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Hexatonic systems and dual groups in mathematical music theory

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Motivated by the music-theoretical work of Richard Cohn and David Clampitt on late-nineteenth century harmony, we mathematically prove that the *PL*-group of a hexatonic cycle is dual (in the sense of Lewin) to its T/I-stabilizer. Our points of departure are Cohn's notions of maximal smoothness and hexatonic cycle, and the symmetry group of the 12-gon; we do *not* make use of the duality between the T/I-group and *PLR*-group. We also discuss how some ideas in the present paper could be used in the proof of T/I-*PLR* duality by Crans, Fiore, and Satyendra (*Amer. Math. Monthly* **116**:6 (2009), 479–495).

1. Introduction: hexatonic cycles and associated dual groups

Why did late nineteenth century composers, such as Franck, Liszt, Mahler, and Wagner, continue to favor consonant triads over other tone collections, while simultaneously moving away from the diatonic scale and classical tonality?

Richard Cohn [1996] proposed an answer, independent of acoustic consonance: major and minor triads are preferred because they can form *maximally smooth cycles*. Consider for instance the following sequence of consonant triads, called a *hexatonic cycle* by Cohn:

$$E\flat,\ e\flat,\ B,\ b,\ G,\ g,\ E\flat. \tag{1}$$

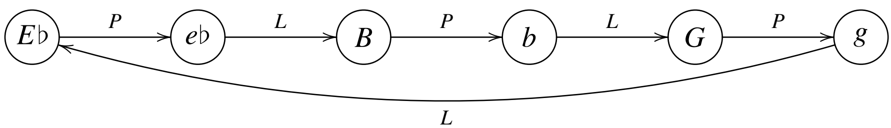
We have indicated major chords with capital letters and minor chords with lowercase letters. Although the motion from a major chord to its parallel minor, e.g., $E\flat$ to $e\flat$, B to b, and G to g, is distinctly nondiatonic, this sequence has cogent properties of importance to late-Romantic composers, as axiomatized in Cohn's notion of maximally smooth cycle [1996, page 15]:

MSC2010: 20-XX.

Keywords: mathematical music theory, dual groups, hexatonic cycle, maximally smooth cycle, triad, transposition, inversion, simple transitivity, centralizer, PLR-group, neo-Riemannian group, transformational analysis, Parsifal.

- It is a *cycle* in the sense that the first and last chords are the same but all others are different. A *cycle* is required to contain more than three chords.
- All of the chords are in one “set class”; in this case each chord is a consonant triad.
- Every transition is *maximally smooth* in the sense that two notes stay the same while the third moves by the smallest possible interval: a semitone.

Cohn considered movement along this sequence transformationally as an action by a cyclic group of order 6. Additionally, David Clampitt [1998] considered movement along this sequence via P and L, and also via certain rotations and reflections. As usual, we denote by P the “parallel” transformation that sends a major or minor chord to its parallel minor or major chord, respectively. We denote by L the “leading tone exchange” transformation, which moves the root of a major chord down a semitone and the fifth of a minor chord up a semitone, so the L sends consonant triads $e\flat$ to B, and b to G, and g to $E\flat$. The hexatonic cycle (1) is then positioned in the network



(2)

with alternating P and L transformations between the nodes.

Wagner’s Grail motive in *Parsifal* can be interpreted in terms of network (2), as proposed by David Clampitt [1998]. A small part of Clampitt’s analysis of the first four chords is pictured in Figure 1. Clampitt includes the final $D\flat$ chord, which lies outside of the hexatonic cycle (1), in his interpretation via a conjugation-modulation applied to a certain subsystem. A third interpretation, in addition to the cyclic one of Cohn [1996, Example 5] and the PL-interpretation in Figure 1, was also proposed by Clampitt, this time in terms of the transpositions and inversions $\{T_0, T_4, T_8, I_1, I_5, I_9\}$. Clampitt observes that this group and the PL-group form *dual groups in the sense of Lewin* [1987], via their actions on the hexatonic set of chords in (1). The perceptual basis of all three groups is explained in [Clampitt 1998].

The contribution of the present article is to directly prove that the PL-group and the group $\{T_0, T_4, T_8, I_1, I_5, I_9\}$ in Clampitt’s article are dual groups acting on (1). Our points of departure are the hexatonic cycle (1), the standard action of the dihedral group of order 24 on the 12-gon, and the Orbit-Stabilizer Theorem. We do *not* use the duality of the T/I-group and PLR-group. Some arguments in Section 3 are similar to arguments of Crans, Fiore, and Satyendra [Crans et al. 2009], but there are important differences; see Remark 3.10.

