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On computable classes of equidistant sets: finite focal sets

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The equidistant set of two nonempty subsets K and L in the Euclidean plane is the set of all points that have the same distance from K and L. Since the classical conics can be also given in this way, equidistant sets can be considered as one of their generalizations: K and L are called the focal sets. The points of an equidistant set are difficult to determine in general because there are no simple formulas to compute the distance between a point and a set. As a simplification of the general problem, we are going to investigate equidistant sets with finite focal sets. The main result is the characterization of the equidistant points in terms of computable constants and parametrization. The process is presented by a Maple algorithm. Its motivation is a kind of continuity property of equidistant sets. Therefore we can approximate the equidistant points of K and L with the equidistant points of finite subsets K_n and L_n. Such an approximation can be applied to the computer simulation, as some examples show in the last section.

1. Introduction: notation and preliminaries

Let $K \subset \mathbb{R}^2$ be a subset in the Euclidean coordinate plane. The distance between a point (x, y) and K is measured by the usual infimum formula:

$$d((x, y), K) := \inf\{d((x, y), (a, b)) \mid (a, b) \in K\}.$$

Let us define the equidistant set of K and $L \subset \mathbb{R}^2$ as the set of all points that have the same distance from K and L:

$$\{K = L\} := \{(x, y) \in \mathbb{R}^2 \mid d((x, y), K) = d((x, y), L)\}.$$

The equidistant sets can be considered as a kind of generalization of conics [Ponce and Santibáñez 2014]: K and L are called the focal sets. Equidistant sets are often called midsets. Their investigations were started by Wilker [1975] and Loveland

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[1976]. For another generalization of the classical conics and their applications, see, e.g., [Erdős and Vincze 1958; Melzak and Forsyth 1977] for polyellipses and their applications, and [Gross and Strempel 1998; Nagy and Vincze 2010; Vincze and Nagy 2011; 2012]. "We find equidistant sets as conventionally defined frontiers in territorial domain controversies: for instance, the United Nations Convention on the Law of the Sea (Article 15) establishes that, in absence of any previous agreement, the delimitation of the territorial sea between countries occurs exactly on the median line every point of which is equidistant of the nearest points to each country"; for the citation, see [Ponce and Santibáñez 2014].

Let $R > 0$ be a positive real number. The *parallel body* of a set $K \subset \mathbb{R}^2$ with radius R is the union of the closed disks with radius R centered at the points of K. The infimum of the positive numbers such that L is a subset of the parallel body of K with radius R and vice versa is called the *Hausdorff distance* of K and L. It is well known that the Hausdorff metric makes the family of nonempty closed and bounded (i.e., compact) subsets in the plane a complete metric space; for the details, see, e.g., [Lay 1982; Vincze 2013]. In what follows we are going to characterize the equidistant points of finite focal sets in terms of computable constants and parametrization. The process will be presented by a Maple algorithm. Its motivation is the continuity property of equidistant sets in the sense of the following theorem.

Theorem 1 [Ponce and Santibáñez 2014, Theorem 11]. *If K and L are disjoint compact subsets in the plane, and $K_n \to K$ and $L_n \to L$ are convergent sequences of nonempty compact subsets with respect to the Hausdorff metric then for any $R > 0$ we have*

$$\{K_n = L_n\} \cap \bar{D}(R) \to \{K = L\} \cap \bar{D}(R),$$

where $\bar{D}(R)$ denotes the closed disk with radius R centered at the origin.

Since any compact subset can be approximated by finite subsets with respect to the Hausdorff metric, we can approximate the equidistant points of K and L with the equidistant points of finite subsets K_n and L_n. Such an approximation can be applied to the computer simulation as an alternative to the error estimation process for quasiequidistant points suggested by [Ponce and Santibáñez 2014, §4.2].

2. The main result

Let $K, L \subset \mathbb{R}^2$ be nonempty finite disjoint subsets in the Euclidean coordinate plane:

$$K := \{(a_i, b_i) \mid i = 1, \ldots, p\} \quad \text{and} \quad L := \{(c_k, d_k) \mid k = 1, \ldots, q\},$$

where p and q are positive integers. Since we have only finitely many lines determined by the points of $K \cup L$ we can use the following technical condition without loss of generality:

(H) Each line determined by the points of $K \cup L$ has a slope different from zero; i.e., there are no horizontal "focal lines".

