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2018 vol. 11, no. 2

INVOLVE 11:2 (2018)
dx.doi.org/10.2140/involve.2018.11.297

# The behavior of a population interaction-diffusion equation in its subcritical regime

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(Communicated by Martin J. Bohner)

A model interaction-diffusion equation for population density originally analyzed through terms of third-order in its supercritical parameter range is extended through terms of fifth-order to examine the behavior in its subcritical regime. It is shown that under the proper conditions the two subcritical cases behave in exactly the same manner as the two supercritical ones unlike the outcome for the truncated system. Further, there also exists a region of metastability allowing for the possibility of population outbreaks. These results are then used to offer an explanation for the occurrence of isolated vegetative patches and sparse homogeneous distributions in the relevant ecological parameter range where there is subcriticality for a plant-groundwater model system, as opposed to periodic patterns and dense homogeneous distributions occurring in its supercritical regime.

## 1. Introduction and formulation of the problem

Consider the following interaction-diffusion partial differential equation boundary value problem for $N = N(s, \tau) \equiv$ population density, where $s \equiv$ one-dimensional spatial variable and $\tau \equiv$ time:

$$\frac{\partial N}{\partial \tau} = D_0 \frac{\partial^2 N}{\partial s^2} + R_0 N_e r\left(\frac{N - N_e}{N_e}\right), \quad 0 < s < L, \tag{1-1a}$$

$$N(0, \tau) = N(L, \tau) = N_e, \tag{1-1b}$$

with

$$r(\theta) = \theta + \alpha\theta^3 + \gamma\theta^5 + O(\theta^7). \tag{1-1c}$$

*MSC2010:* 34D20, 35Q56, 92D40.
*Keywords:* interaction-diffusion, Stuart–Watson method, subcritical bifurcation analysis.
Davis was supported by NSF 1029482 Collaborative Research: UBM-Institutional: UI-WSU Program in Undergraduate Mathematics and Biology.

Here, $D_0 \equiv$ dispersal constant, $R_0 \equiv$ interaction rate, $N_e \equiv$ equilibrium population density, and $L \equiv$ territory size, while $\alpha$ and $\gamma$ represent dimensionless interaction coefficients. Note that

$$N(s, \tau) \equiv N_e \tag{1-2}$$

is an exact solution to boundary value problem (1-1).

Introducing the nondimensional variables and parameter

$$z = \frac{\pi s}{L}, \quad t = \frac{D_0 \pi^2 \tau}{L^2}, \quad \theta(z, t) = \frac{N(s, \tau) - N_e}{N_e}, \quad \beta = \frac{R_0 L^2}{D_0 \pi^2}, \tag{1-3}$$

our original problem transforms into

$$\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial z^2} = \beta r(\theta), \quad 0 < z < \pi, \tag{1-4a}$$

$$\theta(0, t) = \theta(\pi, t) = 0. \tag{1-4b}$$

Note that the exact solution (1-2) to the dimensional problem corresponds to

$$\theta(z, t) \equiv 0 \tag{1-5}$$

for our dimensionless one (1-4).

This is an extension to fifth-order of a model equation introduced by Wollkind et al. [1994] to illustrate the Stuart–Watson method of weakly nonlinear stability analysis of prototype reaction-diffusion equations. Asymptotic analyses of this sort are very useful for predicting pattern formation in such nonlinear systems. That analysis requires the expansion of $\theta$ in powers of an unknown function $A(t)$ with spatially dependent coefficients. The pattern-formational aspect of this system can be predicted from the long-time behavior of that amplitude function, which is governed by its Landau ordinary differential equation

$$\frac{dA}{dt} \sim \sigma A - a_1 A^3 - a_3 A^5 = F(A), \tag{1-6}$$

where $\sigma$ is the growth rate of linear stability theory and $a_{1,3}$ are the Landau constants. That long-time behavior is crucially dependent upon the signs of these Landau constants. Wollkind et al. [1994] concentrated on the special case for which $r(\theta) = \sin(\theta)$, employed by Matkowsky [1970] to develop his two-time method of weakly nonlinear stability theory, since their main concern was to compare the results obtained from the application of the Stuart–Watson method with those he deduced. Then $a_1 > 0$, identically (see below), and it is only necessary to include terms through third-order in $r(\theta)$ to make pattern formation predictions for this problem. In that event, there are two solutions of the truncated system: the first, a homogeneous one that is stable for $\sigma < 0$ and the second, a supercritical re-equilibrated pattern forming one that exists and is stable for $\sigma > 0$. These results can be directly applied to our problem for its generalized $r(\theta)$ in the parameter range where $a_1 > 0$. In the range where $a_1 < 0$ and there is so-called subcriticality, the solutions to the truncated problem can grow without bound, and one must take the fifth-order terms into account in order to determine the long-time behavior of the system. Then we shall show that, if there is a parameter range over which the other Landau constant $a_3$ satisfies $a_3 > 0$, the pattern formation properties of our system can be ascertained without having to resort to considering even higher-order terms in $r(\theta)$. That requires the development of a formula for this Landau constant and an examination of its sign as a function of $\alpha$ and $\gamma$.