Just how special are the consonant triads with regard to the maximal smoothness property? According to [Cohn 1996], only six categories of tone collections support

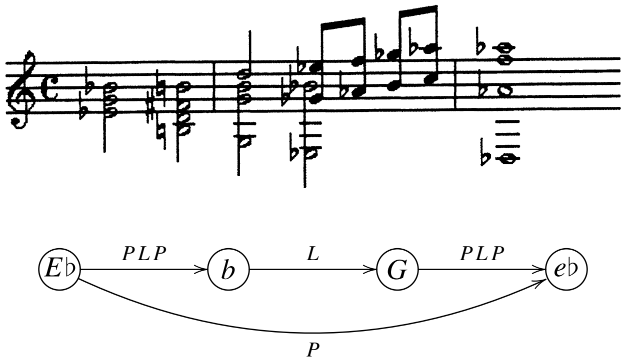


Figure 1. Top: Grail motive from Wagner, *Parsifal*, Act 3, measures 1098–1100, reproduced from [Clampitt 1998, Example 1]. Bottom: First four chords of the Grail motive in a hexatonic *PL*-network of Clampitt. Notice that the bottom arrow is the composite of the three top arrows, and goes in the opposite direction of the bottom arrow of diagram (2).

maximally smooth cycles: singletons, consonant triads, pentatonic sets, diatonic sets, complements of consonant triads, and 11-note sets. Clearly the singletons and 11-note sets do not give musically significant cycles. The pentatonic sets and the diatonic sets each support only one long cycle, which exhausts all 12 of their respective exemplars. The consonant triads and their complements, *on the other hand,* support short cycles that do not exhaust all of their transpositions and inversions. The maximally smooth cycles of consonant triads are enumerated as sets as follows:

$$\{E\flat, e\flat, B, b, G, g\}, \tag{3}$$

$$\{E, e, C, c, A\flat, a\flat\}, \tag{4}$$

$$\{F, f, C\sharp, c\sharp, A, a\}, \tag{5}$$

$$\{F\sharp, f\sharp, D, d, B\flat, b\flat\}. \tag{6}$$

These are the four *hexatonic cycles* of Cohn [1996, page 17]. They (and their reverses and complements) are the only short maximally smooth cycles that exist in the Western chromatic scale.

2. Mathematical and musical preliminaries: standard dihedral group action on consonant triads and the Orbit-Stabilizer Theorem

We quickly recall the standard preliminaries about consonant triads, transposition, inversion, P, L, and the Orbit-Stabilizer Theorem. A good introduction to this very

major triads	minor triads
$C = \langle 0, 4, 7\rangle$	$\langle 0, 8, 5\rangle = f$
$C\sharp = D\flat = \langle 1, 5, 8\rangle$	$\langle 1, 9, 6\rangle = f\sharp = g\flat$
$D = \langle 2, 6, 9\rangle$	$\langle 2, 10, 7\rangle = g$
$D\sharp = E\flat = \langle 3, 7, 10\rangle$	$\langle 3, 11, 8\rangle = g\sharp = a\flat$
$E = \langle 4, 8, 11\rangle$	$\langle 4, 0, 9\rangle = a$
$F = \langle 5, 9, 0\rangle$	$\langle 5, 1, 10\rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6, 10, 1\rangle$	$\langle 6, 2, 11\rangle = b$
$G = \langle 7, 11, 2\rangle$	$\langle 7, 3, 0\rangle = c$
$G\sharp = A\flat = \langle 8, 0, 3\rangle$	$\langle 8, 4, 1\rangle = c\sharp = d\flat$
$A = \langle 9, 1, 4\rangle$	$\langle 9, 5, 2\rangle = d$
$A\sharp = B\flat = \langle 10, 2, 5\rangle$	$\langle 10, 6, 3\rangle = d\sharp = e\flat$
$B = \langle 11, 3, 6\rangle$	$\langle 11, 7, 4\rangle = e$

Table 1. The set of consonant triads, denoted Triads, as displayed on page 483 of [Crans et al. 2009].

well-known background material is [Crans et al. 2009]. Since this background has been treated in many places, we merely rapidly introduce the notation and indicate a few sources.

Consonant triads. We encode pitch classes using the standard $\mathbb{Z}_{12}$ model, where $C = 0,\ C\sharp = D\flat = 1$, and so on, up to $B = 11$. Via this bijection we freely refer to elements of $\mathbb{Z}_{12}$ as *pitch classes*. Major chords are indicated as ordered 3-tuples in $\mathbb{Z}_{12}$ of the form $\langle x, x+4, x+7\rangle$, where x ranges through $\mathbb{Z}_{12}$. Minor chords are indicated as 3-tuples $\langle x+7, x+3, x\rangle$ with $x \in \mathbb{Z}_{12}$. We choose these orderings to make simple formulas for P and L; this is not a restriction for applications, as the framework was extended in [Fiore et al. 2013a] to allow any orderings. We call the set of 24 major and minor triads Triads, this is the set of *consonant triads*. The letter names are indicated in Table 1.

Transposition and inversion, and P and L. The 12-tone operations *transposition* $T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ and *inversion* $I_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ are given by

$$T_n(x) = x + n \quad \text{and} \quad I_n(x) = -x + n$$

for $n \in \mathbb{Z}_{12}$. These 24 operations are the symmetries of the regular 12-gon, when we consider 0 through 11 as arranged on the face of a clock. In the music-theory tradition, this group is called the T/I-group (the "/" does *not* indicate any kind of quotient). The unique reflection of the 12-gon which interchanges m and n is I_{m+n}, as can be verified by direct computation.

Many composers, for instance Schoenberg, Berg, and Webern, utilized these mod 12 transpositions and inversions. These functions and their compositional uses have been thoroughly explored by composers, music theorists, and mathematicians; see for example [Babbitt 1955; Forte 1973; Fripertinger and Lackner 2015; Hook 2007; Hook and Peck 2015; McCartin 1998; Mead 2015; Morris 1987; 1991; 2001; 2015; Rahn 1987]. Indeed, the three recent papers [Fripertinger and Lackner 2015; Mead 2015; Morris 2015] together contain over 100 references.

