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# Six variations on a theme: almost planar graphs

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A graph is *apex* if it can be made planar by deleting a vertex, that is, there exists  $v$  such that  $G - v$  is planar. We also define several related notions; a graph is *edge apex* if there exists  $e$  such that  $G - e$  is planar, and *contraction apex* if there exists  $e$  such that  $G/e$  is planar. Additionally we define the analogues with a universal quantifier: for all  $v$ ,  $G - v$  is planar; for all  $e$ ,  $G - e$  is planar; and for all  $e$ ,  $G/e$  is planar. The graph minor theorem of Robertson and Seymour ensures that each of these six notions gives rise to a finite set of obstruction graphs. For the three definitions with universal quantifiers we determine this set. For the remaining properties, apex, edge apex, and contraction apex, we show there are at least 36, 55, and 82 obstruction graphs respectively. We give two similar approaches to almost nonplanar (there exists  $e$  such that  $G + e$  is nonplanar, and for all  $e$ ,  $G + e$  is nonplanar) and determine the corresponding minor minimal graphs.

## 1. Introduction

Kuratowski [1930] showed that the set of planar graphs is determined by two obstructions.

**Theorem 1.1** [Kuratowski 1930; Wagner 1937]. *A graph is planar if and only if it has neither  $K_5$  nor  $K_{3,3}$  as a minor.*

We give the formulation in terms of minors due to Wagner [1937] to make the connection with Robertson and Seymour's [2004] graph minor theorem. We say  $H$  is a *minor* of graph  $G$  if it can be obtained by contracting edges in a subgraph of  $G$ . We can state the graph minor theorem as follows.

**Theorem 1.2** [Robertson and Seymour 2004]. *In any infinite set of graphs, there is a pair such that one is a minor of the other.*

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This has two useful consequences. We say  $G$  is *minor minimal*  $\mathcal{P}$  (or  $\text{MMP}$ ) if  $G$  has property  $\mathcal{P}$  but no proper minor does.

**Corollary 1.3.** *For any graph property  $\mathcal{P}$ , there is a corresponding finite set of minor minimal  $\mathcal{P}$  graphs.*

**Corollary 1.4.** *Let  $\mathcal{P}$  be a graph property that is closed under taking minors. Then there is a finite set of minor minimal non- $\mathcal{P}$  graphs  $S$  such that for any graph  $G$ ,  $G$  satisfies  $\mathcal{P}$  if and only if  $G$  has no minor in  $S$ .*

When  $\mathcal{P}$  is minor closed, we say that  $S$  is the *Kuratowski set* for  $\mathcal{P}$ . For example,  $\{K_5, K_{3,3}\}$  is the Kuratowski set for planarity.

The graph minor theorem is not constructive, so there are only a few graph properties  $\mathcal{P}$  for which we know the finite set of  $\text{MMP}$  graphs. In particular, there are several graph properties closely related to planarity for which this set is unknown. Our goal in this paper is to investigate the minor minimal sets for the following eight graph properties.

**Definition 1.5.** A planar graph is *almost nonplanar* (AN) if there exist two nonadjacent vertices such that adding an edge between the vertices yields a nonplanar graph. A planar graph is *completely almost nonplanar* (CAN) if it is not complete and adding an edge between any pair of nonadjacent vertices yields a nonplanar graph.

Let  $G - v$  denote the graph resulting from deletion of vertex  $v$  and its edges in  $G$ , let  $G - e$  denote the graph resulting from the deletion of edge  $e$  in  $G$ , and let  $G/e$  denote the graph resulting from the contraction of edge  $e$  in  $G$ .

**Definition 1.6.** A graph is *not apex* (NA) if, for all vertices  $v$ ,  $G - v$  is nonplanar. Similarly, a graph is *not edge apex* (NE) if, for all edges  $e$ ,  $G - e$  is nonplanar and *not contraction apex* (NC) if, for all edges  $e$ ,  $G/e$  is nonplanar.

**Definition 1.7.** A graph  $G$  is *incompletely apex* (IA) if there is a vertex  $v$  such that  $G - v$  is nonplanar, *incompletely edge apex* (IE) if there is an edge  $e$  such that  $G - e$  is nonplanar, and *incompletely contraction apex* (IC) if there is an edge  $e$  such that  $G/e$  is nonplanar.

We call these last three properties “incomplete” in contrast to their negations. For example, we think of a graph as “completely” apex if  $G - v$  is planar for every vertex  $v$ . Table 1 gives a summary of our eight definitions.

We summarize our results in Table 2. Four of the properties give Kuratowski sets (as their negation generates a minor closed set) and with the exception of NA, NE, and NC, we determine the finite set of  $\text{MMP}$  graphs. For the remaining three properties we give a lower bound, which is simply the number of  $\text{MMP}$  graphs we have found, so far.

Our paper is organized as follows. Below we conclude this introduction with a survey of the literature and provide some preliminary notions used throughout the

property	definition
AN	$\exists e$ such that $G + e$ is nonplanar, where $G$ is planar
CAN	$\forall e$ , $G + e$ is nonplanar, where $G$ is planar, not complete
NA	$\forall v$ , $G - v$ is nonplanar
NE	$\forall e$ , $G - e$ is nonplanar
NC	$\forall e$ , $G/e$ is nonplanar
IA	$\exists v$ such that $G - v$ is nonplanar
IE	$\exists e$ such that $G - e$ is nonplanar
IC	$\exists e$ such that $G/e$ is nonplanar

**Table 1.** Comparison of the eight definitions.

graph property $\mathcal{P}$	AN	CAN	NA	NE	NC	IA	IE	IC
Is (not $\mathcal{P}$ ) minor closed?	no	no	yes	no	no	yes	yes	yes
number of $\text{MM}\mathcal{P}$ graphs	2	1	$\geq 36$	$\geq 55$	$\geq 82$	2	5	7

**Table 2.** Results for the eight graph properties.

paper. In [Section 2](#) we determine the MMAN and MMCAN graphs and show that neither is a Kuratowski set. In [Section 3](#) we give our classification of the MMIA, MMIE, and MMIC graphs, all three of which we show are Kuratowski. In [Section 4](#) we give an overview of the MMNA graphs, which is a Kuratowski set. We classify graphs in this family of connectivity at most 1. For graphs of connectivity 2, with  $\{a, b\}$  a 2-cut, we classify those for which  $ab \in E(G)$ , as well as those for which a component of  $G - a, b$  is nonplanar. We also prove that an MMNA graph has connectivity at most 5. In total, we give explicit constructions for 36 MMNA graphs. Finally, in [Section 5](#) we discuss MMNE and MMNC graphs, first showing these are not Kuratowski. We classify graphs of connectivity at most 1 in these two families and discuss computer searches, complete through graphs of order 9 or size 19, that yielded 55 MMNE and 82 MMNA graphs.