## 2. The Stuart–Watson method of nonlinear stability theory

Toward that end, we seek a Stuart–Watson expansion for the solution of our model equation of the form [Wollkind et al. 1994]

$$\theta(z,t) \sim A(t)\sin(z) + A^3(t)[\theta_{31}\sin(z) + \theta_{33}\sin(3z)] + A^5(t)[\theta_{51}\sin(z) + \theta_{53}\sin(3z) + \theta_{55}\sin(5z)]. \quad (2\text{-}1)$$

Note that the spatial terms in expansion (2-1) satisfy our boundary conditions (1-4b) at $z = 0$ and $\pi$, identically. Then, expanding $r(\theta)$ in powers of $A(t)$, employing the relevant trigonometric identities for the resulting products of sine functions contained in its coefficients, and making use of the Landau amplitude equation (1-6), we obtain a series of problems, one for each term appearing explicitly in our expansion of the form $A^n(t)\sin(mz)$, given by

$$\begin{aligned}
A(t)\sin(z):&\quad \sigma + 1 = \beta,\\
A^3(t)\sin(z):&\quad 3\sigma\theta_{31} - a_1 + \theta_{31} = \beta\big(\theta_{31} + \tfrac{3}{4}\alpha\big),\\
A^3(t)\sin(3z):&\quad 3\sigma\theta_{33} + 9\theta_{33} = \beta\big(\theta_{33} - \tfrac{1}{4}\alpha\big),\\
A^5(t)\sin(z):&\quad 5\sigma\theta_{51} - a_3 - 3a_1\theta_{31} + \theta_{51} = \beta\big(\theta_{51} + \tfrac{9}{4}\alpha\theta_{31} - \tfrac{3}{4}\alpha\theta_{33} + \tfrac{5}{8}\gamma\big).
\end{aligned}$$

Although there are also two other $A^5(t)$ problems, they have not been cataloged above since only the one proportional to $\sin(z)$ which involves $a_3$ is required for our purposes. Here, while $\sigma$ and the $\theta_{nm}$ are being considered as functions of $\beta$, the coefficients $a_{1,3}$ are assumed to be independent of that bifurcation parameter and hence the use of the terminology Landau *constants*. That assumption is critical for their determination as solvability conditions, which is developed below.

We now solve these problems sequentially. Then, from the ones not involving these Landau constants, we obtain in a straightforward manner that

$$\sigma(\beta) = \beta - 1, \quad (2\text{-}2a)$$

and

$$\theta_{33}(\beta) = -\frac{\alpha\beta}{8(\beta+3)}, \tag{2-2b}$$

while the other two problems yield

$$2\sigma(\beta)\theta_{31}(\beta) = a_1 + \tfrac{3}{4}\alpha\beta \tag{2-2c}$$

and

$$4\sigma(\beta)\theta_{51}(\beta) = a_3 + 3\theta_{31}(\beta)\big(a_1 + \tfrac{3}{4}\alpha\beta\big) - \tfrac{3}{4}\alpha\beta\theta_{33}(\beta) + \tfrac{5}{8}\gamma\beta. \tag{2-2d}$$

(i) Assuming that $\theta_{31}(\beta)$ is well behaved at the critical bifurcation value of $\beta = 1$ and taking the limit of this first relation as $\beta \to 1$, while noting that $\sigma(\beta) = \beta - 1 \to 0$ in this limit, we obtain the solvability condition

$$a_1 = -\tfrac{3}{4}\alpha \tag{2-3a}$$

and, upon substitution of this back into (2-2c), the solution

$$\theta_{31}(\beta) \equiv \theta_{31} = \tfrac{3}{8}\alpha. \tag{2-3b}$$

Hence, we can deduce that

$$a_1 > 0 \quad \text{for } \alpha < 0 \qquad \text{and} \qquad a_1 < 0 \quad \text{for } \alpha > 0. \tag{2-4}$$

