - n$ rows. Letting Y be the analysis matrix of the orthonormal sequence $\{\omega_1, \dots, \omega_s\}$, extending it by one vector requires solving the equation

$$\begin{pmatrix} V \\ Y \\ \iota_k^* \end{pmatrix} \omega_{s+1} = \begin{pmatrix} 0_{k-n} \\ 0_s \\ 1 \end{pmatrix}. \tag{4-1}$$

In order to avoid producing an inconsistent equation during the induction process, we strengthen the induction assumption by the requirement that ι_k^* is not in the span of the rows of the matrix formed by V and Y and conclude in each step that ι_k^* is not in the span of the rows of the matrix formed by V and Y and ω_{s+1}^*. As before, this is obtained by the fact that $VY^* = 0$, so if $\iota_k = \sum_{i=1}^{s+1} c_i\omega_i + v$ with v being in the span of the columns of V^*, then $Yv = 0$ and orthonormality forces $c_i = 1$ for all $i \in \{1, 2, \dots, s+1\}$. The solutions of (4-1) form an affine subspace of dimension $k-(k-n)-s-1 = n-s-1$, so if $s \leq n-2$, then there are at least two solutions, one of which does not satisfy the identity $\iota_k = \sum_{i=1}^{s+1} c_i\omega_i + v$.

Having constructed the sequence $\{\omega_1, \dots, \omega_{n-1}\}$, in the remaining step the unique solution to (4-1) for $s = n-1$ completes the orthonormal sequence. $\square$

5. Binary cyclic frames and circulant Gram matrices

Next, we examine a special type of frame whose Gram matrices are circulants. We recall that a cyclic subspace V of $\mathbb{Z}_2^k$ has the property that it is closed under cyclic shifts; that is, the cyclic shift S, which is characterized by $Se_j = e_{j+1 \pmod k}$, leaves V invariant.

Definition 5.1. A frame $\mathcal{F} = \{f_1, f_2, \dots, f_k\}$ for $\mathbb{Z}_2^n$ is called a binary cyclic frame if the range of the analysis operator is invariant under the cyclic shift S. If $\mathcal{F}$ is also Parseval, then we say that is a binary cyclic (k, n)-frame.

Since the range of the Gram matrix G belonging to a Parseval frame is identical to the set of eigenvectors corresponding to eigenvalue 1, we have a simple characterization of Gram matrices of binary cyclic Parseval frames.

Theorem 5.2. *A binary frame $\mathcal{F} = \{f_1, f_2, \dots, f_k\}$ for $\mathbb{Z}_2^n$ is a cyclic Parseval frame if and only if its Gram matrix $G_{\mathcal{F}}$ is a symmetric idempotent circulant matrix* (*that is*, $G_{\mathcal{F}} = G_{\mathcal{F}}^* = G_{\mathcal{F}}^2$ *and* $SG_{\mathcal{F}}S^* = G_{\mathcal{F}}$), *with only odd column vectors.*

Proof. If $G_{\mathcal{F}}$ is the Gram matrix of a binary cyclic Parseval frame, then from the Parseval property, we know that $G_{\mathcal{F}} = G_{\mathcal{F}}^* = G_{\mathcal{F}}^2$. Moreover, by the cyclicity of the frame, the eigenspace corresponding to eigenvalue 1 of $G_{\mathcal{F}}$ is invariant under S, and thus if $x = G_{\mathcal{F}}x$, then $Sx = SG_{\mathcal{F}}x = G_{\mathcal{F}}Sx$. Using this identity repeatedly and writing $y = S^{k-1}x = S^*x$ gives $y = SG_{\mathcal{F}}S^*y$ for all y in the range of $G_{\mathcal{F}}$. By the symmetry of $G_{\mathcal{F}}$, the range of $G_{\mathcal{F}}$ is identical to that of $G_{\mathcal{F}}^*$, so $\langle G_{\mathcal{F}}x, y\rangle = \langle SG_{\mathcal{F}}S^*x, y\rangle$ for all x, y in the range of $G_{\mathcal{F}}$ establishes the circulant property $G_{\mathcal{F}} = SG_{\mathcal{F}}S^*$. If $G_{\mathcal{F}}$ is a circulant, then each column vector generates all the others by applying powers of the cyclic shift to it. Thus, if one column vector is odd, so are all the other column vectors. Applying Theorem 4.1 then yields that the Gram matrices of binary cyclic Parseval frames are symmetric idempotent circulant matrices with only odd column vectors.

k	first row of matrix
3	100
	111
4	1000
5	10000
	11111
6	100000
	101010
7	1000000
	1111111
8	10000000
9	100000000
	100100100
	111011011
	111111111
10	1000000000
	1010101010
11	10000000000
	11111111111
12	100000000000
	100010001000
13	1000000000000
	1111111111111
14	10000000000000
	10101010101010
15	100000000000000
	100001000010000
	100100100100100
	100101100110100
	111010011001011
	111011011011011
	111110111101111
	111111111111111
16	1000000000000000
17	10000000000000000
	10010111001110100
	11101000110001011
	11111111111111111
18	100000000000000000
	100000100000100000
	101010001010001010
	101010101010101010
19	1000000000000000000
	1111111111111111111
20	10000000000000000000
	10001000100010001000

Table 2. For k ranging from 3 to 20, the table gives the first row of the circulant $k \times k$ Gram matrix of each binary cyclic (k, n)-frame.

