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Zero divisor graphs of commutative graded rings

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We study a natural generalization of the zero divisor graph introduced by Anderson and Livingston to commutative rings graded by abelian groups, considering only homogeneous zero divisors. We develop a basic theory for graded zero divisor graphs and present many examples. Finally, we examine classes of graphs that are realizable as graded zero divisor graphs and close with some open questions.

1. Introduction

Zero divisor graphs of commutative rings have been well-studied since their introduction by Beck [1988], and there have also been many generalizations, from noncommutative rings to semigroups. Anderson and Livingston [1999] began studying the graph created from just the nonzero zero divisors. We focus on a generalization of their graph to graded rings. In this way we are able to realize significantly more graphs as graded zero divisor graphs. While the class of realizable graphs is expanded, some of the same restrictions still exist in the graded case. For other types of graphs associated to graded rings, see [Khosh-Ahang and Nazari-Moghadam 2016]. For examples of other graphs associated to commutative rings, see [Anderson and Badawi 2012; Ashrafi et al. 2010; Badawi 2014; 2015; Behboodi and Rakeei 2011]. For more examples in the commutative case and characterizations based on numbers of zero divisors, among other things, see [Anderson and Badawi 2008].

In Section 2 we summarize the basic notation, terminology, and necessary facts for graded rings. We also define the graded zero divisor graph and give some basic examples.

Section 3 contains the basic properties and theory of graded zero divisor graphs. As mentioned, many of the familiar properties from the nongraded case hold true in the graded case: the graded zero divisor graph is connected with diameter less than or equal to 3, the girth is less than or equal to 4 (when finite), and the graph is finite if and only if the ring is finite.

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Keywords: graded ring, zero divisor graph.

The final section is devoted to realizability of various graphs and classes of graphs. We show that all but one of the connected graphs on four vertices are realizable as graded zero divisor graphs, and we completely classify the connected graphs on five vertices. Further, we show that every star, complete, and complete bipartite graph is realizable, a marked difference from the nongraded case. We also include some interesting open questions.

Throughout the paper, all rings are assumed to be commutative with identity, and G will always represent an abelian group.

2. Preliminaries

We now summarize some basic language and notation relating to rings graded by abelian groups as well as zero divisor graphs associated with such rings. For more details on graded commutative rings, the reader is referred to [Johnson 2012]. For a more general treatment, see [Năstăsescu and Van Oystaeyen 2004].

Graded rings. Let G be an abelian group. A *G-graded ring* R is a ring R with a family of subgroups $\{R_g \mid g \in G\}$ of R such that $R = \bigoplus_{g\in G} R_g$ (as abelian groups) and $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$. At times, we may refer simply to the "graded ring R" if G is understood. If $r \in R$ then there exist unique elements $r_g \in R_g$ for each $g \in G$, all but finitely many of which are zero, such that $r = \sum_{g\in G} r_g$. If $r = r_g$ for some $g \in G$ then r is called *G-homogeneous of degree g* (or simply "homogeneous"). An ideal $I \subseteq R$ is G-homogeneous (again, "homogeneous" when appropriate) provided $I = \bigoplus_{g\in G} I_g$ for some family of subgroups $\{I_g \mid g \in G\}$. Equivalently, we only need know that I has a generating set consisting of homogeneous elements.

When defining some basic ring-theoretic properties in terms of only homogeneous elements, we incorporate the grading group to simplify language and avoid confusion. For example, a G-graded ring R is called a *G-field* (respectively, *G-domain*) if every nonzero G-homogeneous element of R is a unit (respectively, not a zero divisor). Note that when we refer to a property holding under the trivial grading, or 0-grading, we will not write "R is a 0-field," but rather "R is a field."

The following lemma is interesting on its own. It says that to decompose a graded ring as a (graded) direct product, it is enough to write the ring as a direct product of subrings. We use it later in our analysis of realizable graphs.

Lemma 2.1. *Suppose R is a G-graded ring, and $R = S \times T$ for subrings S and T of R. Then S and T are G-graded subrings of R, and R is the (graded) direct product of S and T.*

Proof. As above, suppose $R = S \times T$. Define $S_g := \{s \in S \mid (s, 0) \in R_g\}$ and $T_g := \{t \in T \mid (0, t) \in R_g\}$. This defines a G-grading on S and T, and so it only remains to be shown that R is their graded direct product.

By Remark 1.2.3 in [Năstăsescu and Van Oystaeyen 2004], we are done. □

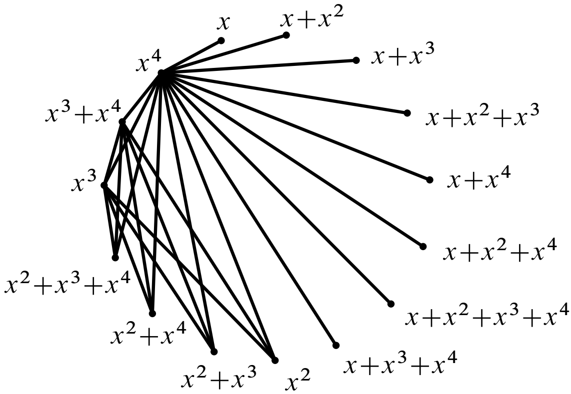


Figure 1. $\Gamma(R)$.