Thus, as mentioned earlier,

$$r(\theta) = \sin(\theta) = \theta - \tfrac{1}{6}\theta^3 + O(\theta^5) \quad \Longrightarrow \quad \alpha = -\tfrac{1}{6} \quad \Longrightarrow \quad a_1 = \tfrac{1}{8}. \tag{2-5}$$

Now, in this case, defining

$$\varepsilon^2 = \frac{\sigma(\beta)}{a_1} \quad \text{or} \quad \beta = 1 + \tfrac{1}{8}\varepsilon^2 \tag{2-6a}$$

and introducing the rescaled variables

$$\eta = \sigma t, \quad \mathcal{A}(\eta) = \frac{A(t)}{\varepsilon} \tag{2-6b}$$

into the truncated amplitude equation

$$\frac{dA}{dt} = \sigma A - a_1 A^3 + O(A^5), \tag{2-6c}$$

we obtain

$$\frac{d\mathcal{A}}{d\eta} = \mathcal{A} - \mathcal{A}^3 + O(\varepsilon^2), \tag{2-6d}$$

which justifies that truncation procedure. Now multiplying the truncated amplitude equation by $A(t)$ and rewriting it as

$$\frac{1}{2}\frac{dA^2}{dt} = \sigma A^2 - a_1 A^4 = \sigma A^2\left(1 - \frac{A^2}{\sigma/a_1}\right) = f_3(A^2), \tag{2-7}$$

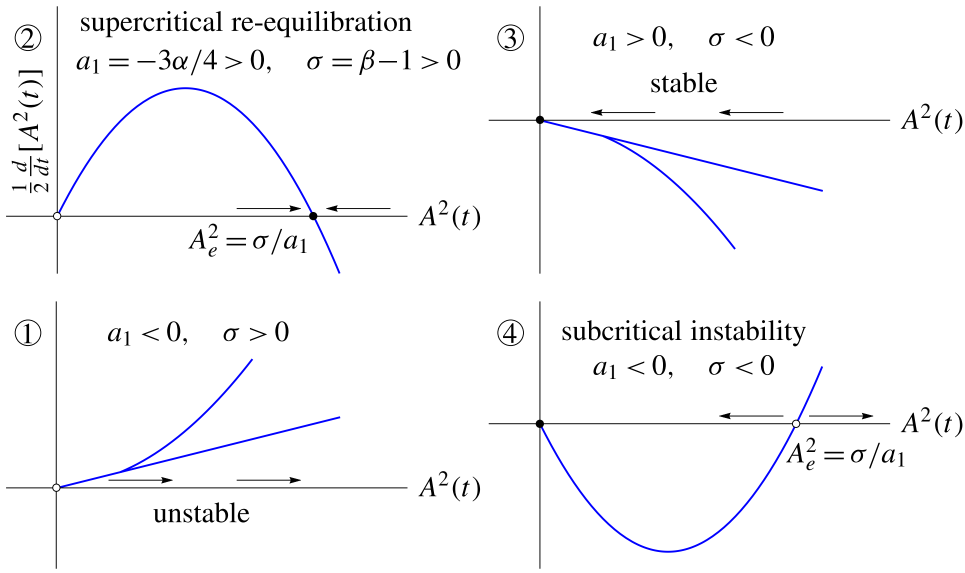


**Figure 1.** Plots of $f_3(A^2)$ for the third-order truncated amplitude equation with $\sigma = \beta - 1$ and $a_1 = -\frac{3}{4}\alpha$. Here the circled numbers correspond to the quadrants in the $\alpha\beta$-space of Figure 5 with horizontal axis $\beta = 1$ and vertical axis $\alpha = 0$.

we can easily deduce its long-time behavior by means of the four phase-plane plots of

$$\frac{1}{2}\frac{dA^2}{dt} = f_3(A^2)$$

that constitute Figure 1, which catalogs the four qualitatively different cases corresponding to the possibility of $\sigma$ and $a_1$ being either positive or negative. These serve as graphical representations of the cases discussed in Section 1 for the truncated version of our amplitude equation.