Apex graphs are well-studied, including results on MMNA graphs in [[Ayala 2014](#); [Barsotti and Mattman 2016](#); [Pierce 2014](#)]. Note that [[Pierce 2014](#)] reports on a computer search that yields 157 MMNA graphs, including all graphs through order 10 or size 21 and most of the 36 graphs we describe here. Different authors have used terms like “almost planar” or “near planar” in various ways. Here is how our definitions relate to others in the literature. Cabello and Mohar [[2013](#)] say that a graph is *near-planar* if it can be obtained from a planar graph by adding an edge. This corresponds to our definition of edge apex. Wagner [[1967](#)] defined *nearly planar* (*Fastplättbare*), which corresponds to our idea of completely apex

or not IA. Two further notions of almost planar are not directly related to the properties we have defined. For Gubser [1996], a graph  $G$  is *almost planar* if for every edge  $e$ , either  $G - e$  or  $G/e$  is planar. In characterizing graphs with no  $K^{\infty_0}$ , Diestel, Robertson, Seymour, and Thomas say a graph  $G$  is *nearly planar* if deleting a bounded number of vertices makes  $G$  planar except for a subgraph of bounded linear width sewn onto the unique cuff of  $S^2 - 1$ ; see [Diestel 2010, Section 12.4]. Finally, our notion of CAN is also known as *maximally planar*; see [Diestel 2010].

We conclude this introductory section with some notation and definitions, as well as a lemma, used throughout. For us, graphs are simple (no loops or double edges) and undirected. We use  $V(G)$  and  $E(G)$  to denote the vertices and edges of a graph. The *order* of a graph is  $|V(G)|$  and  $|E(G)|$  is its *size*. We use  $\delta(G)$  to denote the *minimum degree* of all the vertices in  $G$ .

As mentioned earlier,  $G - v$ ,  $G - e$ , and  $G/e$  denote the results of vertex deletion, edge deletion, and edge contraction, respectively. For  $v, w \in V(G)$ , the graph  $G - v, w$  is the result of deleting two vertices and their edges. Similarly, for  $e, f \in E(G)$ , we define as  $G - e, f$  the result of deleting two edges and  $G/e, f$  the result of contracting two edges. Note that the order of deletion or contraction is arbitrary. Contracting an edge may result in a double edge. We will assume that one of the doubled edges is deleted so that  $G/e$  is again a simple graph. We use  $G_1 \sqcup G_2$  to denote the disjoint union of two graphs and  $G_1 \dot{\cup} G_2$  for the union identified on a single vertex. Similarly,  $G_1 \ddot{\cup} G_2$  denotes the union of two graphs identified on two vertices.

In light of Kuratowski's theorem, we call  $K_5$  and  $K_{3,3}$  the *Kuratowski graphs* and also refer to them as minor minimal nonplanar or MMNP. A *Kuratowski subgraph* or *K-subgraph* of  $G$  is one homeomorphic to a Kuratowski graph. A *cut set* of graph  $G$  is a set  $U \subset V(G)$  such that deleting the vertices of  $U$  and their edges results in a disconnected graph. If  $|U| = k$ , we call  $U$  a *k-cut*. We say  $G$  has *connectivity*  $k$  and write  $\kappa(G) = k$  if  $k$  is the largest integer such that  $|V(G)| > k$  and  $G$  has no  $l$ -cut for  $l < k$ . In particular,  $\kappa(K_n) = n - 1$ .

We conclude this introduction with a useful lemma. In the case that  $\kappa(G) = 2$ , we have  $G - a, b = G'_1 \sqcup G'_2$ , where  $\{a, b\}$  is a 2-cut. We will use  $G_i$  to denote the induced subgraph on  $V(G'_i) \cup \{a, b\}$ . In the literature, e.g., [Mohar and Thomassen 2001], the pair  $(G_1, G_2)$  is called a separation of order 2 (since  $|G_1 \cap G_2| = 2$ ).

**Lemma 1.8.** *If  $G$  is homeomorphic to  $K_5$  or  $K_{3,3}$  with cut set  $\{a, b\}$  such that  $G - a, b = G'_1 \sqcup G'_2$ , then one of  $G_1$  and  $G_2$  is an  $a$ - $b$ -path.*

*Proof.* Since,  $\kappa(K_5) = 4$  and  $\kappa(K_{3,3}) = 3$ ,  $G$  must be a proper subdivision of a Kuratowski graph and, since they disconnect the graph,  $a$  and  $b$  are vertices on a subdivided edge of the underlying  $K_5$  or  $K_{3,3}$ . This means that one of the components is simply an  $a$ - $b$ -path.  $\square$

## 2. Almost nonplanar: MMAN and MMCAN graphs

In this section we classify the MMAN and MMCAN graphs. Let  $K_5 - e$  denote the complete graph on five vertices with an edge deleted and  $K_{3,3} - e$  the result of deleting an edge in the complete bipartite graph  $K_{3,3}$ . The unique MMCAN graph is  $K_5 - e$  and there are two MMAN graphs,  $K_5 - e$  and  $K_{3,3} - e$ . Neither of these are Kuratowski sets, since, for example,  $K_5$  is a nonplanar graph (hence neither AN nor CAN) that contains the MMAN and MMCAN graph  $K_5 - e$  as a minor.

Our classification of the minor minimal CAN graphs makes use of a theorem due to Mader.

**Theorem 2.1** [Mader 1998]. *Any graph with  $n$  vertices and at least  $3n - 5$  edges contains a subdivision of  $K_5$ .*

In [Diestel 2010], CAN is called *maximally planar*, and it is proved equivalent to a graph admitting a plane triangulation in Proposition 4.2.8 of that text.