Indeed, an infinitesimal rotation about the origin provides the configuration we need to satisfy condition (H). On the other hand, the inverse rotation takes the equidistant points of the rotated sets into the equidistant points of the original ones. Let

$$K_i := \{(x, y) \in \mathbb{R}^2 \mid d((x, y), K) = d((x, y), (a_i, b_i))\} \quad (i = 1, \dots, p).$$

It is clear that

$$K_i = \bigcap_{\substack{j=1,\dots,m \\ j \neq i}} F_{ij},$$

where the closed half-planes F_{ij} $(i \neq j)$ are determined by the perpendicular bisector

$$(a_i - a_j)x + (b_i - b_j)y = \frac{a_i^2 - a_j^2}{2} + \frac{b_i^2 - b_j^2}{2} \tag{1}$$

of the segment (a_i, b_i) and (a_j, b_j) such that $(a_i, b_i) \in F_{ij}$. For any index i, the set K_i is closed and convex as the intersection of finitely many closed half-planes. It is nonempty because $(a_i, b_i) \in K_i$. Since $K_i \cap K_j$ $(i \neq j)$ is a subset of the perpendicular bisector (1) of the corresponding focal points in K, we can conclude that $\operatorname{int} K_i \cap \operatorname{int} K_j = \varnothing$ for any $i \neq j$. Finally,

$$\mathbb{R}^2 = \bigcup_{i=1}^{p} K_i;$$

i.e., we have a partitioning of the plane into (nonempty, closed and convex) regions with pairwise disjoint interiors based on the distance to points in a specific subset. It is called the *Voronoi decomposition*.

Exercise 1. Prove that K_i is bounded if and only if (a_i, b_i) is in the interior of the convex hull of K.

The Voronoi decomposition of the plane with respect to the points of K means that the plane is divided into (nonempty, closed and convex) regions with pairwise disjoint interiors such that the distance of $(x, y) \in K_i$ to the focal set K can be measured as the distance of $(x, y) \in K_i$ to $(a_i, b_i) \in K$. In terms of inequalities,

$$K_i : (a_i - a_j)x + (b_i - b_j)y \geq \frac{a_i^2 - a_j^2}{2} + \frac{b_i^2 - b_j^2}{2}, \tag{2}$$

where j runs from 1 to p but $i \neq j$. Using condition (H) we can reformulate the system of inequalities as

$$\begin{aligned} K_i : y &\geq \alpha_{ij}x + \beta_{ij} \quad (b_i - b_j > 0), \\ y &\leq \alpha_{ij}x + \beta_{ij} \quad (b_i - b_j < 0), \end{aligned} \tag{3}$$

where

$$\alpha_{ij} = -\frac{a_i - a_j}{b_i - b_j}, \quad \beta_{ij} = \frac{1}{b_i - b_j}\left(\frac{a_i^2 - a_j^2}{2} + \frac{b_i^2 - b_j^2}{2}\right) \quad \text{and} \quad i \neq j.$$

In a similar way consider the Voronoi decomposition of the plane with respect to the points of L:

$$L_k := \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; d((x, y), L) = d((x, y), (c_k, d_k))\right\} \quad (k = 1, \dots, q),$$

$$L_k = \bigcap_{\substack{l=1,\dots,q \\ l \neq k}} F_{kl},$$

where the closed half-planes F_{kl} $(k \neq l)$ are determined by the perpendicular bisector

$$(c_k - c_l)x + (d_k - d_l)y = \frac{c_k^2 - c_l^2}{2} + \frac{d_k^2 - d_l^2}{2} \tag{4}$$

of the segment (c_k, d_k) and (c_l, d_l) such that $(c_k, d_k) \in F_{kl}$,

$$\mathbb{R}^2 = \bigcup_{k=1}^{q} L_k.$$

In terms of inequalities,

$$L_k : (c_k - c_l)x + (d_k - d_l)y \geq \frac{c_k^2 - c_l^2}{2} + \frac{d_k^2 - d_l^2}{2}, \tag{5}$$

where l runs from 1 to q but $k \neq l$. Using condition (H) we can reformulate the system of inequalities as

$$\begin{aligned} L_k : y &\geq \gamma_{kl}x + \delta_{kl} \quad (d_k - d_l > 0), \\ y &\leq \gamma_{kl}x + \delta_{kl} \quad (d_k - d_l < 0), \end{aligned} \tag{6}$$

where

$$\gamma_{kl} = -\frac{c_k - c_l}{d_k - d_l}, \quad \delta_{kl} = \frac{1}{d_k - d_l}\left(\frac{c_k^2 - c_l^2}{2} + \frac{d_k^2 - d_l^2}{2}\right) \quad \text{and} \quad k \neq l.$$