In particular, for the supercritical re-equilibration case of $\sigma, a_1 > 0$, we have

$$\lim_{t\to\infty} A(t) = A_e = \varepsilon, \tag{2-8a}$$

and hence

$$\lim_{t\to\infty} \theta(z,t) \sim \theta_e(z) = \delta \sin(z) \quad \text{as } \delta \to 0 \tag{2-8b}$$

since

$$\begin{aligned}\lim_{t\to\infty} \theta(z,t) &= \varepsilon \sin(z) + \varepsilon^3[\theta_{31}\sin(z) + \theta_{33}(\beta)\sin(3z)] + O(\varepsilon^5)\\ &= (\varepsilon + \theta_{31}\varepsilon^3)\sin(z) + \varepsilon^3\theta_{33}(1)\sin(3z) + O(\varepsilon^5)\\ &= \delta\sin(z) + \tfrac{1}{192}\delta^3\sin(3z) + O(\delta^5) \sim \delta\sin(z) \quad \text{as } \delta \to 0,\end{aligned} \tag{2-8c}$$

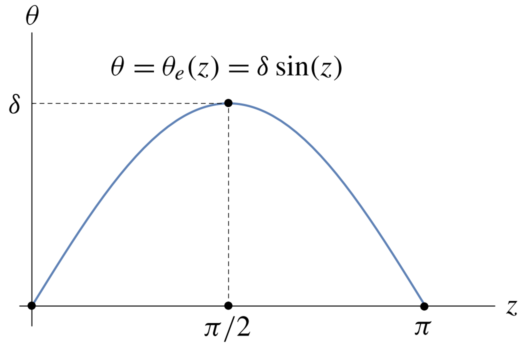


**Figure 2.** Plot of the arch solution $\theta_e(z)$ for $0 \le z \le \pi$.

where $\delta = \varepsilon + \varepsilon^3\theta_{31} > 0$. This equilibrium state, plotted in Figure 2, is an arch-type pattern formed from one-cycle of a sine curve with its maximum amplitude $\delta$ occurring at $z = \frac{1}{2}\pi$.

(ii) We next proceed to analyze the second Landau constant relation (2-2d) involving $a_3$ and $\theta_{51}$ in an analogous manner to that just employed to evaluate $a_1$ and $\theta_{31}$. Thus, assuming $\theta_{51}(\beta)$ to be well behaved at $\beta = 1$ and taking the limit of this relation as $\beta \to 1$, we obtain the solvability condition

$$a_3 = -\tfrac{5}{8}\gamma - 3\theta_{31}\left(a_1 + \tfrac{3}{4}\alpha\right) + \tfrac{3}{4}\alpha\theta_{33}(1) = -\tfrac{5}{8}\gamma - \tfrac{3}{128}\alpha^2 \tag{2-9a}$$

and, upon substitution of this back into (2-2d), the solution

$$\theta_{51}(\beta) = \tfrac{5}{32}\gamma + \tfrac{9}{16}\alpha\theta_{31} + \frac{3\alpha^2(4\beta+3)}{512(\beta+3)}. \tag{2-9b}$$

Observe that, by virtue of the value of $a_1$, we have $a_3$ is independent of $\theta_{31}$. Also observe that, unlike this quantity, $\theta_{51}$ is a function of $\beta$. Finally note, in addition, should we have assumed that the Stuart–Watson expansion for $\theta(z, t)$ and the Landau equation for $dA/dt$ contained even powers of $A(t)$, then the solvability conditions and solutions for their coefficients would have shown them to be zero. Hence our implicit assumption that these quantities only contained odd powers was made without loss of generality and follows as a direct consequence of the form of $r(\theta)$.

Having determined its coefficients, we shall examine the truncated amplitude equation (1-6) through terms of fifth-order, i.e.,

$$\frac{dA}{dt} = F(A), \tag{2-10}$$

and defer until after this examination has been completed a justification for that truncation. We seek conditions under which the inclusion of fifth-order terms will re-equilibrate the growing solutions predicted through third-order when $a_1 < 0$. Hence we assume a parameter range in which $a_1 < 0$ or $\alpha > 0$. Further, anticipating