We consider these bijective functions on $\mathbb{Z}_{12}$ also as bijective functions on Triads via their componentwise evaluation on consonant triads:

$$T_n\langle x_1, x_2, x_3\rangle = \langle T_n x_1, T_n x_2, T_n x_3\rangle \quad \text{and} \quad I_n\langle x_1, x_2, x_3\rangle = \langle I_n x_1, I_n x_2, I_n x_3\rangle. \tag{7}$$

Also on the set Triads of consonant triads (with the indicated ordering), but not on the level of individual pitch classes, we have the bijective functions P and L defined by

$$P\langle x_1, x_2, x_3\rangle = I_{x_1+x_3}\langle x_1, x_2, x_3\rangle \quad \text{and} \quad L\langle x_1, x_2, x_3\rangle = I_{x_2+x_3}\langle x_1, x_2, x_3\rangle. \tag{8}$$

As remarked above, P stands for "parallel" and L stands for "leading tone exchange".

We consider T_n, I_n, P, and L as elements of the symmetric group Sym(Triads).

Proposition 2.1. *The bijections P and L commute with T_n and I_n as elements of the symmetric group* Sym(Triads).

Proof. This is a straightforward computation using equations (7) and (8). This computation has been discussed in broader contexts in [Fiore et al. 2013b] and [Fiore and Satyendra 2005]. □

Orbit-Stabilizer Theorem. Suppose S is a set with a left group action by a group G (all group actions in this paper are *left* group actions). Recall that the *orbit of an element* $Y \in S$ is

$$\text{orbit of } Y := \{gY \mid g \in G\}.$$

The *stabilizer group of an element* $Y \in S$ is

$$G_Y := \{g \in G \mid gY = Y\}.$$

Theorem 2.2 (Orbit-Stabilizer Theorem). *Let G be a group with an action on a set S. Neither G nor S is assumed to be finite. Then the assignment*

$$G/G_Y \to \textit{orbit of } Y,$$
$$gG_Y \mapsto gY,$$

is a bijection. In particular, if G is finite, then each orbit is finite, and

$$|G|/|G_Y| = |\textit{orbit of } Y|. \tag{9}$$

Simple transitivity. A group action of a group G on a set S is said to be *simply transitive* if for any $Y, Z \in S$ there is a unique $g \in G$ such that $gY = Z$. Informally, we also say the group G is *simply transitive* if the sole action under consideration is simply transitive.

Proposition 2.3. (1) *An action of a group G on a set S is simply transitive if and only if it is transitive and every stabilizer G_Y is trivial.*

(2) *Suppose G is a finite group that acts on a set S. Then G is simply transitive if and only if any two of the following three hold*:

  (a) *G is transitive.*
  (b) *Every stabilizer G_Y is trivial.*
  (c) *G and S have the same cardinality.*

In this case, *the third condition also holds.*

Another way to read this "if and only if" statement is: *assuming G is finite and any one of the conditions holds*, *G is simply transitive if and only if another one of the conditions holds.*

(3) *Suppose a* (*not necessarily finite*) *group H_1 acts simply transitively on a set S*, *and a subgroup H_2 of H_1 acts transitively on S via its subaction. Then $H_1 = H_2$.*

Proof. (1) If the action is simply transitive, then it acts transitively and for each $Y \in S$, there is only one $g \in G$ with $gY = Y$, and hence each G_Y is trivial.

Suppose G acts transitively and for every $Y \in S$, the group G_Y is trivial. Suppose $Y, Z \in S$ and $g_1, g_2 \in G$ satisfy $g_1 Y = Z$ and $g_2 Y = Z$. Then $Y = g_2^{-1} Z$ and $g_2^{-1} g_1 Y = Y$, so $g_2^{-1} g_1 \in G_Y = \{e\}$, and finally $g_1 = g_2$.

(2) We first prove that any two of the conditions implies the third and implies simple transitivity.

(a)(b) $\Rightarrow$ (c): G is simply transitive by (1), and equation (9) says $|G|/1 = |S|$, so $|G| = |S|$ and (c) holds.

(b)(c) $\Rightarrow$ (a): Equation (9) says $|S| = |G|/1 = |\text{orbit of } Y|$, so $S = \text{orbit of } Y$, and G is transitive and (a) holds, so G is simply transitive by (1).

(a)(c) $\Rightarrow$ (b): Equation (9) says $|G|/|G_Y| = |G|$, so $|G_Y| = 1$ and (b) holds, and G is simply transitive by (1).

Now that we have shown any two of the conditions implies the third and simple transitivity, we want to see that simply transitivity implies all three conditions. From (1), simple transitivity implies (a) and (b), and we have already seen (a) and (b) imply (c).

(3) Suppose H_1 properly contains H_2, and $h_1 \in H_1 \backslash H_2$. Fix a $Y \in S$ and define $Z := h_1 Y$. Then by the transitivity of H_2, there is an $h_2 \in H_2$ such that $Z = h_2 Y$. But by the simple transitivity of H_1, we must have $h_1 = h_2$, a contradiction. □

3. Main theorem: Hexatonic Duality

We next review the notion of dual groups, and then turn to the main result, Theorem 3.9 on hexatonic duality. Recall that subgroups G and H of $\mathrm{Sym}(S)$ are *dual in the sense of Lewin* [1987, page 253] if each acts simply transitively on S and each is the centralizer of the other.[1] Recall the *centralizer of* G *in* $\mathrm{Sym}(S)$ is

$$C(G) = \{\sigma \in \mathrm{Sym}(S) \mid \sigma g = g\sigma \text{ for all } g \in G\}.$$

Before turning to the main result, we prove two simultaneous redundancies in the notion of *dual groups*: instead of requiring the two groups to centralize each other, it is sufficient to merely require that they commute, and instead of requiring H to act simply transitively, it is sufficient to merely require H acts transitively.