Lemma 1. *The set of equidistant points is equal to the union of*

$$\bigcup_{i=1}^{p}\bigcup_{k=1}^{q}(K_i \cap L_k \cap l_{ik}),$$

where

$$l_{ik} : (a_i - c_k)x + (b_i - d_k)y = \frac{a_i^2 - c_k^2}{2} + \frac{b_i^2 - d_k^2}{2} \tag{7}$$

is the perpendicular bisector of (a_i, b_i) *and* (c_k, d_k).

In what follows we characterize the sets of the form $K_i \cap L_k \cap l_{ik}$ in terms of a system of linear inequalities. According to condition (H), equation (7) of the perpendicular bisector l_{ik} can be written in the form

$$l_{ik} : y = \mu_{ik} x + \nu_{ik}, \tag{8}$$

where

$$\mu_{ik} := -\frac{a_i - c_k}{b_i - d_k} \quad \text{and} \quad \nu_{ik} = \frac{1}{b_i - d_k}\left(\frac{a_i^2 - c_k^2}{2} + \frac{b_i^2 - d_k^2}{2}\right). \tag{9}$$

This means by (3) and (6) that

$$\begin{aligned} K_i \cap L_k \cap l_{ik} : \mu_{ik} x + \nu_{ik} &\geq \alpha_{ij} x + \beta_{ij} \quad (b_i - b_j > 0), \\ \mu_{ik} x + \nu_{ik} &\leq \alpha_{ij} x + \beta_{ij} \quad (b_i - b_j < 0), \\ \mu_{ik} x + \nu_{ik} &\geq \gamma_{kl} x + \delta_{kl} \quad (d_k - d_l > 0), \\ \mu_{ik} x + \nu_{ik} &\leq \gamma_{kl} x + \delta_{kl} \quad (d_k - d_l < 0), \end{aligned} \tag{10}$$

where j runs from 1 to p but $j \neq i$ and l runs from 1 to q but $l \neq k$. It can be easily seen that the number of inequalities is $p + q - 2$ for any fixed pair of indices (i, k) and the equidistant set is the union of finitely many polygonal chains determined by inequalities of type (10). To reduce the number of possible cases, we formulate necessary and sufficient conditions for the solvability of system (10). Let us introduce the following set of indices:

$$\begin{aligned} P_{ik}^{+} &:= \{j \mid (b_i - b_j)(\mu_{ik} - \alpha_{ij}) > 0\}, \\ P_{ik}^{-} &:= \{j \mid (b_i - b_j)(\mu_{ik} - \alpha_{ij}) < 0\}, \\ P_{ik}^{0+} &:= \{j \mid b_i - b_j > 0 \text{ and } \mu_{ik} - \alpha_{ij} = 0\}, \\ P_{ik}^{0-} &:= \{j \mid b_i - b_j < 0 \text{ and } \mu_{ik} - \alpha_{ij} = 0\}, \\ Q_{ik}^{+} &:= \{l \mid (d_k - d_l)(\mu_{ik} - \gamma_{kl}) > 0\}, \\ Q_{ik}^{-} &:= \{l \mid (d_k - d_l)(\mu_{ik} - \gamma_{kl}) < 0\}, \\ Q_{ik}^{0+} &:= \{l \mid d_k - d_l > 0 \text{ and } \mu_{ik} - \gamma_{kl} = 0\}, \\ Q_{ik}^{0-} &:= \{l \mid d_k - d_l < 0 \text{ and } \mu_{ik} - \gamma_{kl} = 0\}. \end{aligned} \tag{11}$$