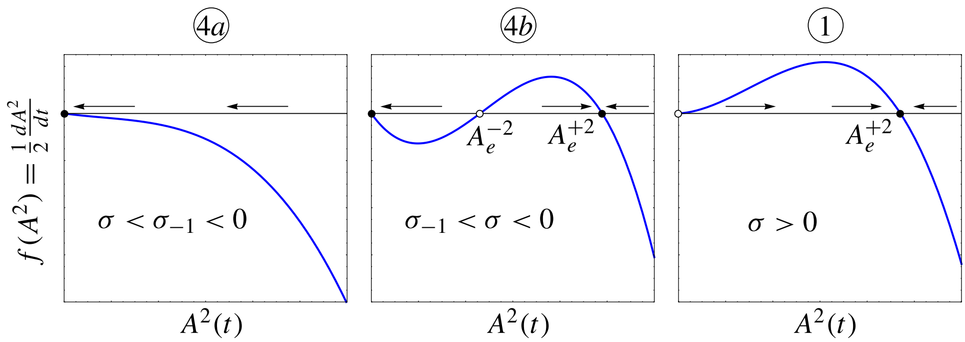


**Figure 3.** Plots of $f(A^2)$ for the fifth-order truncated amplitude equation with $a_1 < 0$; $a_3 > 0$; and $\sigma < \sigma_{-1} = -a_1^2/(4a_3) < 0$, $\sigma_{-1} < \sigma < 0$, and $\sigma > 0$, respectively. Here, the circled numbers correspond to the quadrants in the $\alpha\beta$-space of Figure 5.

our results to be demonstrated below, we assume that $a_3 > 0$, while, as always, $\sigma \in \mathbb{R}$. This equation has three equilibrium points

$$A(t) \equiv A_e \quad \text{such that } F(A_e) = 0 \tag{2-11a}$$

satisfying either

$$A_e = 0 \quad \text{or} \quad 2a_3 A_e^{\pm 2} = \pm\sqrt{a_1^2 + 4a_3\sigma} - a_1. \tag{2-11b}$$

Observe that, since they must be real and positive, $A_e^{+2}$ exists for $\sigma \geq \sigma_{-1} = -a_1^2/(4a_3)$, while $A_e^{-2}$ only exists for $\sigma_{-1} \leq \sigma < 0$. Multiplying our truncated amplitude equation (2-10) by $A(t)$, we obtain

$$\frac{1}{2}\frac{dA^2}{dt} = \sigma A^2 - a_1 A^4 - a_3 A^6 = A^2(A_e^{-2} - A^2)(A^2 - A_e^{+2}) = f(A^2). \tag{2-12}$$

Then we can determine the global stability properties of these equilibrium points by plotting $\frac{1}{2}dA^2/dt = f(A^2)$ for $\sigma < \sigma_{-1} < 0$, $\sigma_{-1} < \sigma < 0$, and $\sigma > 0$, respectively, in the three phase-plane plots of Figure 3. From that figure, we can see that 0 is globally stable for $\sigma < \sigma_{-1} < 0$, $A_e^{+2}$ is globally stable for $\sigma > 0$, and in the overlap region where either can be stable, depending on initial conditions, 0 is stable for $0 < A^2(0) < A_e^{-2}$ and $A_e^{+2}$ is stable for $A^2(0) > A_e^{-2}$, while $A_e^{-2}$, which only exists in that bistability region, is not stable there.

To justify this truncation procedure we consider our Landau equation in the form

$$\frac{dA}{dt} = F(A) + O(A^7), \tag{2-13}$$

define $\varepsilon^2 = -a_1$, assume $a_3 = O(1)$ as $\varepsilon \to 0$, and let $\sigma = O(\varepsilon^4)$. Then $A_e^{+2} = O(\varepsilon^2)$, which implies that $A_e^{+} = O(\varepsilon)$. Note, $\alpha = 10^{-2}$ and $\gamma = -2$ yield Landau constants

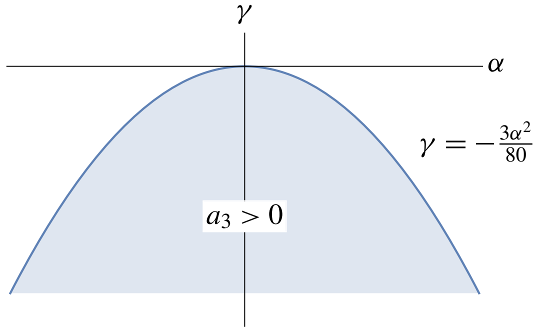


**Figure 4.** Plot of the region in the $\alpha\gamma$-plane, where $a_3 > 0$.

satisfying these conditions. Now, analogous to our approach at third order, we introduce the rescaled variables

$$\eta = \sigma t, \quad \mathcal{A}(\eta) = A(t)/A_e^+, \quad \text{where } \mathcal{A}, \frac{d\mathcal{A}}{d\eta} = O(1) \text{ as } \varepsilon \to 0. \tag{2-14}$$

Since

$$\begin{gathered} \frac{dA}{dt} = \sigma A_e^+ \frac{d\mathcal{A}}{d\eta} = O(\varepsilon^5), \quad \sigma A = \sigma A_e^+ \mathcal{A} = O(\varepsilon^5), \\ a_1 A^3 = a_1 A_e^{+3} \mathcal{A}^3 = O(\varepsilon^5), \quad a_3 A^5 = a_3 A_e^{+5} \mathcal{A}^5 = O(\varepsilon^5), \\ \text{while } O(A^7) = O(A_e^{+7} \mathcal{A}^7) = O(\varepsilon^7) \end{gathered} \tag{2-15}$$

under these conditions, this justifies our truncation procedure at fifth order.