Zero divisor graphs. Let R be a G-graded ring, and let $Z_G^*(R)$ denote the collection of nonzero G-homogeneous zero divisors. Define the *G-graded zero divisor graph* (or just the "graded zero divisor graph" if G is understood) $\Gamma_G(R)$ to be the graph whose vertices are the elements of $Z_G^*(R)$ and which has an edge between distinct elements $x, y \in Z_G^*(R)$ provided $xy = 0$. It is worth mentioning that one could eliminate the restriction that x and y be distinct; the only change is that the graphs now might have loops. However, the graph theory becomes significantly more complicated. See [Vietri 2015] for examples of classifications involving loops.

As in the case of 0-fields, for example, when we consider a trivial grading, we use $Z(R)$, $Z^*(R)$, and $\Gamma(R)$ rather than include the subscript 0.

One interesting result of studying a graded version of zero divisor graphs is that the same ring may have different gradings, leading to distinct graphs from the same underlying ring.

Example 2.2. Let $R = \mathbb{Z}_2[X]/(X^5)$ and use x to denote the image of X in the quotient.

(1) Consider R under a trivial grading. That is, suppose the degree of every element is 0 (so G could be any abelian group, in fact). Since all elements of R are homogeneous, this is the same as the usual zero divisor graph $\Gamma(R)$, as shown in Figure 1.

(2) Now consider R as a $\mathbb{Z}_2$-graded ring under the assignment induced by $\deg(x) = 1$, so the degree of x^i is $i \pmod 2$. This restricts the number of homogeneous elements and homogeneous zero divisors, as shown in Figure 2. For example, $x^2 + x^4$ is homogeneous, but $x^2 + x^3$ is not.

(3) Finally, consider R as a $\mathbb{Z}$-graded ring under the assignment induced by $\deg(x) = 1$, so the degree of x^i is i. This further restricts the number of homogeneous zero divisors, as seen in Figure 3. In fact, the only homogeneous zero divisors are elements of the form x^i, for $i = 1, 2, 3, 4$.

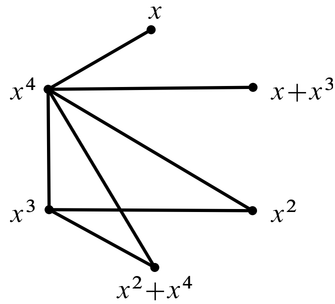


Figure 2. $\Gamma_{\mathbb{Z}_2}(R)$.

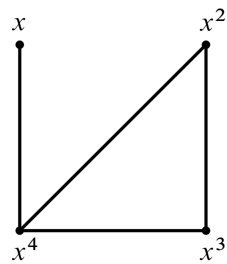


Figure 3. $\Gamma_{\mathbb{Z}}(R)$.

It is worth mentioning that the gradings on the first two rings can be induced from the third ring. In general, given a G-graded ring R and a subgroup H of G, there is a natural grading of R by the quotient G/H, obtained by setting $R_{g+H} = \bigoplus_{h\in H} R_{g+h}$. For instance, to obtain the $\mathbb{Z}_2$-grading of R from the $\mathbb{Z}$-grading, we take $G = \mathbb{Z}$ and $H = 2\mathbb{Z}$, whereas to obtain the trivial grading, we take $G = H = \mathbb{Z}$.

3. Basic properties

Many of the basic properties of $\Gamma(R)$ described by Anderson and Livingston [1999] have analogues for $\Gamma_G(R)$. For example, they show that the zero divisor graph is finite if and only if R is finite or a domain. With modifications we can use a similar proof, combined with the following lemma, to prove a corresponding result for graded rings.

Lemma 3.1. *If $Z_G(R)$ is finite, then for every $x \in Z^*_G(R)$, $\operatorname{ann}(x)$ is finite.*

Proof. Let $I = \operatorname{ann}(x)$. As x is homogeneous, I is homogeneous, and thus $I = \bigoplus_{g\in G} I_g$. Further, $I_g \subseteq Z_G(R)$ for every $g \in G$, so $I_g = 0$ for all but finitely many $g \in G$ and each nonzero I_g is finite. Since there are finitely many nonzero I_g, say $I_{g_1}, \dots, I_{g_k}$, we have $|I| = \left|\bigoplus_{i=1}^k I_{g_i}\right| = \prod_{i=1}^k |I_{g_i}|$. Therefore $|I| < \infty$. $\square$

Theorem 3.2. *Let R be a commutative ring. Then $|\Gamma_G(R)|$ is finite if and only if R is a G-domain or R is finite.*

Proof. Suppose R is not a G-domain and $|Z^*_G(R)|$ is finite. Then there exist nonzero homogeneous $x, y \in R$ with $xy = 0$. Let $I = \operatorname{ann}(x)$. By Lemma 3.1, I is finite. Also, $ry \in I$ for all $r \in R$. If R is infinite, then there exists $i \in I$ with $B = \{r \in R \mid ry = i\}$ infinite. For any $r, s \in B$, we have $(r - s)y = 0$, so $\operatorname{ann}(y)$ is infinite, contradicting Lemma 3.1. Thus R must be finite. □

Because there is no "graded" version of the ring being finite, we get an interesting corollary.