Then we have that $(x, y) \in K_i \cap L_k \cap l_{ik}$ if and only if the following conditions are satisfied:

$$x \geq \frac{\beta_{ij} - \nu_{ik}}{\mu_{ik} - \alpha_{ij}} \quad (j \in P_{ik}^{+}) \qquad \text{and} \qquad x \geq \frac{\delta_{kl} - \nu_{ik}}{\mu_{ik} - \gamma_{kl}} \quad (l \in Q_{ik}^{+}), \tag{12}$$

$$x \leq \frac{\beta_{ij} - \nu_{ik}}{\mu_{ik} - \alpha_{ij}} \quad (j \in P_{ik}^{-}) \qquad \text{and} \qquad x \leq \frac{\delta_{kl} - \nu_{ik}}{\mu_{ik} - \gamma_{kl}} \quad (l \in Q_{ik}^{-}), \tag{13}$$

$$\nu_{ik} \geq \beta_{ij} \qquad (j \in P_{ik}^{0+}) \qquad \text{and} \qquad \nu_{ik} \geq \delta_{kl} \qquad (l \in Q_{ik}^{0+}), \tag{14}$$

$$\nu_{ik} \leq \beta_{ij} \qquad (j \in P_{ik}^{0-}) \qquad \text{and} \qquad \nu_{ik} \leq \delta_{kl} \qquad (l \in Q_{ik}^{0-}). \tag{15}$$

Therefore we can formulate the sufficient and necessary conditions in terms of the following constants:

$$m_{ik}^K := \begin{cases} -\infty & \text{if } P_{ik}^+ = \varnothing, \\ \sup_{j \in P_{ik}^+} (\beta_{ij} - \nu_{ik})/(\mu_{ik} - \alpha_{ij}) & \text{otherwise,} \end{cases} \tag{16}$$

$$m_{ik}^L := \begin{cases} -\infty & \text{if } Q_{ik}^+ = \varnothing, \\ \sup_{l \in Q_{ik}^+} (\delta_{kl} - \nu_{ik})/(\mu_{ik} - \gamma_{kl}) & \text{otherwise,} \end{cases} \tag{17}$$

$$M_{ik}^K := \begin{cases} \infty & \text{if } P_{ik}^- = \varnothing, \\ \inf_{j \in P_{ik}^-} (\beta_{ij} - \nu_{ik})/(\mu_{ik} - \alpha_{ij}) & \text{otherwise,} \end{cases} \tag{18}$$

$$M_{ik}^L := \begin{cases} \infty & \text{if } Q_{ik}^- = \varnothing, \\ \inf_{l \in Q_{ik}^-} (\delta_{kl} - \nu_{ik})/(\mu_{ik} - \gamma_{kl}) & \text{otherwise,} \end{cases} \tag{19}$$

$$r_{ik}^K := \begin{cases} -\infty & \text{if } P_{ik}^{0+} = \varnothing, \\ \sup_{j \in P_{ik}^{0+}} \beta_{ij} & \text{otherwise,} \end{cases} \tag{20}$$

$$r_{ik}^L := \begin{cases} -\infty & \text{if } Q_{ik}^{0+} = \varnothing, \\ \sup_{l \in Q_{ik}^{0+}} \delta_{kl} & \text{otherwise,} \end{cases} \tag{21}$$

$$R_{ik}^K := \begin{cases} \infty & \text{if } P_{ik}^{0-} = \varnothing, \\ \inf_{j \in P_{ik}^{0-}} \beta_{ij} & \text{otherwise,} \end{cases} \tag{22}$$

$$R_{ik}^L := \begin{cases} \infty & \text{if } Q_{ik}^{0-} = \varnothing, \\ \inf_{l \in Q_{ik}^{0-}} \delta_{kl} & \text{otherwise,} \end{cases} \tag{23}$$

$$m_{ik} := \sup\{m_{ik}^K, m_{ik}^L\}, \qquad M_{ik} := \inf\{M_{ik}^K, M_{ik}^L\}, \tag{24}$$

$$r_{ik} := \sup\{r_{ik}^K, r_{ik}^L\}, \qquad R_{ik} := \inf\{R_{ik}^K, R_{ik}^L\}. \tag{25}$$

Theorem 2. *If K and L are disjoint finite subsets satisfying condition (H) then for any pair (i,k) of indices, $K_i \cap L_k$ contains equidistant points if and only if*

$$m_{ik} \leq M_{ik} \quad \textit{and} \quad r_{ik} \leq \nu_{ik} \leq R_{ik}.$$