Finally, when $\sigma > 0$, we have the same type of equilibrium solution as depicted in Figure 2, except in this case

$$\delta = \varepsilon_0 + \theta_{31}(1)\varepsilon_0^3 + \theta_{51}(1)\varepsilon_0^5, \quad \text{where } A_e^+ = A_0\varepsilon = \varepsilon_0 \text{ with } A_0 = O(1) \text{ as } \varepsilon \to 0. \tag{2-16}$$

This result depends upon

$$a_3 > 0 \implies \gamma < -\tfrac{3}{80}\alpha^2. \tag{2-17}$$

Recall that, in addition, we have already taken $\alpha > 0$ to guarantee that $a_1 = -\frac{3}{4}\alpha < 0$. That region is plotted in the fourth quadrant of the $\alpha\gamma$-plane of Figure 4. In this context, note from Figure 3 that, unlike the situation depicted in Figure 1 for $\alpha > 0$, all the solutions remain bounded when the fifth-order terms in $r(\theta)$ are retained.

## 3. Bifurcation diagram, ecological interpretations, and conclusions

Should there exist a parameter range in a dynamical systems model of a given phenomenon for which the third-order Landau constant $a_1$ satisfies $a_1 < 0$ and hence the bifurcation is subcritical, the weakly nonlinear stability analysis must be pushed to fifth order as originally pointed out by DiPrima et al. [1971]. This has been standard operating procedure particularly over the last five years when practitioners of the Palermo nonlinear stability theory group began considering fifth-order terms in the Landau equation during their investigation of subcritical bifurcation for a variety of two-component reaction-diffusion systems [Gambino et al. 2010; 2012; Tulumello et al. 2014]. By necessity, such calculations are long and technically complicated. Thus, when surveying the theory, there is some merit in introducing a simple model equation that preserves all the salient features of a more complex system but considerably reduces the labor involved in determining the Landau constants. This was our rationale for considering the generalized Matkowsky equation under investigation. That was also the rationale for Drazin and Reid's [1981] employment of their nondimensionalized version of the Matkowsky equation in order to develop weakly nonlinear theory relevant to hydrodynamic stability. Matkowsky [1970] regarded his problem as a mathematical model for temperature distribution in a finite bar with a nonlinear source term, the ends of which were maintained at the ambient, while Drazin and Reid [1981] offered their corresponding version as a phenomenological model of parallel flow in a channel. Hence, they both envisioned their instabilities to be rate-driven by considering the bifurcation parameter $\beta \sim R_0$. For ecological applications, it is often more relevant to envision these instabilities to be *territory-size* driven by considering $\beta \sim L^2$ and then the instability criterion describes the evolution of spatially heterogeneous structure in a specific domain.

Given that the fifth-order extensions referenced above primarily concentrated only on the subcritical regime, we begin this section by synthesizing our fifth-order results of Figure 3 valid for $a_1 < 0$ or, equivalently, $\alpha > 0$, and $a_3 > 0$ or, equivalently, $3\alpha^2 + 80\gamma < 0$, with those valid for $a_1 > 0$ or, equivalently, $\alpha < 0$, and $a_3 > 0$, as well. Note, that under these conditions, $A_e^{+2} > 0$ for $\sigma > 0$ and $A_e^{-2} < 0$, identically. If we plot a figure analogous to the supercritical cases of Figure 1, it is obvious that the qualitative morphological behavior of those cases is preserved at fifth order with the only change being now $A_e^2 = A_e^{+2}$. We accomplish this synthesis by means of Figure 5, a bifurcation diagram in $\alpha\beta$-space, where the relevant regions associated with these predicted morphological identifications are represented graphically. Since those results also depend on the behavior of $\sigma$, while $\sigma = 0$ and $\sigma = \sigma_{-1}$ are the critical loci for that quantity in this regard, it is necessary for us to generate loci equivalent to them in $\alpha\beta$-space. In this context, using our previous solvability conditions and definitions, we can deduce the following equivalences:

$$\begin{aligned} \sigma = \beta - 1 = 0 \quad &\Longleftrightarrow \quad \beta = 1, \\ \sigma = \beta - 1 = \sigma_{-1} = -\frac{a_1^2}{4a_3} = \frac{18\alpha^2}{3\alpha^2 + 80\gamma} \quad &\Longleftrightarrow \quad \beta = 1 + \frac{18\alpha^2}{3\alpha^2 + 80\gamma}, \end{aligned} \tag{3-1}$$