Corollary 3.3. *If $1 \le |Z^*_G(R)| < \infty$, then $1 \le |Z^*(R)| < \infty$.*

Proof. Suppose $1 \le |Z^*_G(R)| < \infty$. If $|Z^*(R)| = \infty$, then R is not finite. Therefore, R must be a G-domain, so $|Z^*_G(R)| = 0$, a contradiction. If $|Z^*_G(R)| \ge 1$, clearly $|Z^*(R)| \ge 1$. □

Note. The converse of Corollary 3.3 is also true for the upper bounds, but fails when the lower bound 1 is added, as the following example shows.

Example 3.4. Consider

$$R := \frac{\mathbb{Z}_3[X]}{(X^2 - 1)} = \mathbb{Z}_3 \oplus \mathbb{Z}_3 x,$$

where x is the image of X in the quotient ring. This has a natural grading by $\mathbb{Z}_2$, where $\deg(x^i) = i \pmod 2$. One easily verifies that this grading makes R a $\mathbb{Z}_2$-field. However, $(x + 1)(x - 1) = x^2 - 1 = 0$, so $|Z^*_G(R)| = 0$, yet $|Z^*(R)| \ge 1$.

Another obvious consequence of the finiteness result above is that we can assume a ring with a finite graded zero divisor graph is graded by a finitely generated group. Moreover, it can be shown that the grading group can be chosen to be finite. For example, if such a ring R is graded by $\mathbb{Z}$, say $R = \bigoplus_{n \in \mathbb{Z}} R_n$, we can form the quotient group $G = \mathbb{Z}/k\mathbb{Z}$, where $k = \max\{m - \ell \mid R_m \neq 0 \text{ and } R_\ell \neq 0\}$. This argument can be extended to any finitely generated group by applying it in each component of the free part of the grading group as necessary.

Other well known facts about zero divisor graphs concern connectedness, diameter, and girth. None of these theorems change in the graded setting.

Theorem 3.5. Let G be an abelian group and R a G-graded ring. Then $\Gamma_G(R)$ is connected and $\operatorname{diam}(\Gamma_G(R)) \le 3$.

Proof. The proof given in [Anderson and Livingston 1999, Theorem 2.3] can be used if one simply adds that each zero divisor chosen is homogeneous. □

Similarly, the following well-known result can be obtained by modifying the proof given by Axtell, Coykendall, and Stickles [Axtell et al. 2005], insisting that each choice of a zero divisor is homogeneous.

Theorem 3.6. *Suppose G is an abelian group and R is a G-graded ring. If $\Gamma_G(R)$ contains a cycle, then the girth of $\Gamma_G(R)$ is less than or equal to* 4.

Some of the previous facts can also be obtained by results on zero divisor graphs of semigroups found in [DeMeyer et al. 2002]. Indeed, the homogeneous elements (together with 0) in a ring are closed under the ring multiplication.

4. Realizability of Graphs

There has been ample study on which graphs are realizable as zero divisor graphs of commutative rings; for example, see [Axtell et al. 2009; LaGrange 2008; Redmond 2007]. Certainly, any graph realizable as $\Gamma(R)$ for a ring R is realizable as $\Gamma_G(R)$ for the same ring under a trivial grading (by any group G). It turns out that there are significantly more graphs realizable as graded zero divisor graphs. We begin with graphs on four vertices, but every connected graph on one, two, or three vertices is realizable as the (nongraded) zero divisor graph of a commutative ring. Therefore, there is nothing to show in the graded case for these.

Connected graphs on four vertices. Anderson and Livingston [1999] indicate that of the six connected graphs on four vertices, only those shown in Figure 4 may be realized as $\Gamma(R)$. Their proofs that the other three graphs seen in Figure 5 are not realizable all have a similar flavor. One uses the fact that certain sums or products must be annihilated by another element in the graph, and therefore must also be vertices in the zero divisor graph. This breaks down (often) in the graded case. Even though all of the vertices represent homogeneous elements and the sum of elements may still be annihilated, unless we know that both (homogeneous) elements are of *the same degree*, this sum no longer needs to be another vertex in the graded zero divisor graph.

For zero divisor graphs of graded rings, the three graphs in Figure 4 are still realizable, but we can also produce two more.