The parametrization of the line segment of the equidistant points in $K_i \cap L_k$ is

$$y = \mu_{ik} x + \nu_{ik} \quad (m_{ik} \leq x \leq M_{ik}).$$

Proof. It is clear that in the case where

$$m_{ik} \leq M_{ik} \quad \text{and} \quad r_{ik} \leq \nu_{ik} \leq R_{ik},$$

conditions (12)–(15) are satisfied for any $m_{ik} \leq x \leq M_{ik}$. □

3. A Maple algorithm

Our algorithm, available in the online supplement, is implemented in Maple. The input data are the lists of K and L containing the points of the focal sets, respectively. $K[i][1]$ and $K[i][2]$ denote the coordinates of the i-th point in the focal set K for each $i \in \{1, 2, \dots, p\}$, and $L[k][1]$ and $L[k][2]$ denote the coordinates of the k-th point in the focal set L for each $k \in \{1, 2, \dots, q\}$. The main procedure `equidistant` creates a plot of the equidistant set with focal sets K and L. The Maple command `LinearUnivariateSystem` produces the solution of system (10).

The procedure `equidistant_ini` takes the lists K and L as input and returns the following six objects as output:

- a is a list containing the slopes $a[i][j] := \alpha_{ij}$ for each $i, j \in \{1, 2, \dots, p\}$ and $i \neq j$, while $a[i][j] = 0$ when $i = j$.
- g is a list containing the slopes $g[k][l] := \gamma_{kl}$ for each $k, l \in \{1, 2, \dots, q\}$ and $k \neq l$, while $g[k][l] = 0$ when $k = l$.
- b is a list containing the constants $b[i][j] := \beta_{ij}$ for each $i, j \in \{1, 2, \dots, p\}$ and $i \neq j$, while $b[i][j] = 0$ when $i = j$.
- d is a list containing the constants $d[k][l] := \delta_{kl}$ for each $k, l \in \{1, 2, \dots, q\}$ and $k \neq l$, while $d[k][l] = 0$ when $k = l$.
- m is a list containing the slopes $m[i][k] := \mu_{ik}$ for each $i \in \{1, 2, \dots, p\}$ and $k \in \{1, 2, \dots, q\}$.
- n is a list containing the constants $n[i][k] := \nu_{ik}$ for each $i \in \{1, 2, \dots, p\}$ and $k \in \{1, 2, \dots, q\}$.

The procedure `xmaxmin` returns the maximal and the minimal values of the first coordinates of the points in the focal sets K and L, respectively. They appear in the range option of the "plot" command.

The procedure `equidistant_system` creates the system of inequalities (10) for each $i \in \{1, 2, \dots, p\}$ and $k \in \{1, 2, \dots, q\}$. In the input data, a, g, b, d, m, n are the objects created by `equidistant_ini`.

The procedure `inequality_range` defines the ranges for the next procedure, `equidistant_grafikon`. The equidistant set can be considered as the graph of a piecewise linear, continuous real function. The domain of such a function can be split into a finite number of disjoint intervals such that the function is linear over each interval. The procedure defines the endpoints of such intervals. The operands of the local variable T containing the solution of a linear univariate system of inequalities are of the forms $c_1 < x$ and $x < c_2$. If T has at least (and, consequently, exactly) two operands of the forms $c_1 < x$ and $x < c_2$, respectively, then we have both lower and upper bounds for the solution. Otherwise we have only a lower or

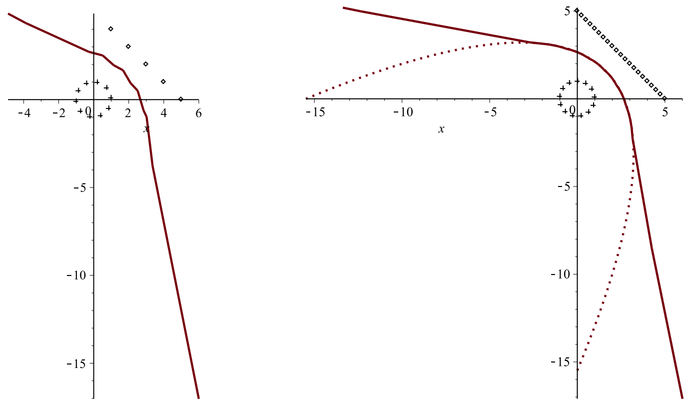

Figure 1. Examples 1 (left) and 2 (right).

an upper bound of the form $c_1 < x$ or $x < c_2$. In the case of $c_1 < x$, we choose the variable of numeric type as the lower bound for the range: c_1. The upper bound for the range is defined as the maximum of $c_1 + 1$ and xmax $+ 1$.