**Theorem 2.2.** *Every plane triangulation with at least five vertices has  $K_5 - e$  as a minor.*

*Proof.* Let  $G$  be a plane triangulation on at least five vertices. By Euler's formula,  $|E(G)| = 3(|V(G)| - 2)$ . Let  $G'$  be a nonplanar graph obtained by adding edge  $ab$  to  $G$ . Then  $|E(G')| = |E(G)| + 1 = 3|V(G)| - 5$ . By Mader's theorem  $G'$  has a subgraph  $H$  homeomorphic to  $K_5$ . Note that we must have  $ab \in E(H)$ , else  $H$  would be planar. Since  $H$  is homeomorphic to  $K_5$ , contracting appropriate edges in  $H - ab$  will result in  $K_5 - e$ , showing that  $K_5 - e$  is a minor of  $G$ .  $\square$

**Corollary 2.3.** *The only MMCAN graph is  $K_5 - e$ .*

**Theorem 2.4.** *The MMAN graphs are  $K_5 - e$  and  $K_{3,3} - e$ .*

*Proof.* First note that these two graphs are MMAN. Let  $G$  be AN and let  $ab$  be the edge that is added to form the nonplanar  $G'$ . By Kuratowski's theorem  $G'$  contains a subdivision  $H$  of  $K_5$  or  $K_{3,3}$  and  $ab \in E(H)$ . By contracting edges,  $H$  gives  $K_5 - e$  or  $K_{3,3} - e$  as a minor of  $G$ . So  $G$  is MMAN only if it is one of these two.  $\square$

## 3. Incomplete properties: MMIA, MMIE, and MMIC graphs

In this section we classify the MMIA, MMIE, and MMIC graphs. Note that each is a Kuratowski set since the corresponding "complete" property is minor closed. In the case of the IA graphs, for example, suppose  $G$  is not IA and let  $H$  be a subgraph of  $G$ . Then for any  $v \in V(H)$ , the graph  $H - v$  is planar since it is a subgraph of the planar graph  $G - v$ . Similarly if  $G$  is not IA, let  $H = G/f$  for some  $f \in E(G)$ . Then for any  $v \in V(H)$ , the graph  $H - v$  is planar since it is a minor of the planar graph  $G - v$ . This shows that the property *not* IA (also known as the completely apex property) is minor closed. Similar arguments show that not IE and not IC are also minor closed.

We next show there are exactly two MMIA graphs,  $K_1 \sqcup K_5$  and  $K_1 \sqcup K_{3,3}$ . We begin by classifying the disconnected graphs.

**Theorem 3.1.** *If  $G$  is not connected and MMIA, then  $G = K_1 \sqcup G_2$ , where  $G_2 \in \{K_5, K_{3,3}\}$ .*

*Proof.* Note that both  $K_1 \sqcup K_5$  and  $K_1 \sqcup K_{3,3}$  are MMIA. If  $G = G_1 \sqcup G_2$  is nonplanar with neither component empty, then  $K_5$ , or  $K_{3,3}$  is a minor of one of  $G_1$  and  $G_2$ . By minor minimality this means one of  $G_1$  and  $G_2$  is a Kuratowski graph, and, again by minimality, the other component can have no nontrivial proper minors, so must be simply a vertex. □

**Theorem 3.2.** *There are no connected MMIA graphs.*

*Proof.* Suppose instead that  $G$  is a connected MMIA graph. Then there is a vertex,  $v$ , such that  $G - v$  is nonplanar. However, since  $G$  is connected,  $v$  must have at least one edge,  $e$ . Since when deleting a vertex we also delete all of its edges,  $G - e$  must be a proper, nonplanar minor of  $G$ . However, deleting  $v \in V(G - e)$  is again nonplanar so that  $G - e$  is IA. This contradicts the property that  $G$  is MMIA and therefore cannot happen. □

**Corollary 3.3.** *There are two MMIA graphs:  $K_1 \sqcup K_5$  and  $K_1 \sqcup K_{3,3}$ .*

Next we show there are five MMIE graphs. We begin with the disconnected examples. Note that if  $G$  has distinct edges  $e, e'$  such that  $G - e, e'$  is nonplanar, then  $G$  is not MMIE. Indeed,  $G - e$  is an IE proper minor.

**Theorem 3.4.** *If  $G$  is not connected and MMIE, then  $G = K_2 \sqcup G_2$ , where  $G_2 \in \{K_5, K_{3,3}\}$ .*

*Proof.* The proof is similar to that of [Theorem 3.1](#), but now the planar component is minor minimal among graphs with edges, so  $K_2$ . □

Recall that  $G_1 \dot{\cup} G_2$  denotes the union of  $G_1$  and  $G_2$  with one vertex in common.

**Theorem 3.5.** *If  $G$  is connected, MMIE, and has a cut vertex, then  $G = K_2 \dot{\cup} G_2$ , where  $G_2 \in \{K_5, K_{3,3}\}$ .*

*Proof.* Let  $G$  be a connected MMIE graph such that  $G - v = G'_1 \sqcup G'_2$ . Let  $G_i$  denote the induced subgraph on  $V(G'_i) \cup \{v\}$ . If both  $G_1$  and  $G_2$  are nonplanar, then  $G$  would not be MMIE since, for example, there are two distinct edges  $e, e' \in E(G_2)$  such that  $G - e, e'$  contains  $G_1$  and is therefore nonplanar. If both subgraphs were planar, then  $G$  would also be planar and therefore not MMIE. So one of  $G_1$  and  $G_2$  is nonplanar, say  $G_1$ , and the other,  $G_2$ , is planar.

By minor minimality of  $G$ , the nonplanar  $G_2$  is, in fact, a Kuratowski graph, and the planar  $G_1$  is minimal among graphs with edges, i.e.,  $K_2$ . □

**Theorem 3.6.** *If  $G$  is MMIE, then there is a unique edge  $e$  such that  $G - e$  is nonplanar.*

*Proof.* Assume, for the sake of contradiction, that there are  $e, e' \in E(G)$  such that  $e \neq e'$  but  $G - e$  and  $G - e'$  are nonplanar. If  $G - e$  is nonplanar, then there is a subgraph of  $G - e$ , say  $H$ , with  $e \notin E(H)$ , that has a  $K_5$  or  $K_{3,3}$  minor. Likewise, if  $G - e'$  is nonplanar, then it has a nonplanar subgraph  $H'$  with  $e' \notin E(H')$ . If  $H' = H$ , then  $e' = e$ . Otherwise,  $G - e, e'$  would be nonplanar, contradicting that  $G$  is MMIE. So  $H' \neq H$ . If  $e \notin H'$ , then  $G - e, e'$  contains  $H'$  and will be nonplanar, contradicting that  $G$  is MMIE.