The graph on the left in Figure 5 is realized using the ring $\mathbb{Z}_2[X, Y]/(XY, X^2, Y^4)$ under the $\mathbb{Z}_2 \oplus \mathbb{Z}_4$-grading defined by $\deg(x) = (1 \pmod 2), 0 \pmod 4))$ and $\deg(y) = (0 \pmod 2), 1 \pmod 4))$, where x and y represent the images of X and Y in the quotient.

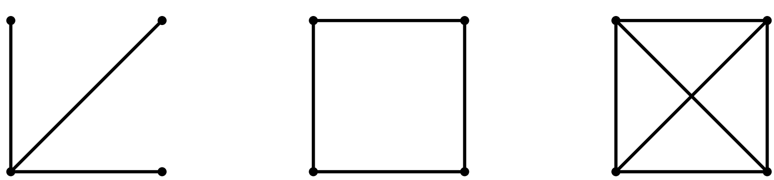


Figure 4. The three connected graphs on four vertices realizable as $\Gamma(R)$.

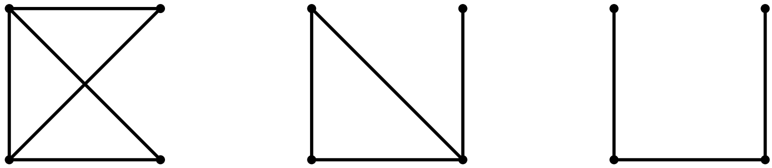


Figure 5. Two additional graphs realizable as $\Gamma_G(R)$ (left, middle) and an unrealizable (right) connected graph on four vertices.

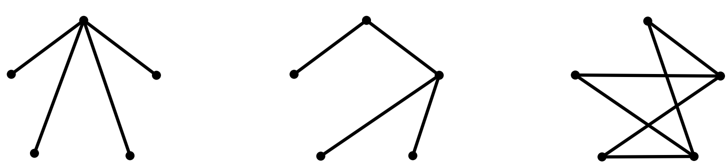
The graph in the middle is realized with the ring $\mathbb{Z}_2[X]/(X^5)$ under the $\mathbb{Z}$-grading defined by $\deg(x) = 1$, where x is the image of X in the quotient. We could also obtain the same graph using a $\mathbb{Z}_5$-grading and setting $\deg(x^i) = i \pmod 5$.

The final graph on the right in Figure 5 remains unrealizable as $\Gamma_G(R)$ for any group G. It can be proven that each of the four zero divisors must be (homogeneous) of the same degree, and thus the proof provided by Anderson and Livingston can be used.

Connected graphs on five vertices. An interesting fact is that while there are 21 connected graphs on five vertices, there are still only three of these graphs realizable as $\Gamma(R)$. This can be proved using a mix of results from [Anderson and Livingston 1999] and direct analysis of adding and/or multiplying certain zero divisors together to reach a contradiction; alternatively, this is shown in [Redmond 2003]. These three graphs and the rings used to construct them are shown in Figure 6. Here, $\mathbb{F}_4$ represents a finite field with four elements.

As before, we are able to construct more of these graphs in the graded setting (in addition to those in Figure 6). Figure 7 summarizes the additional graphs we are able to realize, while Table 1 summarizes the grading used on each ring. In the table we use x and y to denote the images of X and Y in factor rings, while e_i denotes the i-th basis vector, which has a $1 \pmod n$ (for the appropriate n) in the i-th position and 0s elsewhere.

Not every connected graph on five vertices is realizable as a graded zero divisor graph. Figure 8 contains the graphs unrealizable as graded zero divisor graphs.



$G_2 = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)$ $G_3 = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$ $G_{11} = \Gamma(\mathbb{Z}_3 \times \mathbb{F}_4)$

Figure 6. Connected graphs on five vertices realizable as (non-graded) zero divisor graphs.

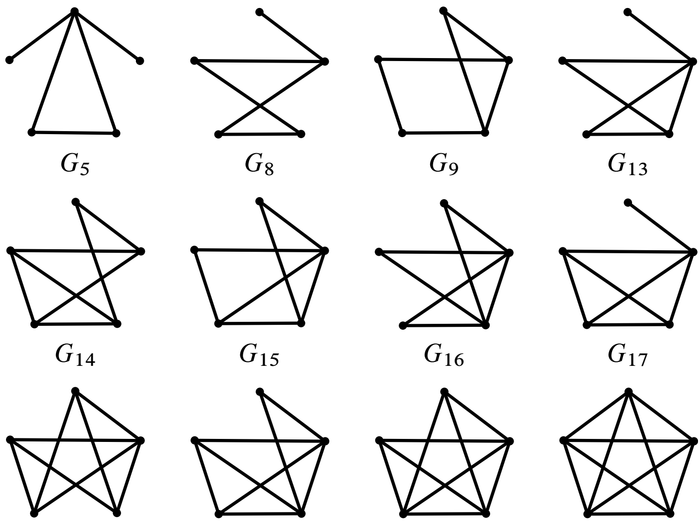


Figure 7. Additional connected graphs on five vertices realizable as graded zero divisor graphs.