1	1	1	0	1	1	0	1	1		1	0	0	1	1	1	1
1	1	1	1	0	1	1	0	1		1	1	1	0	0	1	1
1	1	1	1	1	0	1	1	0		1	1	1	1	1	0	0
0	1	1	1	1	1	0	1	1		0	1	0	1	1	1	1
1	0	1	1	1	1	1	0	1		0	0	0	1	0	1	1
1	1	0	1	1	1	1	1	0		0	0	0	0	0	1	0
0	1	1	0	1	1	1	1	1		0	0	1	1	1	1	1
1	0	1	1	0	1	1	1	1		0	0	0	0	1	1	1
1	1	0	1	1	0	1	1	1		0	0	0	0	0	0	1

Table 3. Circulant Gram matrix (left) and analysis matrix (right) of the unique binary cyclic (9, 7)-frame with nonrepeating vectors.

Conversely, if G is a symmetric idempotent circulant and each column vector is odd, then Theorem 4.1 again yields that it is the Gram matrix of a binary Parseval frame $\mathcal{F}$ with $G = \Theta_{\mathcal{F}}\Theta_{\mathcal{F}}^*$. Moreover, the range of G is invariant under the cyclic shift, because one column vector generates all the others by applying powers of the cyclic shift to it. Since the range of G is identical to that of $\Theta_{\mathcal{F}}$, we have that $\mathcal{F}$ is a cyclic binary Parseval frame. $\square$

$n=7$	
111010011001011	1010111
111101001100101	1101101
111110100110010	0101100
011111010011001	1011011
101111101001100	1101110
010111110100110	1100100
001011111010011	0110001
100101111101001	1000011
110010111110100	1101000
011001011111010	0110010
001100101111101	0001011
100110010111110	0000010
010011001011111	0011111
101001100101111	0000111
110100110010111	0000001

$n=13$	
111011011011011	1001111001111
111101101101101	1110011110011
111110110110110	1111100111100
011111011011011	0101111001111
101111101101101	0001011110011
110111110110110	0000010111100
011011111011011	0011111001111
101101111101101	0000111110011
110110111110110	0000001111100
011011011111011	0000000101111
101101101111101	0000000001011
110110110111110	0000000000010
011011011011111	0000000011111
101101101101111	0000000000111
110110110110111	0000000000001

$n=9$	
100101100110100	101000100
010010110011010	010010010
001001011001101	100000101
100100101100110	111000110
010010010110011	111110011
101001001011001	011011001
110100100101100	101110100
011010010010110	011111110
001101001001011	000111011
100110100100101	000001101
110011010010010	001100010
011001101001001	000011001
101100110100100	000000100
010110011010010	000000010
001011001101001	000000001

$n=11$	
111110111101111	10011111111
111111011110111	11100111111
111111101111011	11111001111
111111110111101	11111110011
111111111011110	11111111100
011111111101111	01011111111
101111111110111	00010111111
110111111111011	00000101111
111011111111101	00000001011
111101111111110	00000000010
011110111111111	00111111111
101111011111111	00001111111
110111101111111	00000011111
111011110111111	00000000111
111101111011111	00000000001

Table 4. Circulant Gram matrix (first matrix of each pair) and analysis matrix (second of pair) of binary cyclic $(15, n)$-frames, $n < k$, whose vectors do not repeat.

Since adding the identity matrix changes odd columns of G to even columns, we conclude that complementary Parseval frames do not exist for binary cyclic Parseval frames.

Corollary 5.3. *If $\mathcal{F}$ is a binary cyclic Parseval frame, then it has no complementary Parseval frame.*

In Table 2, we provide an exhaustive list of the Gram matrices of cyclic binary Parseval frames with $3 \le k \le 20$. Factoring these into the corresponding analysis and synthesis matrices shows that many of these examples contain repeated frame vectors. In an earlier paper [Bodmann et al. 2009], such repeated vectors were associated with a trivial form of redundancy incorporated in the analysis matrix $\Theta_{\mathcal{F}}$. Tables 3 and 4 list the circulant Gram matrices of rank $n < k \leq 20$, paired with $k \times n$ analysis matrices, for which no repetition of frame vectors occurs.

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