So,  $e \in H'$  and, similarly,  $e' \in H$ . If  $H$  and  $H'$  have empty intersection, then let  $e_1, e_2 \in E(H')$ . This means  $G - e_1, e_2$  contains  $H$  and is nonplanar. This contradicts that  $G$  is MMIE. So,  $H$  and  $H'$  have nonempty intersection. If their intersection is nonplanar, then removing  $e$  and  $e'$  will not change this intersection, and  $G$  is not MMIE. If their intersection is planar, then there must be more than one edge in  $H'$  that is not in  $H$  besides  $e$ . But, if  $H'$  has more edges besides  $e$  that are not in  $H$  it would be possible to remove another edge,  $f \neq e$ , without changing  $H$ . This means that  $G - f, e$  is nonplanar, and contradicts that  $G$  is MMIE.

Therefore, if  $G$  is MMIE, then there is a unique edge  $e$  such that  $G - e$  is nonplanar.  $\square$

Recall that a K-subgraph is one homeomorphic to  $K_5$  or  $K_{3,3}$ .

**Theorem 3.7.** *If  $G$  is MMIE, then the edge  $e$  such that  $G - e$  is nonplanar is not in a K-subgraph. Furthermore,  $G - e$  is  $K_5$  or  $K_{3,3}$ .*

*Proof.* Assume, for the sake of contradiction, that  $e$  is in a K-subgraph,  $H$ . Since no graph homeomorphic to  $K_5$  or  $K_{3,3}$  is IE,  $G - e$  is planar unless  $G - e$  contains some other K-subgraph,  $H'$ . However, if  $G$  contains two K-subgraphs  $H$  and  $H'$  with empty intersection, then  $G - e$  will leave  $H'$  unchanged. One could then remove a second edge,  $f \in E(H)$ , leaving  $H'$  unchanged so that  $G - e, f$  is nonplanar. This means that  $G$  cannot be MMIE since  $G$  would have an IE minor  $G - e$ . So,  $H$  and  $H'$  have nonempty intersection. But  $H \neq H'$  since  $e$  cannot be an edge in the only K-subgraph, otherwise  $G - e$  is planar.

Next, observe that any proper subgraph of a K-subgraph is planar. This means that for the K-subgraph,  $H'$ , with  $H \neq H'$ , there must be an edge,  $g \neq e$ , with  $g \in E(H')$  and  $g \notin E(H)$ . Then  $G - g$  contains  $H$  and is nonplanar. This contradicts the uniqueness of the edge  $e$  and shows  $e$  is not in a K-subgraph.

Following the same argument as above,  $G$  cannot contain more than one K-subgraph. Indeed, if there were distinct K-subgraphs  $H$  and  $H'$ , then either the intersection is empty or it is not, and we achieve similar contradictions as in the previous argument. So,  $G$  contains exactly one K-subgraph.



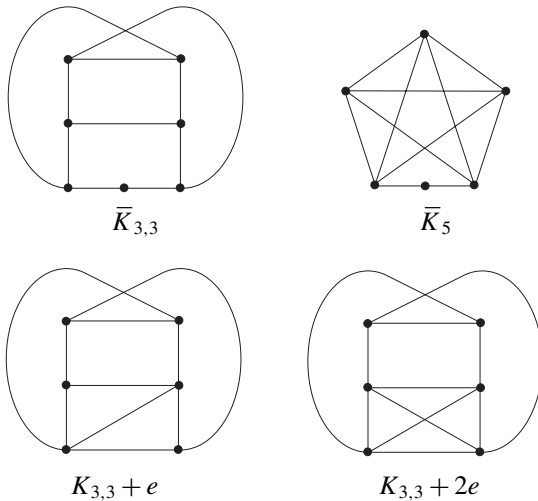
Finally, the only possible K-subgraph contained in  $G$ , call it  $N$ , must contain all edges besides  $e$ . If not, then there is an edge  $e' \neq e$  such that  $G - e'$  is nonplanar. This contradicts the uniqueness of  $e$ . Also, the K-subgraph  $N$  in  $G - e$  must be either  $K_5$  or  $K_{3,3}$ . If not, then  $N$  would be a subdivision of either  $K_5$  or  $K_{3,3}$ . But, then there is a proper minor,  $G'$ , of  $G$ , by contracting an edge,  $e_1 \in E(N)$ , which contains a K-subgraph as well. Provided  $e$  remains as an edge of  $G'$ , the graph  $G' - e$  is nonplanar, contradicting that  $G$  is minor minimal. On the other hand, if contracting  $e_1$  removes  $e$ , then there must be another edge  $e_2$  incident to  $e_1$ , with  $e_2 \in E(N)$ , such that  $e$  is incident to both  $e_1$  and  $e_2$ . Since  $N$  is a subdivision of  $K_5$  or  $K_{3,3}$  and  $G/e_1$  is nonplanar,  $e_1$  and  $e_2$  must be in a path of  $N$  formed by subdividing an edge of the underlying Kuratowski graph. Since  $e$  is incident to both  $e_1$  and  $e_2$ , there exists  $N'$ , another K-subgraph of  $G$  with  $e \in E(N')$ . This contradicts that there is only one K-subgraph of  $G$ .

So, if  $G$  is MMIE then it is made up of either  $K_5$  or  $K_{3,3}$  and an edge that is not in this K-subgraph. □

Aside from the disconnected and connectivity-1 examples above, a final way to add an edge to a K-subgraph is the graph  $K_{3,3} + e$  of Figure 1, formed by adding an edge to the bipartite graph  $K_{3,3}$ .

**Corollary 3.8.** *There are five MMIE graphs:  $K_{3,3} + e$  and  $K_2 \sqcup G_2$ ,  $K_2 \dot{\cup} G_2$ , where  $G_2 \in \{K_5, K_{3,3}\}$ .*

Let  $\bar{K}_5$  and  $\bar{K}_{3,3}$  denote the graphs obtained from  $K_5$  and  $K_{3,3}$  by subdividing a single edge, as in Figure 1. We denote as  $K_{3,3} + 2e$  the graph given by adding two edges to  $K_{3,3}$ , as in Figure 1.



**Figure 1.** MMIE and MMIC graphs.

**Theorem 3.9.** *There are seven MMIC graphs:  $K_{3,3} + 2e$  and  $\bar{K}$ ,  $K_2 \sqcup K$ , and  $K_2 \dot{\cup} K$  with  $K \in \{K_5, K_{3,3}\}$ .*

*Proof.* Observe that these seven graphs are MMIC. If  $G$  is MMIC and disconnected, then  $G$  is  $K_2 \sqcup K$ , with  $K$  a Kuratowski graph. We omit the proof, which is similar to that for MMIE. Note that the remaining five graphs are precisely the graphs that result when a vertex of a Kuratowski graph is split.