graph	ring	group	grading
G_5	$\frac{\mathbb{Z}_3[X]}{(X^2)} \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\deg((x,0)) = 1 \pmod 2$
G_8	$\frac{\mathbb{Z}_2[X]}{((X+1)^2 X^2)}$	$\mathbb{Z}_2$	$\deg(x^i) = i \pmod 2$
G_9	$\frac{\mathbb{Z}_2[X]}{(X^2)} \times \frac{\mathbb{Z}_2[Y]}{(Y^2)}$	$\mathbb{Z}_2$	$\deg((x,0)) = \deg((0,y)) = 1 \pmod 2$
G_{13}	$\frac{\mathbb{Z}_2[X]}{(X^6)}$	$\mathbb{Z}_6$	$\deg(x) = 1 \pmod 6$
G_{14}	$\frac{\mathbb{Z}_2[X]}{(X^3)} \times \mathbb{Z}_3$	$\mathbb{Z}_3$	$\deg((x,0)) = 1 \pmod 3$
G_{15}	$\frac{\mathbb{Z}_2[X,Y]}{(X^3,Y^2)}$	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\deg(x) = e_1,\ \deg(y) = e_2$
G_{16}	$\frac{\mathbb{Z}_2[X,Y]}{(XY,X^2,Y^4)}$	$\mathbb{Z}_4$	$\deg(x) = \deg(y) = 1 \pmod 4$
G_{17}	$\frac{\mathbb{Z}_2[X,Y]}{(X,Y)^2} \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\deg((x,0)) = \deg((y,0)) = 1 \pmod 2$
G_{18}	$\frac{\mathbb{Z}_2[X,Y]}{(XY,X^3-Y^3)}$	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	$\deg(x) = e_1,\ \deg(y) = e_2$
G_{19}	$\frac{\mathbb{Z}_2[X,Y]}{(XY^2,X^2,Y^4)}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	$\deg(x) = e_1,\ \deg(y) = e_2$
G_{20}	$\frac{\mathbb{Z}_2[X,Y]}{(XY,X^3,Y^3)}$	$\mathbb{Z}_3$	$\deg(x) = 1 \pmod 3,\ \deg(y) = 0$
G_{21}	$\frac{\mathbb{Z}_2[X_1,X_2,\dots,X_5]}{(X_i X_j \mid i,j \in \{1,2,\dots,5\})}$	$(\mathbb{Z}_2)^5$	$\deg(x_i) = e_i$

Table 1. Rings and their gradings used to construct the graphs in Figure 7.

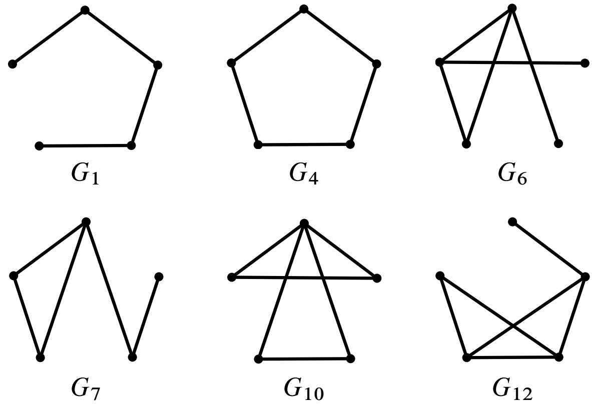


Figure 8. Connected graphs on five vertices unrealizable as a graded zero divisor graph.

Some can be eliminated easily, based on girth or diameter considerations, such as G_1 and G_4. To eliminate others, we used techniques similar to the nongraded case, with some modifications. To indicate the complications that arise, we provide an example.

Example 4.1. To show the graph G_{10} is unrealizable, label the vertices a, b, c, d, and e so that a is the vertex at the top, continuing in alphabetical order clockwise.

From relations in the graph, we get that bc, bd, ce, and de must be (nonzero) zero divisors. It is easily shown that each of these products must be equal to a. This implies $b, e \in R_g$ and $c, d \in R_h$ for some $g, h \in G$; that is, these elements are homogeneous of the same degree. Clearly, $b-e \in R_g$ and $b-e \neq 0$. Similarly, $c-d \in R_h$ and $c-d \neq 0$. As each of these differences is annihilated by a, we have $b-e,\ c-d \in Z_G^*(R)$.

We now simply exhaust all possibilities for $b-e$ and $c-d$. If $b-e=b$, then $e=0$, a contradiction. If $b-e=e$, then $cb-ce=ce$, so that $a-a=a$, a contradiction. Similarly, we reach contradictions if $c-d \in \{c,d\}$. This gives $b-e \in \{a,c,d\}$ and $c-d \in \{a,b,e\}$.