Proposition 3.1. *Let S be a (not necessarily finite) set. Suppose $G \leq \mathrm{Sym}(S)$ acts simply transitively on S and $H \leq \mathrm{Sym}(S)$ acts transitively on S. Suppose G and H commute in the sense that $gh = hg$ for all $g \in G$ and $h \in H$. Then G and H are dual groups. In particular, H also acts simply transitively and G and H centralize one another.*

Proof. We would like to first conclude from the simple transitivity of G, the transitivity of H, and the commutativity of G and H, that the centralizer $C(G)$ acts simply transitively on S.

We claim $C(G)$ acts simply transitively on S. It acts transitively, as $C(G) \supseteq H$ and H acts transitively. So, it suffices by Proposition 2.3(1) to prove that, for each $s \in S$, the only element of $C(G)$ that fixes s is the identity. Let σ be an element of $C(G)$ that fixes s, and g any element of G. Then,

$$\sigma s = s \implies g(\sigma s) = g(s) \implies (g\sigma)s = (gs) \implies (\sigma g)s = (gs) \implies \sigma(gs) = (gs).$$

[1]Lewin [1987, page 253] gave a more general situation that gives rise to examples of dual groups in the sense defined above, though he did not formally make this definition. He starts with a group G, there called *STRANS*, assumed to act simply transitively on a set S, and then makes three claims without proof: (1) the centralizer $C(G)$ in $\mathrm{Sym}(S)$ acts simply transitively on S (the centralizer $C(G)$ is called *STRANS′* there); (2) the double centralizer $C(C(G))$ is contained in G, so actually $C(C(G)) = G$; and (3) the two generalized interval systems with transposition groups G and $C(G)$ respectively have interval-preserving transformation groups, precisely $C(G)$ and G respectively. See Proposition 3.2 for a proof of statements (1) and (2). Statement (3) is a consequence of the first two statements in combination with *COMM-SIMP* duality, which was stated on page 101 of [Lewin 1995] and partially proved in [Lewin 1987, Theorem 3.4.10]. For a review of *COMM-SIMP* duality and another proof, see [Fiore and Satyendra 2005, Section 2 and Appendix]. For the equivalence of generalized interval systems and simply transitive group actions, see pages 157–159 of Lewin's monograph. The equivalence on the level of categories was proved by Fiore, Noll and Satyendra [Fiore et al. 2013b, page 10]. The undergraduate research project of Sternberg [2006] worked out some of the details of Lewin's simply transitive group action associated to a generalized interval system and investigated the Fugue in F from Hindemith's *Ludus Tonalis*.

So, not only does σ fix s, but σ also fixes (gs) for every $g \in G$. That is to say, $\sigma = \mathrm{Id}_S$, and $C(G)$ acts simply transitively on S.

Now we have the transitive subgroup H contained in the simply transitive group $C(G)$ by the assumed commutativity, so by Proposition 2.3(3), $H = C(G)$, and H also acts simply transitively.

To obtain $C(H) = G$, we use the newly achieved simple transitivity of H and repeat the argument with the roles of G and H reversed. □

We may now use a result of Dixon and Mortimer to prove a statement of Lewin [1987, page 253], as suggested by Julian Hook, Robert Peck, and Thomas Noll. Parts (1) and (2) of the following proposition were stated by Lewin.

Proposition 3.2. *Let S be a (not necessarily finite) set. Suppose $G \leq \mathrm{Sym}(S)$ acts simply transitively on S. Then:*

(1) *The centralizer $C(G)$ in $\mathrm{Sym}(S)$ acts simply transitively on S.*

(2) *The centralizer of the centralizer $C(C(G))$ is equal to G.*

(3) *Define $H := C(G)$. Then G and H are dual groups.*

Proof. (1) This follows immediately from [Dixon and Mortimer 1996, Theorem 4.2A(i) and (ii), page 109]. There *semiregular* means point stabilizers are trivial and *regular* means simply transitive.

(2) Since $C(G)$ is simply transitive, we can apply Dixon and Mortimer's result to $C(G)$ to get that the double centralizer $C(C(G))$ simply transitive. But $C(C(G))$ contains the simply transitive group G, so $C(C(G)) = G$ by Proposition 2.3(3).

(3) This follows directly from the preceding two by definition. □

We now turn to the discussion of our main result.

Let Hex be the set of chords in the hexatonic cycle (1) and $\underline{\mathrm{Hex}}$ the set of underlying pitch classes of its chords; that is,

$$\mathrm{Hex} := \{E\flat, e\flat, B, b, G, g\}, \quad \underline{\mathrm{Hex}} := \{2, 3, 6, 7, 10, 11\}.$$

Our goal is to prove that the restriction of the PL-group to Hex and the restriction of $\{T_0, T_4, T_8, I_1, I_5, I_9\}$ to Hex are dual groups, and that each is dihedral of order 6. The strategy is to separately prove the unrestricted groups act simply transitively and are dihedral, and then finally to show that the restricted groups centralize each other. We begin with a characterization of the consonant triads contained in $\underline{\mathrm{Hex}}$.