Suppose  $G$  is MMIC and connected. Then there is an edge  $e$  such that  $G/e$  is nonplanar. Since contracting an edge will not disconnect the graph,  $G/e$  is also connected and has a  $K$ -subgraph  $H$ . If  $H$  is not a Kuratowski graph, then it has  $\bar{K}_5$  or  $\bar{K}_{3,3}$  as a minor, contradicting  $G$  being minor minimal. Therefore,  $H$  is Kuratowski.

If  $V(H) \neq V(G/e)$ , then since  $G/e$  is connected, considering any vertex in  $G/e$  beyond those in  $H$ , along with one of its edges, shows that  $G/e$  contains  $K_2 \sqcup K$  or  $K_2 \dot{\cup} K$ , with  $K$  Kuratowski, contradicting  $G$  being minor minimal. So,  $V(H) = V(G/e)$ .

Now  $G$  is obtained from  $G/e$  by a vertex split. The corresponding vertex split on  $H$  gives rise to a graph  $H'$ , which is one of the five graphs  $K_{3,3} + 2e$ ,  $\bar{K}$ , or  $K_2 \dot{\cup} K$ . Since  $G$  is minor minimal,  $G = H'$  and is one of these five, and hence one of the seven.  $\square$

#### 4. MMNA graphs

In this section we describe several partial results toward a classification of the MMNA graphs, with a focus on graph connectivity. In all, we describe 36 MMNA graphs, including all those of connectivity at most 1 ( $\kappa(G) \leq 1$ ). For graphs with  $\kappa(G) = 2$ , where  $\{a, b\}$  is a 2-cut, we classify the MMNA graphs having  $ab \in E(G)$ , as well as those for which a component of  $G - a, b$  is nonplanar. We also show that  $\kappa(G) \leq 5$  for MMNA graphs, which is a sharp bound. Since the family of apex graphs is minor closed, the MMNA graphs are a Kuratowski set.

We first bound the minimum degree,  $\delta(G)$ , of an MMNA graph and then classify the examples with  $\kappa(G) \leq 1$ .

**Theorem 4.1.** *The minimum vertex degree in an MMNA graph is at least 3.*

*Proof.* The addition or deletion of an isolated vertex or vertex of degree 1 in a planar graph will again result in a planar graph. Similarly, contracting an edge adjacent to a degree-2 vertex will not affect planarity. So if  $G$  is NA with  $\delta(G) < 3$ , then removing a vertex of small degree will result in a NA graph; hence  $G$  is not MMNA.  $\square$

**Theorem 4.2.** *There are three disconnected MMNA graphs:  $K_5 \sqcup K_5$ ,  $K_5 \sqcup K_{3,3}$ , and  $K_{3,3} \sqcup K_{3,3}$ .*

*Proof.* First observe that these three graphs are all MMNA. On the other hand, if  $G = G_1 \sqcup G_2$  is MMNA, both components must be nonplanar. Otherwise if  $G_1$  is planar, then  $G_2$  must be NA and is a proper minor of  $G$ , contradicting  $G$  being MMNA. So each component  $G_i$  has a  $K_5$  or  $K_{3,3}$  minor and  $G$  has one of the three candidates as a minor. Since  $G$  is minor minimal, it must be one of the three candidates.  $\square$

**Theorem 4.3.** *There are no MMNA graphs of connectivity 1.*

*Proof.* Suppose instead  $G$  is MMNA with cut vertex  $a$ . Then  $G - a = G'_1 \sqcup G'_2$ . If both  $G'_1$  and  $G'_2$  are planar, then  $G - a$  is planar, contradicting that  $G$  is NA. If both are nonplanar, then  $G$  has one of the disconnected MMNA graphs as a proper minor and is not minor minimal. So, one of  $G'_1$  and  $G'_2$ , say  $G'_1$ , is planar, and the other,  $G'_2$ , is not. Let  $G_i$  denote the induced graph on  $V(G'_i) \cup \{a\}$ . If  $G_1$  is nonplanar, then together with  $G'_2$  this gives one of the three disconnected MMNA graphs as a proper minor of  $G$ , contradicting that  $G$  is minor minimal. So  $G_1$  is planar. But then  $G_2$  must be NA, which again contradicts  $G$  being minor minimal.  $\square$

We can also give an upper bound on the connectivity of an MMNA graph. We first bound the minimum degree  $\delta(G)$ .

**Theorem 4.4.** *If  $G$  is MMNA, then  $\delta(G) \leq 5$ .*

*Proof.* Suppose  $G$  is MMNA and, for a contradiction, that  $\delta(G) \geq 6$ . Let  $D$  be the largest integer so that there are two vertices  $a, b \in V(G)$  both of degree at least  $D$ . Surely,  $D \geq 6$ . We will argue that there are two vertices with degree at least  $D + 2$ , contradicting our choice of  $D$ . Let  $v = |V(G)|$  be the number of vertices of  $G$ . There will be  $v - 2$  vertices of degree at least 6 and two vertices of degree at least  $D$ . A lower bound on the number of edges of  $G$  is then  $(6(v - 2) + 2D)/2 = 3v - 6 + D$ .

Since  $G$  is MMNA, we can form a planar graph by deleting an edge (to get a proper minor) and then an apex vertex, which is not adjacent to the deleted edge. For if it were adjacent to the edge, the vertex deletion would also remove the edge, making  $G$  apex, a contradiction.

After deleting an edge,  $G - e$  has at least  $3v - 7 + D$  edges. Next delete a vertex,  $a \in V(G)$  of degree  $d$ . Then the lower bound on the number of edges in the resulting planar graph is  $3v - 7 + D - d$ . As this graph is planar on  $v - 1$  vertices, an upper bound on the number of edges is  $3(v - 1) - 6$ , the number of edges in a triangulation. Thus  $3v - 7 + D - d \leq 3(v - 1) - 6$ , which implies  $d \geq D + 2$ .

This means the degree of  $a$  is at least  $D + 2$ . However, following the argument above, if we first delete an edge incident to  $a$ , we deduce that there is a second vertex  $b$  that is again of degree at least  $D + 2$ . This is a contradiction since  $D$  was assumed to be the maximum such that two vertices have degree at least  $D$ . Therefore, if  $\delta(G) \geq 6$ , then  $G$  is not MMNA.  $\square$

Since  $\kappa(G) \leq \delta(G)$ , we have a bound on connectivity as an immediate corollary.