Suppose $b-e=a$. Then $b,e,a \in R_g$. Thus, if $c-d \in \{a,b,e\}$, then $c,d \in R_g$, and the following statement holds:

(†) All five vertices are of the same degree, and $de=a$ (for example) implies this is degree 0. This implies $\Gamma_G(R)=\Gamma(R_0)$, but we know this graph cannot be realized as the usual zero divisor graph of any ring.

Now suppose $b-e=c$. Then $b,e,c,d \in R_g$. If $c-d=a$, then (†) applies again. If $c-d=b$, then $c-d-e=c$, so $d=-e$. This contradicts (for example) the fact that $ce \neq 0$. We obtain a similar contradiction if $c-d=e$.

Finally, suppose $b-e=d$. Then $b,e,c,d \in R_g$. Again, if $c-d=a$, (†) applies. If $c-d=b$, then $c-d-e=d$, so $bc-bd-be=bd$ gives us $a=0$, a contradiction. If $c-d=e$, then $b-c+d=d$, so $b=c$, a contradiction. It follows that G_{10} cannot be realized as $\Gamma_G(R)$.

Complete graphs. A central result of Anderson and Livingston [1999, Theorem 2.5] in their classification of realizable complete graphs (and in their classification of realizable star graphs, in fact) states that $\Gamma(R)$ has a vertex adjacent to every other vertex if and only if $R \cong \mathbb{Z}_2 \times A$, where A is an integral domain, or $Z(R)$ is an annihilator ideal (and hence is prime). We prove a similar result in Theorem 4.3, using the following lemma.

Lemma 4.2. *Suppose R is a G-graded ring and $a \in R$ is homogeneous. If* $\operatorname{ann}(a)$ *is maximal among annihilators of homogeneous elements, then* $\operatorname{ann}(a)$ *is G-prime.*

Proof. Suppose x and y are homogeneous and $xy \in \operatorname{ann}(a)$, but $x \notin \operatorname{ann}(a)$. We have $xa \neq 0$, but $xya = 0$. Thus $y \in \operatorname{ann}(xa)$. However, $\operatorname{ann}(xa) \subseteq \operatorname{ann}(a)$ implies $\operatorname{ann}(xa) = \operatorname{ann}(a)$. This implies $y \in \operatorname{ann}(a)$, and thus $\operatorname{ann}(a)$ is a G-prime ideal. □

Because $Z_G(R)$ is very often not an ideal in the graded setting, we will end up considering $(Z_G(R))$, the ideal generated by the homogeneous zero divisors, in the theorem below.

Theorem 4.3. *Suppose R is a G-graded ring. Then there is a vertex of $\Gamma_G(R)$ adjacent to every other vertex if and only if $R \cong \mathbb{Z}_2 \times A$, where $\mathbb{Z}_2$ and A are G-graded and A is a G-domain, or* $(Z_G(R)) = \operatorname{ann}(x)$ *for some nonzero homogeneous $x \in R$.*

Proof. ($\Leftarrow$) If $(Z_G(R)) = \operatorname{ann}(x)$, then x is adjacent to every other vertex. If $R \cong \mathbb{Z}_2 \times A$, where A is a G-domain, then $(1,0)$ is adjacent to everything in $Z^*_G(R)$, except $(1,0)$.

($\Rightarrow$) Suppose $(Z_G(R)) \neq \operatorname{ann}(x)$ for all nonzero homogeneous $x \in R$. Also, suppose there exists a such that $0 \neq a \in Z_G(R)$ with a adjacent to every other vertex.

If $a \in \operatorname{ann}(a)$, then $ax = 0$ for all $x \in Z_G(R)$. This implies $(Z_G(R)) \subseteq \operatorname{ann}(a)$. Also, $\operatorname{ann}(a)$ is homogeneous, so every homogeneous generator of $\operatorname{ann}(a)$ is in $Z_G(R)$. Thus $\operatorname{ann}(a) \subseteq (Z_G(R))$. So $\operatorname{ann}(a) = (Z_G(R))$, a contradiction. Therefore $a \notin \operatorname{ann}(a)$.

We claim $\operatorname{ann}(a)$ is maximal among those $\operatorname{ann}(x)$ such that x is homogeneous. To see this, note that a is adjacent to every other homogeneous zero divisor, yet $a \notin \operatorname{ann}(a)$.

By Lemma 4.2, $\operatorname{ann}(a)$ is G-prime. Since a is a zero divisor, a^2 is also a homogeneous zero divisor. But $a \notin \operatorname{ann}(a)$, so $a^2 \neq 0$. If $a^2 \neq a$, then $a^2 \in \operatorname{ann}(a)$, but $\operatorname{ann}(a)$ is G-prime, so $a \in \operatorname{ann}(a)$, a contradiction. Therefore $a^2 = a$; that is, a is a nontrivial (homogeneous) idempotent of degree 0.