Lemma 3.3. *The only consonant triads of Table 1 contained in* $\underline{\mathrm{Hex}}$ *as subsets are the elements of* Hex.

Proof. We first identify the available perfect fifths in $\underline{\mathrm{Hex}}$ (pairs with difference 7), and then check if the corresponding major/minor thirds are in $\underline{\mathrm{Hex}}$.

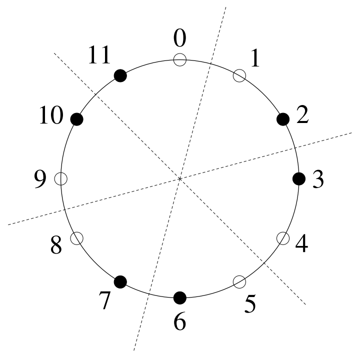


Figure 2. The solid circles represent the subset Hex of $\mathbb{Z}_{12}$. The symmetry of the subset makes apparent that the only rotations which preserve Hex are T_0, T_4, and T_8. The geometric locations of the solid circles also imply that the reflections across the dashed lines are the only reflections which preserve Hex.

The only pairs of the form $\langle x, x+7\rangle$ are $\langle 3, 10\rangle$, $\langle 7, 2\rangle$, and $\langle 11, 6\rangle$, and we see that $x+4$ is contained in Hex in each case; that is, 7, 11, and 3 are in Hex. Thus we have the three major chords $E\flat$, G, and B, and no others.

The only pairs of the form $\langle x+7, x\rangle = \langle y, y+5\rangle$ are $\langle 2, 7\rangle$, $\langle 6, 11\rangle$, and $\langle 10, 3\rangle$, and we see that $x+3 = y+8$ is contained in Hex in each case; that is, 10, 2, and 6 are in Hex. Thus we have the three minor chords g, b, and $e\flat$, and no others. □

Proposition 3.4. (1) *The only elements of the T/I-group that preserve* Hex *as a set are* $\{T_0, T_4, T_8, I_1, I_5, I_9\}$, *so they form a group, which we will denote by H.*

(2) $H := \{T_0, T_4, T_8, I_1, I_5, I_9\}$ *is dihedral of order* 6.

Proof. (1) If an element of the T/I-group preserves Hex as a set, then it must also preserve the collection Hex of underlying pitch classes as a set. Geometric inspection of the plot of Hex in Figure 2 reveals that the only rotations that preserve Hex are T_0, T_4, and T_8.

Again looking at Figure 2, we see that the three reflections which interchange $2 \leftrightarrow 3$ or $6 \leftrightarrow 7$ or $10 \leftrightarrow 11$ preserve Hex. By a comment on page 256, these are

$$I_{2+3} = I_5, \quad I_{6+7} = I_1, \quad I_{10+11} = I_9.$$

No other reflections preserve Hex, as we can see geometrically from its limited reflection symmetry.

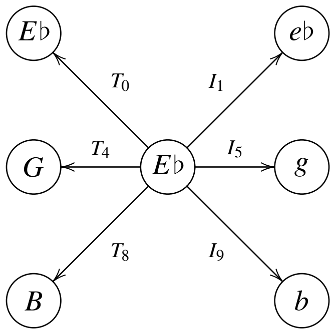
Since $H := \{T_0, T_4, T_8, I_1, I_5, I_9\}$ is a setwise stabilizer of Hex, it is a group.

From Lemma 3.3 we see that $\{T_0, T_4, T_8, I_1, I_5, I_9\}$ must also stabilize the chord collection Hex as a set. No other transpositions or inversions stabilize Hex by the argument at the outset of this proof.

(2) The only noncommutative group of order 6 is the symmetric group on three elements, denoted Sym(3), which is isomorphic to the dihedral group of order 6. The group under consideration is noncommutative, because $T_4 I_1(x) = -x + 5$ while $I_1 T_4(x) = -x - 3$. □

Proposition 3.5. *The setwise stabilizer H acts simply transitively on* Hex.

Proof. The H-orbit of $E\flat$ is all of Hex, as the following diagram shows.



We have $|H| = 6 = |\text{orbit of } Y|$ so the Orbit-Stabilizer Theorem

$$|H|/|H_Y| = |\text{orbit of } Y|$$

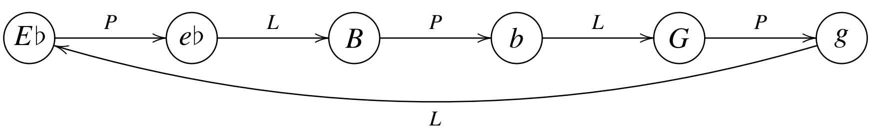
implies $|H_Y| = 1$. See Proposition 2.3(2). □

Next we can investigate the subgroup of Sym(Triads) generated by P and L, which is called the *PL-group*.

Proposition 3.6. *The subgroup $\langle P, L\rangle$ of* Sym(Triads) *is dihedral of order* 6.

Proof. We first observe that P and L are involutions; that is, $P^2 = \text{Id}_{\text{Triads}}$ and $L^2 = \text{Id}_{\text{Triads}}$. A musical justification comes from the definitions of "parallel" and "leading tone exchange". Direct computations of P^2 and L^2 using the formulas in (8) provide a mathematical justification.