**Corollary 4.5.** *If  $G$  is MMNA, then  $\kappa(G) \leq 5$ .*

Note that  $K_6$  is an MMNA graph of connectivity 5, so this bound is sharp. Indeed,  $K_6$  is part of the Petersen family, a family of seven graphs shown to be MMNA by Barsotti and Mattman [2016]. Other graphs in this family provide examples of graphs of connectivity 4 ( $K_{3,3,1}$ ) and connectivity 3 ( $K_{4,4} - e$  and the Petersen graph) and the computer search of [Pierce 2014] unearthed numerous further examples with connectivity greater than 2.

Nonetheless, in the remainder of this section, we restrict attention to MMNA graphs of connectivity 2. Let us fix some notation for this situation. For  $G$  MMNA with cut set  $\{a, b\}$ , we have  $G - a, b = G'_1 \sqcup G'_2$ . Let  $G_i$  denote the induced subgraph on  $V(G'_i) \cup \{a, b\}$  so that  $(G_1, G_2)$  is a separation of order 2.

**Theorem 4.6.** *Let  $G$  be an MMNA graph where  $\kappa(G) = 2$ , with cut set  $\{a, b\}$ . If  $G - a, b = G'_1 \sqcup G'_2$ , then  $G'_1$  and  $G'_2$  are not both nonplanar.*

*Proof.* Let  $c_a$  be an apex of  $G - a$ . By the assumption that  $G$  is MMNA,  $G - a, c_a$  is planar. If  $c_a = b$ , we are done because  $G'_1 \sqcup G'_2 = G - a, b = G - a, c_a$ , which would imply both  $G'_1$  and  $G'_2$  are planar.

Without loss of generality, assume  $c_a \in V(G'_1)$ . Since none of the edges of  $G'_2$  are in  $G'_1$  and  $a, c_a \notin V(G'_2)$ , it follows that  $G'_2$  is a subgraph of the planar graph  $G - a, c_a$ . Thus,  $G'_2$  is planar.  $\square$

**Theorem 4.7.** *If  $G$  is MMNA and  $\kappa(G) = 2$  such that  $G - a, b = G'_1 \sqcup G'_2$ , then, up to relabeling,  $G'_1 + a, G'_1 + b$  are planar, and  $G'_2 + a, G'_2 + b$  are nonplanar.*

We prove this with two lemmas.

**Lemma 4.8.**  *$G'_1 + a$  and  $G'_2 + a$  cannot both be planar.*

*Proof.* Let  $G$  be as described. Suppose both  $G'_1 + a$  and  $G'_2 + a$  are planar. Since  $G'_1$  and  $G'_2$  are otherwise disjoint,  $G - b = (G'_1 + a) \cup (G'_2 + a)$  is the union of two planar graphs at only one vertex, with no new edges. Thus,  $G - b$  is planar, which is a contradiction. So it cannot be that both  $G'_1 + a$  and  $G'_2 + a$  are planar. A similar argument could be made for  $b$ .  $\square$

**Lemma 4.9.**  *$G'_1 + a$  and  $G'_2 + b$  cannot both be nonplanar (up to relabeling).*

*Proof.* Let  $G$  be as described. Suppose both  $G'_1 + a$  and  $G'_2 + b$  are nonplanar. Let  $e$  be an edge between a vertex in  $G'_1$  and the vertex  $b$ . Since  $G$  is MMNA,  $G - e$  is apex. So there is a vertex  $v$  such that  $(G - e) - v$  is planar. If  $v = a$  then  $G'_2 + b$  is a subgraph of  $(G - e) - v$ , which is a contradiction since  $G'_2 + b$  is nonplanar. If  $v \in V(G'_1)$  then again  $G'_2 + b$  is a subgraph of  $(G - e) - v$ , which is a contradiction since  $G'_2 + b$  is nonplanar. If  $v = b$  then  $(G - e) - v = G - v$ , which implies  $(G - e) - v$  is nonplanar since  $G$  is NA, so this is a contradiction. If  $v \in V(G'_2)$

then  $G'_1 + a$  is a subgraph of  $(G - e) - v$ , which is a contradiction since  $G'_1 + a$  is nonplanar. Therefore there is no apex for  $G - e$ , which is a contradiction. So our assumption was wrong and one of  $G'_1 + a$  and  $G'_2 + b$  must be planar.  $\square$

We can now prove [Theorem 4.7](#).

*Proof of [Theorem 4.7](#).* Let  $G$  be as described. By the first lemma we know that at least one of  $G'_1 + a$  and  $G'_2 + a$  must be nonplanar. Without loss of generality suppose  $G'_2 + a$  is nonplanar. Since  $G'_2 + a$  is nonplanar, we know that  $G'_1 + b$  must be planar by the second lemma. Since  $G'_1 + b$  is planar, by the first lemma we know that  $G'_2 + b$  is nonplanar. By the second lemma this implies that  $G'_1 + a$  must be planar. Therefore, up to relabeling,  $G'_1 + a$  and  $G'_1 + b$  are both planar, and  $G'_2 + a$  and  $G'_2 + b$  are both nonplanar.  $\square$

Going forward, we adopt the convention suggested by [Theorem 4.7](#) and label  $G'_1$  and  $G'_2$  such that  $G'_1 + a$ ,  $G'_1 + b$  are planar and  $G'_2 + a$ ,  $G'_2 + b$  are not. Let  $G$  be MMNA with cut set  $\{a, b\}$ . Our next goal is to classify such graphs in the case that  $ab$  is an edge of the graph.

**Theorem 4.10.** *If  $G$  is MMNA and  $\kappa(G) = 2$  with cut set  $\{a, b\}$  such that  $ab \in E(G)$ , then  $G_1$  and  $G_2$  are nonplanar.*

*Proof.* Let  $G_i$  denote the induced subgraph on  $V(G'_i) \cup \{a, b\}$ . By [Theorem 4.7](#),  $G_2$  is nonplanar. For the sake of contradiction, assume  $G_1$  is planar. Since  $G_2$  is a proper subgraph of  $G$ , there is a vertex  $v \in V(G_2)$  such that  $G_2 - v$  is planar. But this means  $G - v$  is planar and contradicts that  $G$  is NA.