By Lemma 2.1, $R = S \times T$ (as graded rings). Without loss of generality, let $a = (1,0)$. Then $R = \mathbb{Z}_2 \times A$, where A is a G-domain. □

As we have seen in the examples above, we can construct graded zero divisor graphs that are complete for both four and five vertices. This already contrasts with the nongraded case, as Anderson and Livingston [1999, Theorem 2.10] show that only complete graphs on $p^n - 1$ vertices, where p is prime and $n \geq 1$, are realizable as the zero divisor graph of a ring. In fact, in the graded case, we can realize every complete graph as a graded zero divisor graph. While we assume the graph is finite, the proof can easily be extended to infinite complete graphs.

Theorem 4.4. *A complete graph of any size is realizable as $\Gamma_G(R)$ for some abelian group G and G-graded ring R.*

Proof. Consider K_n, the complete graph on n vertices, where $n \geq 1$. Define the ring S to be $\mathbb{Z}_2[X_1, \ldots, X_n]$, where the X_i are indeterminates. This has an obvious grading by the group $G := \mathbb{Z}^n$, where we define the degree of X_i to be e_i, the i-th basis vector in G (which has a 1 in the i-th position and 0s elsewhere).

Let $I = (X_i X_j \mid i, j \in \{1, \ldots, n\})$ be the ideal generated by all products of two (not necessarily distinct) variables. As each generator is homogeneous, I is a homogeneous ideal, and $R := S/I$ is also a G-graded ring.

One can now verify that $\Gamma_G(R) = K_n$ by noting that the only homogeneous elements in R are the images of the X_i, all of which annihilate each other. □

Star graphs and complete bipartite graphs. Another well-studied class of graphs is the class of star graphs. A *star graph* is the complete bipartite graph $K_{1,k}$ for some $k \geq 0$. Except for the case $k = 0$, it can be thought of as having one vertex adjacent to all other vertices with no additional edges. Anderson and Livingston [1999, Theorem 2.13] completely characterized which star graphs are realizable for finite commutative rings. Star graphs were also studied by Coykendall, Sather-Wagstaff, Sheppardson, and Spiroff [Coykendall et al. 2012], but they focused on a different construction introduced by Mulay [2002], based on equivalence classes of zero divisors, denoted by $\Gamma_E(R)$.

For nongraded rings, it is only possible to realize the star graphs with p^n vertices, where p is a prime and $n \geq 0$. As with complete graphs, we can construct all (finite) star graphs in the graded setting. The following theorem is an obvious corollary of Theorem 4.6, and we omit the proof.

Theorem 4.5. *A star graph of any (finite) size is realizable as $\Gamma_G(R)$ for some abelian group G and G-graded ring R.*

Not only can we realize all star graphs as graded zero divisor graphs, we can also realize *every* complete bipartite graph.

Theorem 4.6. *A complete bipartite graph of any (finite) size is realizable as $\Gamma_G(R)$ for some abelian group G and G-graded ring R.*

Proof. Consider the graph $K_{m,n}$ and the rings defined by $S = \mathbb{Z}_2[X]/(X^m - 1)$ and $T = \mathbb{Z}_2[Y]/(Y^n - 1)$. Use x and y, respectively, to denote the images of X and Y in S and T. Define $L = \operatorname{lcm}(m, n)$. Set $G = \mathbb{Z}_L$ and define $\mathbb{Z}_L$-gradings on S and T, respectively, by setting $\deg(x) = \frac{L}{m}$ and $\deg(y) = \frac{L}{n}$. It is a straightforward exercise to show that each of these rings is now a Z_L-field under its respective grading.

Form the graded direct product $R := S \times T$ (where $R_i = S_i \times T_i$). Notice that every nonzero element of R of the form $(s, 0)$ or $(0, t)$, where $s \in S$ and $t \in T$ are homogeneous, is a vertex in $\Gamma_G(R)$. Also, each such element $(s, 0)$ is adjacent to each element $(0, t)$. Further, we claim these are the only vertices and edges in $\Gamma_G(R)$. To see this, suppose (s_1, t_1) and (s_2, t_2) are two elements of $Z^*_G(R)$. Because S and T are $\mathbb{Z}_L$-fields, and the s_i and t_i must be homogeneous, we can only have

$$(s_1, t_1)(s_2, t_2) = (0, 0)$$

when the elements on the left are of the form $(s_1, 0)$ and $(0, t_2)$ or $(0, t_1)$ and $(s_2, 0)$. □

Open questions.

Question 4.7. Notice that for the constructions above, each ring is graded by a different abelian group. Another interesting question to consider is whether this is necessary. For example, for a fixed group G, can we still realize all complete graphs? If not, which graphs can we realize for a specific group?

Question 4.8. Theorem 4.3 is a step toward characterizing the graded rings that give rise to graded zero divisor graphs that are stars or complete graphs. A further avenue of study would be to determine if one can classify, completely or in part, the (graded) rings that give rise to star and/or complete graphs.