Since P and L are involutions, every nontrivial element of $\langle P, L\rangle$ can be expressed as an alternating word in the letters P and L. The six functions $\text{Id}_{\text{Triads}}$, P, LP, PLP, $LPLP$, and $PLPLP$ are all distinct by evaluating at $E\flat$ using the following diagram from the Introduction.



(10)

From diagram (10) we also see that $(LP)^3(E\flat) = E\flat$, and for any $Y \in \{E\flat, B, G\}$, $(LP)^3(Y) = Y$. Similarly, by reading the diagram backwards (recall P and L are involutions), we see $(LP)^3(Y) = Y$ for any minor triad $Y \in \{e\flat, b, g\}$. We have similar *PL*-diagrams and considerations for the cycles in (4), (5), and (6), and therefore $(LP)^3 = \mathrm{Id}_{\mathrm{Triads}}$ on the entire set Triads of consonant triads. Another way to see that $(LP)^3 = \mathrm{Id}_{\mathrm{Triads}}$ is to combine the observation $(LP)^3(E\flat) = E\flat$ from diagram (10) with Proposition 2.1 and the fact that Triads is the T/I-orbit of $E\flat$.

We next show via a word-theoretic argument that $\langle P, L\rangle$ consists only of the six functions $\mathrm{Id}_{\mathrm{Triads}}$, P, LP, PLP, $LPLP$, and $PLPLP$ discussed above. From $(LP)^3 = \mathrm{Id}_{\mathrm{Triads}}$, we express PL in terms of LP. Namely,

$$(LP)^3{=}\mathrm{Id}_{\mathrm{Triads}} \implies (LP)^3(PL){=}(PL) \implies (LP)^2{=}PL.$$

Consider any alternating word in P and L. If the rightmost letter is P, then we can use $(LP)^3 = \mathrm{Id}_{\mathrm{Triads}}$ to achieve an equality with one of the six functions we already have. If the rightmost letter is L, then we replace each PL by $(LP)^2$ and use $L^2 = \mathrm{Id}_{\mathrm{Triads}}$ if LL results on the far left. Then we have an equal function with rightmost letter P, which we can then reduce to one of the six above using $(LP)^3 = \mathrm{Id}_{\mathrm{Triads}}$, as we did in the first case of rightmost letter P. Thus $\langle P, L\rangle = \{\mathrm{Id}_{\mathrm{Triads}}, P, LP, PLP, LPLP, PLPLP\}$.

This group is noncommutative, as $PL \neq LP$; hence it is isomorphic to Sym(3), the only noncommutative group of order 6. But Sym(3) is dihedral of order 6.

Instead of the previous paragraph, we can show $\langle P, L\rangle$ is dihedral of order 6 using a presentation. Let $t := L$ and $s := LP$; then $s^3 = e$, $t^2 = e$, and $tst = s^{-1}$. The dihedral group of order 6 is the largest group with elements s and t such that $s^3 = e$, $t^2 = e$, and $tst = s^{-1}$. But we observed from diagram (10) that $\langle P, L\rangle$ has at least six distinct elements. Hence, $\langle P, L\rangle$ is dihedral of order 6. □

Proposition 3.7. *The PL-group $\langle P, L\rangle$ acts simply transitively on* Hex.

Proof. From diagram (10) we see that $\langle P, L\rangle$ acts transitively on Hex. Since $\langle P, L\rangle$ and Hex have the same cardinality, the Orbit-Stabilizer Theorem implies that every stabilizer must be trivial. See Proposition 2.3(2). □

Lemma 3.8. *Let S be a set and suppose $G \leq$* Sym(S)*. Suppose G acts simply transitively on an orbit $\bar{S}$, and $\overline{G}$ is the restriction of G to the orbit $\bar{S}$. Then the restriction homomorphism $G \to \overline{G}$ is an isomorphism, and $\overline{G}$ also acts simply transitively.*

Proof. Suppose $g \in G$ has restriction $\bar{g}$ with $\bar{g}\bar{s} = \bar{s}$ for all $\bar{s} \in \bar{S}$. Then g also has $g\bar{s} = \bar{s}$ for all $\bar{s} \in \bar{S}$, so $g = \mathrm{Id}_S$ by simple transitivity, and the kernel of the surjective homomorphism $G \to \overline{G}$ is trivial. The transitivity of $\overline{G}$ is clear: for any $\bar{s}, \bar{t} \in \bar{S}$ there exists $g \in G$ such that $g\bar{s} = \bar{t}$, so also $\bar{g}\bar{s} = \bar{t}$ with $\bar{g} \in \overline{G}$. The uniqueness of $\bar{g} \in \overline{G}$ is also clear: if $\bar{h} \in \overline{G}$ also satisfies $\bar{h}\bar{s} = \bar{t}$, then so do g and h, so $g = h$ by the simple transitivity of G acting on $\bar{S}$, so $\bar{g} = \bar{h}$. □

Theorem 3.9 (Hexatonic Duality). *The restrictions of the PL-group and the group $H = \{T_0, T_4, T_8, I_1, I_5, I_9\}$ to* Hex *are dual groups in* Sym(Hex), *and both are dihedral of order* 6.

Proof. Let $\overline{G}$ be the restriction of the PL-group to Hex, and let $\overline{H}$ be the restriction of $H = \{T_0, T_4, T_8, I_1, I_5, I_9\}$ to Hex.