So if  $G$  is MMNA with cut set  $\{a, b\} \subset V(G)$  such that  $ab \in E(G)$ , then  $G_1$  and  $G_2$  are nonplanar.  $\square$

**Theorem 4.11.** *If  $G$  is MMNA and  $\kappa(G) = 2$  with cut set  $\{a, b\}$  such that  $ab \in E(G)$ , then  $G'_1$  and  $G'_2$  are both planar.*

*Proof.* By [Theorem 4.10](#),  $G_1$  is nonplanar. By [Theorem 4.6](#), without loss of generality,  $G'_1$  is planar. Suppose  $G'_2$  is nonplanar. Then  $G_1 \sqcup G'_2$  is a proper subgraph of  $G$ . Since  $G_1$  and  $G'_2$  are both nonplanar,  $G_1 \sqcup G'_2$  has a disconnected MMNA minor, contradicting that  $G$  is minor minimal.  $\square$

**Theorem 4.12.** *If  $G$  is MMNA with cut set  $\{a, b\}$  such that  $ab \in E(G)$ , then  $G_1 \in \{K_5, K_{3,3}\}$ .*

*Proof.* First observe that for any  $e \in E(G_1)$ , the graph  $G_1 - e$  must be planar. Suppose instead that there is  $e' \in E(G_1)$  such that  $G_1 - e'$  is nonplanar. Since  $G - e'$  is apex, there is a vertex  $v \in V(G)$  such that  $(G - e') - v$  is planar. However,  $v \notin \{a, b\}$  since  $G'_2 + a$  and  $G'_2 + b$  are nonplanar by [Theorem 4.7](#). If  $v \in V(G'_1)$ , then  $G_2$  is a subgraph of  $(G - e') - v$ . By [Theorem 4.10](#), since  $G_2$  is nonplanar,  $(G - e') - v$  is also nonplanar. If  $v \in V(G'_2)$ , then  $G_1 - e'$  is a subgraph of

$(G - e') - v$ , and since  $G_1 - e'$  is nonplanar,  $(G - e') - v$  is nonplanar. So we have a contradiction and deduce that for all  $e \in E(G_1)$ , the graph  $G_1 - e$  must be planar.

Since  $G_1$  is nonplanar by [Theorem 4.10](#), and since  $G_1 - e$  is planar for all  $e \in G_1$ , it follows that  $G_1$  consists of a  $K$ -subgraph along with some number (possibly zero) of isolated vertices. However, if  $G_1$  is anything other than  $K_5$  or  $K_{3,3}$ , then  $G_1$  has a proper minor  $N \in \{K_5, K_{3,3}\}$  formed by deleting isolated vertices or contracting edges in the  $K$ -subgraph. Then  $G$  has a proper minor  $G'$  such that  $N$  is a subgraph of  $G'$ . Since  $G$  is MMNA, there exists vertex  $v \in V(G')$  that is an apex. Since  $N$  and  $G_2$  are subgraphs of  $G'$  and both  $N$  and  $G_2$  are nonplanar, we have that  $v \in V(N) \cap V(G_2) \subset \{a, b\}$ . However,  $G_2 - a = G'_2 + b$  and  $G_2 - b = G'_2 + a$  are both nonplanar ([Theorem 4.7](#)) and therefore  $G$  has a proper NA minor. This contradicts  $G$  being minor minimal.

Therefore if  $G$  is MMNA with cut set  $\{a, b\}$  such that  $ab \in E(G)$ , then  $G_1 \in \{K_5, K_{3,3}\}$ .  $\square$

**Theorem 4.13.** *If  $G$  is MMNA with cut set  $\{a, b\}$  such that  $ab \in E(G)$ , then there is a vertex  $c \in V(G'_2)$  such that every  $a$ - $b$ -path in  $G_2 - ab$  passes through  $c$ .*

*Proof.* Assume for the sake of contradiction that there is no such vertex  $c$ . Since  $G$  is MMNA,  $G - ab$  must have some apex  $v$ . If  $v \in \{a, b\}$ , then  $(G - ab) - v = G - v$ . This would mean that  $G$  has an apex, and contradicts that  $G$  is NA. If  $v \in V(G'_1)$ , then  $(G - ab) - v$  is nonplanar as it contains  $G'_2 + a$ , which is nonplanar by [Theorem 4.7](#). So it must be that  $v \in V(G'_2)$ . Then  $G_1 - ab$  is a subgraph of  $(G - ab) - v$ . Note that  $G_1 - ab \in \{K_5 - e, K_{3,3} - e\}$  since  $G_1 \in \{K_5, K_{3,3}\}$  by [Theorem 4.12](#).

Since there is no  $c$  vertex as described in the statement of the theorem, there remains an  $a$ - $b$ -path in  $(G_2 - ab) - v$ . Together with  $G_1 - ab$ , this constitutes a nonplanar subgraph of  $(G - ab) - v$ , contradicting the definition of  $v$  as an apex for  $G - ab$ . Thus, if  $G$  is MMNA with  $ab \in E(G)$ , then there is a vertex  $c$  such that every  $a$ - $b$ -path of  $G_2 - ab$  passes through  $c$ .  $\square$

**Theorem 4.14.** *Let  $G$  be MMNA with cut set  $\{a, b\}$  and  $ab \in E(G)$  and let  $c \in V(G_2)$  be such that every  $a$ - $b$ -path of  $G_2 - ab$  passes through  $c$ . Then  $\{a, c\}$  and  $\{b, c\}$  are also cut sets.*

*Proof.* First we show there exists some  $v_2 \in V(G'_2)$  such that  $v_2 \neq c$ , but  $v_2$  is adjacent to  $a$ . Suppose instead that  $c$  is the only vertex in  $G'_2$  adjacent to  $a$ . Since  $G'_2$  is planar by [Theorem 4.11](#), and since  $G'_2 + a$  has only one more edge than  $G'_2$ ,  $G'_2 + a$  is also planar. However, this contradicts [Theorem 4.7](#), where  $G'_2 + a$  is shown to be nonplanar.

So let  $v_2$  be a vertex of  $G'_2$  that is adjacent to  $a$ , but is not  $c$ , and take  $v_1 \in V(G'_1)$ . We demonstrate there is no  $v_1$ - $v_2$ -path in  $G - a, c$ . Since any path from a vertex in  $G'_1$  to a vertex in  $G'_2$  must pass through  $a$  or  $b$  by assumption, the supposed path

from  $v_1$  to  $v_2$  must pass through  $b$ , since  $a$  has been deleted. However, there cannot be a path from  $b$  to  $v_2$  that does not pass through  $c$ . Otherwise we would be able to find a path from  $b$  to  $v_2$  and finally to  $a$  without passing through  $c$ , violating our assumption on  $c$ . We conclude that  $G - a, c$  is disconnected. By an analogous argument,  $\{b, c\}$  is also a cut set for  $G$ .  $\square$

In order to classify connectivity-2 MMNA graphs with  $ab \in E(G)$ , we need to describe  $G_1$  in the case that  $ab \notin E(G)$ .