Question 4.9. Is there a generalization, in part or whole, of Theorem 4.6 to n-partite graphs? For example, Akbari, Maimani, and Yassemi [Akbari et al. 2003, Theorem 3.1] determine the rings whose zero divisor graphs are n-partite. They show, in particular, that if $n \geq 3$, at most one partitioning subset of $\Gamma(R)$ can have more than one vertex. As a contrast, graph G_{18} in Figure 7 shows that in the graded case we can construct a complete 3-partite graph with more than one partitioning subset having size greater than 1.

References

[Akbari et al. 2003] S. Akbari, H. R. Maimani, and S. Yassemi, "When a zero-divisor graph is planar or a complete r-partite graph", *J. Algebra* **270**:1 (2003), 169–180. MR Zbl

[Anderson and Badawi 2008] D. F. Anderson and A. Badawi, "On the zero-divisor graph of a ring", *Comm. Algebra* **36**:8 (2008), 3073–3092. MR Zbl

[Anderson and Badawi 2012] D. F. Anderson and A. Badawi, "On the total graph of a commutative ring without the zero element", *J. Algebra Appl.* **11**:4 (2012), art. id. 1250074. MR Zbl

[Anderson and Livingston 1999] D. F. Anderson and P. S. Livingston, "The zero-divisor graph of a commutative ring", *J. Algebra* **217**:2 (1999), 434–447. MR Zbl

[Ashrafi et al. 2010] N. Ashrafi, H. R. Maimani, M. R. Pournaki, and S. Yassemi, "Unit graphs associated with rings", *Comm. Algebra* **38**:8 (2010), 2851–2871. MR Zbl

[Axtell et al. 2005] M. Axtell, J. Coykendall, and J. Stickles, "Zero-divisor graphs of polynomials and power series over commutative rings", *Comm. Algebra* **33**:6 (2005), 2043–2050. MR Zbl

[Axtell et al. 2009] M. Axtell, J. Stickles, and W. Trampbachls, "Zero-divisor ideals and realizable zero-divisor graphs", *Involve* **2**:1 (2009), 17–27. MR Zbl

[Badawi 2014] A. Badawi, "On the annihilator graph of a commutative ring", *Comm. Algebra* **42**:1 (2014), 108–121. MR Zbl

[Badawi 2015] A. Badawi, "On the dot product graph of a commutative ring", *Comm. Algebra* **43**:1 (2015), 43–50. MR Zbl

[Beck 1988] I. Beck, "Coloring of commutative rings", *J. Algebra* **116**:1 (1988), 208–226. MR Zbl

[Behboodi and Rakeei 2011] M. Behboodi and Z. Rakeei, "The annihilating-ideal graph of commutative rings, I", *J. Algebra Appl.* **10**:4 (2011), 727–739. MR Zbl

[Coykendall et al. 2012] J. Coykendall, S. Sather-Wagstaff, L. Sheppardson, and S. Spiroff, "On zero divisor graphs", pp. 241–299 in *Progress in commutative algebra, II*, edited by C. Francisco et al., de Gruyter, Berlin, 2012. MR Zbl

[DeMeyer et al. 2002] F. R. DeMeyer, T. McKenzie, and K. Schneider, "The zero-divisor graph of a commutative semigroup", *Semigroup Forum* **65**:2 (2002), 206–214. MR Zbl

[Johnson 2012] B. P. Johnson, *Commutative rings graded by abelian groups*, Ph.D. thesis, University of Nebraska–Lincoln, 2012, available at http://search.proquest.com/docview/1038955241.

[Khosh-Ahang and Nazari-Moghadam 2016] F. Khosh-Ahang and S. Nazari-Moghadam, "An associated graph to a graded ring", *Publ. Math. Debrecen* **88**:3-4 (2016), 401–416. MR Zbl

[LaGrange 2008] J. D. LaGrange, "On realizing zero-divisor graphs", *Comm. Algebra* **36**:12 (2008), 4509–4520. MR Zbl

[Mulay 2002] S. B. Mulay, "Cycles and symmetries of zero-divisors", *Comm. Algebra* **30**:7 (2002), 3533–3558. MR Zbl

[Năstăsescu and Van Oystaeyen 2004] C. Năstăsescu and F. Van Oystaeyen, *Methods of graded rings*, Lecture Notes in Mathematics **1836**, Springer, 2004. MR Zbl

[Redmond 2003] S. P. Redmond, "An ideal-based zero-divisor graph of a commutative ring", *Comm. Algebra* **31**:9 (2003), 4425–4443. MR Zbl

[Redmond 2007] S. P. Redmond, "On zero-divisor graphs of small finite commutative rings", *Discrete Math.* **307**:9-10 (2007), 1155–1166. MR Zbl

[Vietri 2015] A. Vietri, "A new zero-divisor graph contradicting Beck's conjecture, and the classification for a family of polynomial quotients", *Graphs Combin.* **31**:6 (2015), 2413–2423. MR Zbl

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