The procedure `equidistant_ grafikon` generates the list of plots of the graph of the linear functions, which represent the equidistant set with focal sets K and L. In the input data, m, n are created by `equidistant_ini` and S is a list containing the list of ranges created by `inequality_range`.

3.1. ***Examples.*** We present some examples generated by the algorithm above. The code for Examples 1, 2, and 4 can be found in the online supplement.

Example 1. The focal set K contains the points of a regular 10-gon inscribed in the unit circle; it is rotated by a small angle 0.1 to satisfy condition (H). The focal set L contains the points $(1,4)$, $(2,3)$, $(3,2)$, $(4,1)$ and $(5,0)$. They are lying on the same line segment $y = -x + 5$ $(0 \leq x \leq 5)$. See Figure 1.

Example 2. This case, shown in Figure 1, illustrates what happens when increasing the number of the focal points in Example 1.

The limit shape is a parabolic arc,

$$r(\varphi) = \frac{1 + 5/\sqrt{2}}{1 + \cos\left(\varphi - \frac{1}{4}\pi\right)}, \tag{26}$$

provided that the polar angle belongs to the interval

$$-\arcsin\frac{12\sqrt{2}}{26+5\sqrt{2}} \leq \varphi \leq \frac{\pi}{2} + \arcsin\frac{12\sqrt{2}}{26+5\sqrt{2}}$$

because the line segment can be substituted by the entire line without changing the equidistancy in this region. Otherwise we have hyperbolic arcs because the distance to the line segment reduces to the distance from one of its endpoints.

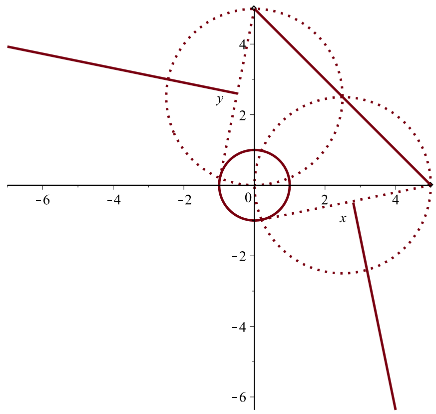

Figure 2. The asymptotic ends.

The asymptotic "ends", shown in Figure 2, are the bisectors of the points $P(0, 5)$, T_2 and $Q(5, 0)$, T_4, where T_2, T_4 denote the touching points of the tangent lines passing through P and Q in the second and the fourth quadrants, respectively. For the asymptotic behavior of the equidistant sets, see [Ponce and Santibáñez 2014, Theorem 12].

Example 3. The focal set K contains the points $(-2, 1)$, $(-1, 1.3)$, $(0, 0)$, $(1, -2)$, $(2, -2.2)$, $(3, 1.5)$ and $(4, 3.5)$ and the focal set L consists of $(1, 2)$, $(-1, 3)$, $(2, 4)$ and $(3, 5)$; see Figure 3.

Example 4. The focal set K contains the points of a regular 7-gon inscribed in the circle of radius $\frac{1}{3}$ centered at the origin; it is also rotated by a small angle 0.1 to

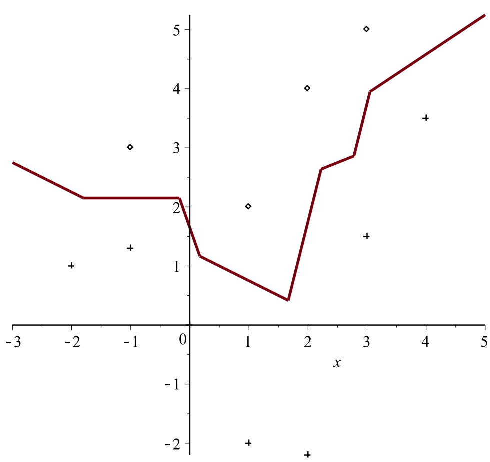

Figure 3. Example 3.