We already know that $\overline{G}$ and $\overline{H}$ are dihedral of order 6 by Propositions 3.4 and 3.6 and Lemma 3.8.

We also already know that $\overline{G}$ and $\overline{H}$ each act simply transitively on Hex by Propositions 3.5 and 3.6 and Lemma 3.8. We even already know that the groups $\overline{G}$ and $\overline{H}$ commute by Proposition 2.1. Finally, Proposition 3.1 guarantees that $\overline{G}$ and $\overline{H}$ centralize one another. □

Remark 3.10 (Comparison with the proof strategy of Crans, Fiore, and Satyendra). There are several differences between the proof strategy of hexatonic duality in the present Theorem 3.9 and the proof strategy of T/I-PLR duality in Theorem 6.1 of [Crans et al. 2009]. In the present paper, we first proved that the concerned groups act simply transitively, and determined their structure, and only then showed that the groups exactly centralize each other. In [Crans et al. 2009], on the other hand, the determination of the size of the PLR-group was postponed until after the centralizer $C(T/I)$ was seen to act simply, i.e., that each stabilizer $C(T/I)_Y$ is trivial. Then, from these trivial stabilizers, the Orbit-Stabilizer Theorem, the earlier observation that $24 \leq |PLR\text{-group}|$, and the consequence

$$24 \leq |PLR\text{-group}| \leq |C(T/I)| \leq |\text{orbit of } Y| \leq 24$$

on page 492, the authors of [Crans et al. 2009] simultaneously conclude that the PLR-group has 24 elements and is the centralizer of T/I.

A slight simplification of the aforementioned inequality would be an argument like the one in the present paper: observe that the PLR-group acts transitively on the 24 consonant triads because of the Cohn LR-sequence, recalled on page 487 of [Crans et al. 2009]; then $C(T/I)$ must act transitively as it contains the PLR-group, and then the Orbit-Stabilizer Theorem and the trivial stabilizers imply that $|C(T/I)|$ must be 24, so the PLR-group also has 24 elements. Also, instead of postponing the proof that the PLR-group has exactly 24 elements from Theorem 5.1 of [Crans et al. 2009] until the aforementioned inequality in Theorem 6.1, one could do a word-theoretic argument in Theorem 5.1 to see that the PLR-group has exactly 24 elements, similar to the present argument in Proposition 3.6.

Remark 3.11. For an explicit computation of the four hexatonic cycles as orbits of the PL-group, see [Oshita 2009], which was also an undergraduate research project with the second author of the present article. That preprint includes a sketch that $\langle P, L\rangle \cong \text{Sym}(3)$.

Remark 3.12 (Alternative derivation using the Sub Dual Group Theorem). Hexatonic Duality, Theorem 3.9, can also be proved using the Sub Dual Group Theorem of Fiore and Noll [2011, Theorem 3.1], *if one assumes already* the duality of the

k	kHex	kHk^{-1} = dual group to PL-group on kHex
$\mathrm{Id}_{\mathrm{Triads}}$	$\{E\flat, e\flat, B, b, G, g\}$	$H = \{T_0, T_4, T_8, I_1, I_5, I_9\}$
T_1	$\{E, e, C, c, A\flat, a\flat\}$	$\{T_0, T_4, T_8, I_3, I_7, I_{11}\}$
T_2	$\{F, f, C\sharp, c\sharp, A, a\}$	$\{T_0, T_4, T_8, I_5, I_9, I_1\}$
T_3	$\{F\sharp, f\sharp, D, d, B\flat, b\flat\}$	$\{T_0, T_4, T_8, I_7, I_{11}, I_3\}$

Table 2. The four hexatonic cycles as PL-orbits and the respective dual groups determined as conjugations of H via the Sub Dual Group Theorem of Fiore and Noll.

T/I-group and PLR-group (maximal smoothness is not discussed in that paper). In Section 3.1 of the same paper, they apply the Sub Dual Group Theorem to the construction of dual groups on the hexatonic cycles. The method is to select G_0 to be the PL-group, select $s_0 = E\flat$, and compute $S_0 := G_0 s_0 = \text{Hex}$, and then the dual group will consist of the restriction of those elements of the T/I-group that map $E\flat$ into S_0.

Notice that in the present paper, on the other hand, we *first* determined which transpositions and inversions preserve Hex in Proposition 3.4, and then proved duality, whereas the application of the Sub Dual Group Theorem of Fiore and Noll starts with the PL-group and determines from it the dual group as (the restrictions of) those elements of the T/I-group that map $E\flat$ into S_0. Notice also, in the present paper we determined that the PL-group and its dual H are dihedral of order 6, but that the Sub Dual Group Theorem of Fiore and Noll does not specify which group structure is present. In any case, Clampitt [1998] explicitly wrote down all 6 elements of each group in permutation cycle notation.

The present paper is complementary to the work [Fiore and Noll 2011] in that we work very closely with the specific details of the groups and sets involved to determine one pair of dual groups in an illustrative way, rather than appealing to a computationally and conceptually convenient theorem. Fiore and Noll, however, also use their Corollary 3.3 to compute the other hexatonic duals via conjugation, as summarized in Table 2.

The application of the Sub Dual Group Theorem to construct dual groups on octatonic systems is also treated in [Fiore and Noll 2011], and utilized in [Fiore et al. 2013b].