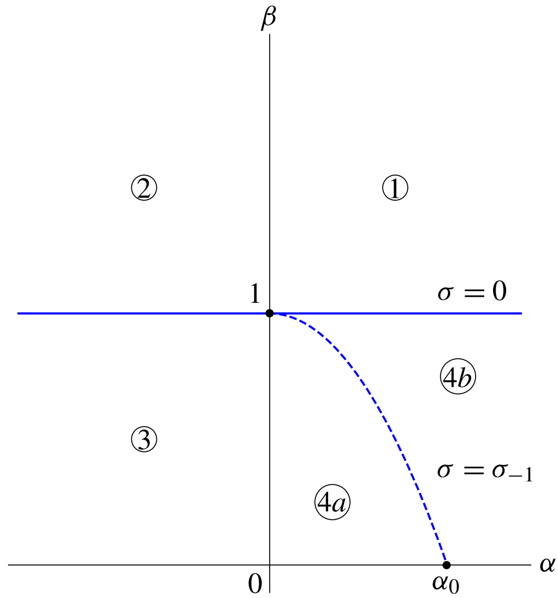


**Figure 5.** Bifurcation diagram in $\alpha\beta$-space with $\sigma_{-1} = -a_1^2/(4a_3)$, $\sigma = \beta - 1$, $a_1 = -\frac{3}{4}\alpha$, and $a_3 = -\frac{5}{8}\gamma - \frac{3}{128}\alpha^2 > 0$, where the circled numbers correspond to the quadrants denoted in Figures 1 and 3.

| quadrant | 1 | 2 | 3 | 4a | 4b |
|---|---|---|---|---|---|
| stable equilibrium point | $A_e^{+2}$ | $A_e^{+2}$ | 0 | 0 | 0<br>$A_e^{+2}$ |

**Table 1.** Stable equilibrium points for $A^2$ in the quadrants of Figure 5.

which are plotted in Figure 5. Here, that first locus is a horizontal line parallel to the $\alpha$-axis which divides our $\alpha\beta$-space into the four quadrants formed by it and the $\beta$-axis, while the second is a concave downward decreasing curve having a horizontal tangent at its $\beta$-intercept of 1 and an $\alpha$-intercept of $\alpha_0 > 0$, where $\alpha_0^2 = -\frac{80}{21}\gamma$, which separates the fourth quadrant of that space into two parts. From an examination of the modification of the supercritical cases of Figure 1 described above and the subcritical cases of Figure 3, we construct Table 1 cataloging the stable equilibrium points for $A^2$ in each of the quadrants of Figure 5.

Note that these fifth-order results for our model equation are much more self-consistent than those obtained in the case of its third-order truncation, in that, the behavior for the subcritical quadrants 1 and 4a now exactly resemble the behavior for the supercritical quadrants 2 and 3, respectively. In the subcritical quadrant 4b, we have what biologists refer to as metastability, in that, the 0 equilibrium point is

| quadrant | 1 | 2 | 3 | 4a | 4b |
|---|---|---|---|---|---|
| stable pattern | arch | arch | dense hom. | sparse hom. | sparse hom.<br>arch |

**Table 2.** Morphological stability predictions for Table 1.

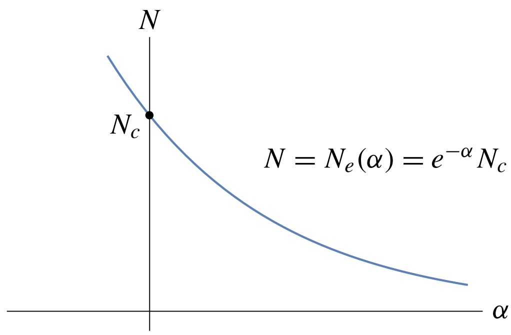


**Figure 6.** Plot of the population equilibrium density $N_e$ versus $\alpha$.

stable to initially small disturbances, but the model will switch to the equilibrium point $A_e^{+2}$ for sufficiently large ones. The existence of such a region of metastability allows our model equation to exhibit outbreak behavior wherein the maximum population level increases several-fold upon a sufficient initial perturbation in amplitude.