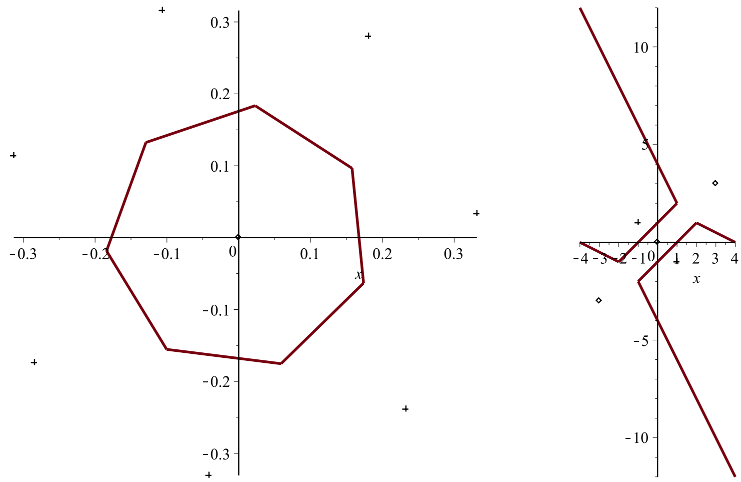

Figure 4. Examples 4 (left) and 5 (right).

satisfy condition (H). The focal set L is a singleton containing the origin. In the same way any regular n-gon can be given as an equidistant set. See Figure 4.

Example 5. In this case, shown in Figure 4, we can see a disconnected case with focal sets K containing the points $(-1, 1)$, $(1, -1)$ and L containing the points $(-3, -3)$, $(0, 0)$, $(3, 3)$, respectively.

3.2. ***Concluding remarks.*** The application of the algorithm for more complicated focal sets is based on the continuity properties of the equidistant sets; see Section 1 and also [Ponce and Santibáñez 2014, Theorem 11].

Examples 1 and 2 represent the approximation of the equidistant set to a circle and a segment as focal sets. The Hausdorff distance can be estimated by comparing the polar distance of the equidistant points and the points of the limit shape as follows; see (26). First, get the polar coordinates of the vertex points of the approximating equidistant set. Since it is a polygonal chain, we have finitely many data depending on the number of the focal points: (r_i, φ_i), where $i < \infty$. By (26) we can compute the exact polar distance $r(\varphi_i)$ belonging to the polar angle φ_i on the limit parabola. Taking

$$D_1 := \max_i |r(\varphi_i) - r_i|,$$

we have an upper bound for the Hausdorff distance between the approximating equidistant set and the polygonal chain inscribed in the limit parabola with vertices of the polar angles φ_i. Indeed, if the polar body of a segment contains the endpoints of another one then it contains the entire line segment too. To estimate the Hausdorff distance of the inscribed polygonal chain and the parabolic arc, it is natural to consider the triangles $\triangle_i$ formed by adjacent vertices V_i and V_{i+1} of the polygonal chain and the intersection of the tangent lines to the arc at V_i and V_{i+1}. If m_i denotes the height of the $\triangle_i$ belonging to the i-th side of the polygonal chain then its maximum D_2 gives an upper bound for the Hausdorff distance between the inscribed polygonal chain and the parabolic arc. Using the triangle inequality, the sum $D := D_1 + D_2$ is an upper bound for the Hausdorff distance between the estimating polygonal chain and the limit parabola.

In the same way, we can approximate the equidistant set to a pair of convex polytopes (the convex hulls of finite sets of points). Taking finitely many convex combinations of the vertices, one can produce finite focal sets to apply the algorithm. In case of general compact subsets we can use their intersections with a sequence of nested grids.

As the limit shape we have a circle by increasing the number of the vertices of the inscribed regular polygon (the focal set K) in Example 4. On the other hand, Example 4 shows a way of presenting regular polygons as equidistant sets. It has an important theoretical consequence in view of Weiszfeld's problem. E. Vázsonyi, also known as E. Weiszfeld, posed the problem of approximating convex plane curves with so-called polyellipses, all of whose points have the same sum of distances from finitely many focal points in the plane. It is the additive version of the approximation of plane curves by polynomial lemniscates, all of whose points have the same product of distances from finitely many focal points in the plane. P. Erdős and I. Vincze [1958] proved that the approximation of a regular triangle with polyellipses has an absolute error even if the number of focuses is increased to the infinity; see also [Varga and Vincze 2008]. This means that the idea of polyellipses gives an essentially different generalization of the classical conics.

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