Remark 3.13 (Other sources on group actions). Music-theoretical group actions on chords have been considered by many, many authors over the past century. In addition to the selected references of Babbitt, Forte, and Morris above, we also mention the expansive and influential work of Mazzola [1985; 1990; 2002] and numerous collaborators. Moreover, issue 42:2 of the *Journal of Music Theory* from

1998 is illuminating obligatory reading on groups in neo-Riemannian theory. That issue contains Clampitt's article [1998], which is the inspiration for the present paper. Clough's article [1998] in that issue illustrates the dihedral group of order 6 and its recombinations with certain centralizer elements in terms of two concentric equilateral triangles (but it does not treat hexatonic systems and duality). The dihedral group of order 6 is a warm-up for his treatment of recombinations of the *Schritt-Wechsel* group with the T/I-group, which are both dihedral of order 24. Peck's article [2010] studies centralizers where the requirement of simple transitivity is relaxed in various ways, covering many examples from music theory. Peck determines the structure of centralizers in several cases.

Remark 3.14 (Discussion of local diatonic containment of hexatonic cycles). No hexatonic cycle is contained entirely in a single diatonic set, as one can see from any of the cycles (3)–(6). However, one can consider a sequence of diatonic sets that changes along with the hexatonic cycle and contains each respective triad, as in [Douthett 2008, Table 4.7]. After transposing and reversing Douthett's table, we see a sequence of diatonic sets such that each diatonic set contains the respective triad of (3).

triad	$E\flat$	$e\flat$	B	b	G	g
in scale	$E\flat$-major	$D\flat$-major	B-major	A-major	G-major	F-major

This sequence of diatonic sets (indicated via major scales) descends by a whole step each time, so is as evenly distributed as possible.

Other diatonic set sequences also contain the hexatonic cycle, though unfortunately there is no maximally smooth cycle of diatonic sets that does the job (recall that the diatonic sets can only form a cycle of length 12). But it is possible to have a maximally smooth sequence of diatonic sets that covers four hexatonic triads. We list all possible diatonic sets containing the respective hexatonic chords.[2]

triad	$E\flat$	$e\flat$	B	b	G	g
in major scales	$E\flat$	$G\flat$	B	D	G	$B\flat$
	$B\flat$	$\boldsymbol{D\flat}$	$F\sharp$	$\boldsymbol{A}$	D	$\boldsymbol{F}$
	$\boldsymbol{A\flat}$	B	$\boldsymbol{E}$	G	$\boldsymbol{C}$	$E\flat$

[2] Recall that major chords only occur with roots on major scale degrees 1, 4, and 5, so we determine in the table the scales containing a given major triad by considering the root, a perfect fourth below the root, and a perfect fifth below the root. Minor chords can only occur with roots on major scale degrees 2, 3, and 6, so we determine in the table the scales containing a given minor triad by considering a major sixth below the root, a whole step below the root, and a major third below the root. This inconsistent major/minor ordering allows us to see (at vertical dividing lines) all three maximally smooth transitions from diatonic sets containing a given a minor triad to a diatonic set containing its subsequent major in a hexatonic cycle.

Vertical dividing lines indicate maximally smooth transitions between consecutive diatonic sets. As indicated by these dividing lines, the transition from a minor triad to its subsequent major in a hexatonic cycle via L is contained in three maximally smooth transitions of diatonic sets. On the other hand, the transition from a major triad to its subsequent minor in a hexatonic cycle via P is contained in only one maximally smooth transition of diatonic sets, as indicated by the bold letters. Altogether, we can trace three maximally smooth chains of four major scales that contain part of the hexatonic cycle (3):

$$B - E - A - D,$$
$$G - C - F - B\flat,$$
$$E\flat - A\flat - D\flat - F\sharp.$$

Local containment of hexatonic cycles in diatonic chains has ramifications for music analysis. Jason Yust [2013; 2015] proposed to include diatonic contexts into analyses involving PL-cycles or PR-cycles, and he provides analytical tools to do so.

4. Conclusion

We began this article with Cohn's proposal that the maximal smoothness of consonant triads is a key factor for their privileged status in late-nineteenth century music. Indeed, consonant triads and their complements are the only tone collections that accommodate short maximally smooth cycles. The four maximally smooth cycles of consonant triads, the so-called hexatonic cycles of Cohn, can be described transformationally as alternating applications of the neo-Riemannian "parallel" and "leading tone exchange" transformations. Cohn interpreted Wagner's Grail motive in terms of a cyclic group action on the hexatonic cycle Hex, whereas Clampitt used the PL-group and the transposition-inversion subgroup we called H in Proposition 3.4. In the present article, we proved the Lewinian duality between these latter two groups, which was discussed by Clampitt [1998].

For perspective, we mention that simply transitive group actions correspond to the *generalized interval systems* of Lewin; see the very influential original source [Lewin 1987], or see [Fiore et al. 2013b, Section 2] for an explanation of some aspects. Dual groups correspond to dual generalized interval systems: the transpositions of one system are the interval-preserving bijections of the other. Clampitt [1998] explained the coherent perceptual basis of the three generalized interval systems associated to the three group actions on Hex by Cohn's cyclic group, the PL-group, and the H group. He employed the coherence of generalized interval systems to incorporate the final $D\flat$ of the Grail motive into his interpretation via a conjugation-modulation of a subsystem.

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