Returning to our original dimensional formulation (1-1), the fact that $A^2 = 0$ represents a globally stable equilibrium point implies that

$$\lim_{\tau \to \infty} N(s, \tau) = N_e. \tag{3-2}$$

Hence this solution represents a homogeneous population. In many actual biological systems, such as the interaction-diffusion plant-groundwater one employed by Chaiya et al. [2015] to model vegetative pattern formation in a flat arid environment, the homogeneous patterns in the subcritical parameter range correspond to relatively sparse distributions, while most of those patterns in the supercritical range correspond to much denser distributions, where the threshold between these two types of distributions occurs at some $N_c$. We can induce this sort of behavior in our model equation by adopting the relationship

$$N_e = N_e(\alpha) = N_c e^{-\alpha}, \tag{3-3}$$

which is plotted in Figure 6. Then from this relation and Table 1 in conjunction with Figure 2, we can deduce the stable pattern predictions given in Table 2 for the quadrants of Figure 5.

In [Chaiya et al. 2015], it was conjectured that the region of parameter space of subcriticality, where $a_1 < 0$, corresponded to isolated vegetative patches when $\sigma > 0$ and low-density homogeneous distributions when $\sigma < 0$, as opposed to the occurrence of periodic patterns for $\sigma > 0$ and high-density homogeneous distributions when $\sigma < 0$, where $a_1 > 0$, which were already predicted by their rhombic-planform two-dimensional nonlinear stability analysis. Such isolated patches are a compromise between periodic patterns and homogeneous stable states that are sparse enough to resemble bare ground. They then associated equilibrium points 0 and $A_e^{+2}$ of quadrants 1 and 4 of Table 1 with the sparse homogeneous state and the isolated patch, respectively, that would occur in a postulated fifth-order extension, should $a_3 > 0$ for this parameter range. Our fifth-order results summarized in Table 2 represent the first step in a conclusive demonstration of the validity of this conjecture.

We conclude by noting that although these results are only strictly asymptotically valid in a neighborhood of the marginal stability curve $\beta = 1$, Boonkorkuea et al. [2010], by comparing their theoretical predictions of this sort with existing numerical simulations of vegetative pattern formation for a model evolution equation, recently showed that the former can often be extrapolated to those regions of parameter space relatively far from the marginal curve. These theoretical predictions also associated that region of parameter space, where numerical simulation generated isolated patches, with $\sigma > 0$ and $a_1 < 0$.

Finally, we close by offering, for the sake of definiteness, a closed-form representation of $r(\theta)$, composed of combinations of common functions that produce Landau constants consistent in sign with our subcriticality assumptions. Recall the following Maclaurin polynomials truncated through terms of fifth order:

$$\sinh(z) \sim z + \tfrac{1}{6}z^3 + \tfrac{1}{120}z^5 \quad \text{and} \quad \arctan(z) \sim z - \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5. \tag{3-4}$$

Then

$$\begin{aligned} 4\sinh\left(\tfrac{1}{2}\theta\right) &\sim 4\left(\tfrac{1}{2}\theta + \tfrac{1}{48}\theta^3 + \tfrac{1}{3840}\theta^5\right) = 2\theta + \tfrac{1}{12}\theta^3 + \tfrac{1}{960}\theta^5, \\ 2\arctan\left(\tfrac{1}{2}\theta\right) &\sim 2\left(\tfrac{1}{2}\theta - \tfrac{1}{24}\theta^3 + \tfrac{1}{160}\theta^5\right) = \theta - \tfrac{1}{12}\theta^3 + \tfrac{1}{80}\theta^5. \end{aligned} \tag{3-5}$$

Now, defining $r(\theta)$ to be the difference between these two functions, we obtain

$$\alpha = \tfrac{1}{6} > 0, \ \gamma = -\tfrac{11}{960} \quad \text{such that} \ \ 80\gamma + 3\alpha^2 = -\tfrac{11}{12} + \tfrac{1}{12} = -\tfrac{5}{6} < 0. \tag{3-6}$$

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Received: 2016-10-15 Revised: 2017-01-27 Accepted: 2017-02-04

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2018 vol. 11 no. 2



1944-4176(2018)11:2;1-7