**Theorem 4.15.** *If  $G$  is MMNA with cut set  $\{a, b\}$  such that  $ab \notin E(G)$ , then  $G_1 \in \{K_5 - e, K_{3,3} - e, K_{3,3}\}$  and  $G_1 + ab$  is nonplanar.*

*Proof.* Let  $G - a, b = G'_1 \sqcup G'_2$  and let  $G_i$  denote the subgraph induced by vertices  $V(G'_i) \cup \{a, b\}$ . If  $G_1$  is nonplanar, then  $G_1$  has a K-subgraph  $N$ . Form a new graph,  $H$ , by replacing  $G_1$  with  $N$ . It is clear that  $a, b \in V(N)$  because if not, then  $G$  contains two disjoint K-subgraphs ( $G'_2 + a$  and  $G'_2 + b$  are nonplanar, [Theorem 4.7](#)) and therefore has a proper MMNA minor.

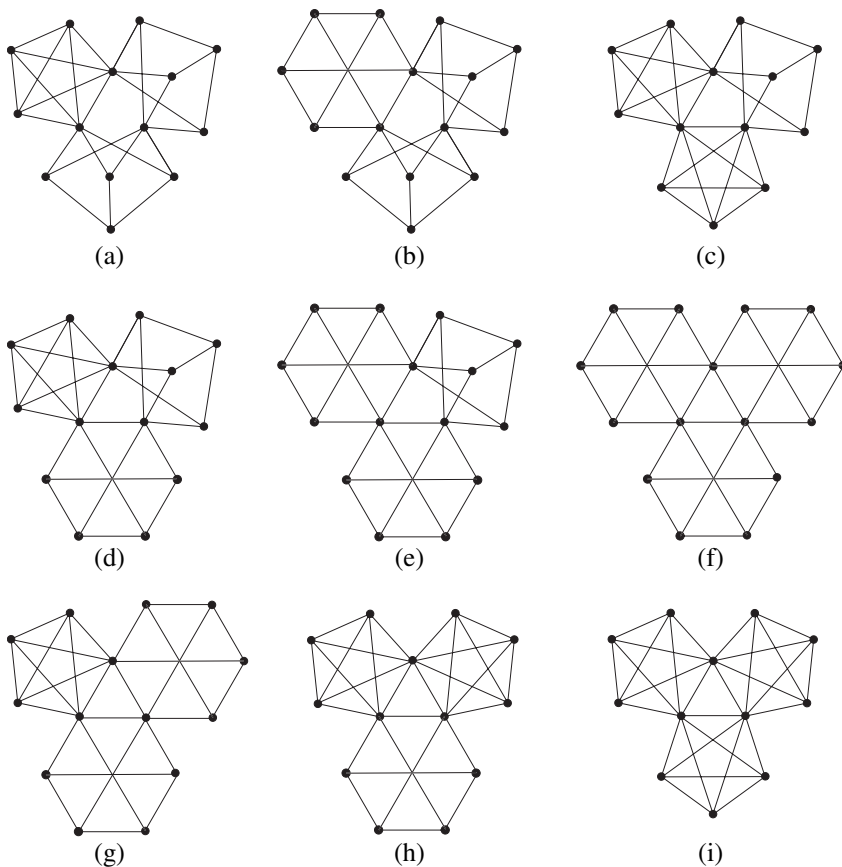
We can see that  $H$  is NA. Take  $v \in V(H)$ . If  $v \in V(N - a, b)$ , then  $G'_2 + a$  is a subgraph of  $H - v$  so  $H - v$  is nonplanar. If  $v \in V(G'_2)$ , then  $N$  is a subgraph of  $H - v$  so  $H - v$  is nonplanar. And if  $v \in \{a, b\}$ , then either  $G'_2 + a$  or  $G'_2 + b$  is a subgraph of  $H - v$  and therefore  $H - v$  is nonplanar. Thus,  $H$  is NA. Since  $G$  is minor minimal,  $G_1 = N$ . As  $G$  is MMNA it has no degree-2 vertices and since  $ab \notin E(G)$ , we have  $G_1 = K_{3,3}$  in this case.

Suppose next that  $G_1$  is planar. Assume for the sake of contradiction  $G_1 + ab$  is planar and replace  $G_1$  with the edge  $ab$  to form a new graph  $H'$ . Equivalently,  $H' = G_2 + ab$ . We observe that for every  $v \in V(H')$ , the graph  $H' - v$  is nonplanar. If  $v \in \{a, b\}$ , then  $G'_2 + a$  or  $G'_2 + b$  is a subgraph of  $H' - v$ , which is then nonplanar. On the other hand if  $v \in V(G'_2)$ , then since  $G$  is NA,  $G - v$  has a K-subgraph  $M$ . However, if  $|\{a, b\} \cap V(M)| < 2$ , then since  $G_1$  is planar,  $M$  lies wholly in  $G_2$  and we may delete  $G'_1$  without changing  $M$ . That is,  $M$  is a subgraph of  $H' - v$ . If  $|\{a, b\} \cap V(M)| = 2$ , then by [Lemma 1.8](#),  $a$  and  $b$  are vertices in a path of  $M$ . Since  $G_1 + ab$  is planar, we may replace  $G_1$  by  $ab$  to create a new K-subgraph  $B$  in  $H' - v$ . Therefore  $H'$  is NA. However, as  $H'$  is a proper minor of  $G$ , this is a contradiction. We conclude  $G_1 + ab$  is nonplanar.

Finally, observe that  $G_1 + ab$  is a K-subgraph. Otherwise, we may replace it with a K-subgraph contained in  $G_1 + ab$  to get a proper minor of  $G$  that is NA. Since an MMNA graph cannot have vertices of degree 2 or less,  $G_1 + ab \in \{K_5, K_{3,3}\}$ .

This shows if  $G$  is MMNA with cut set  $\{a, b\}$  such that  $ab \notin E(G)$ , then we have  $G_1 \in \{K_5 - e, K_{3,3} - e, K_{3,3}\}$ .  $\square$

**Theorem 4.16.** *If  $G$  is MMNA,  $\kappa(G) = 2$  with cut set  $\{a, b\}$ , and  $ab \in E(G)$ , then  $G$  is one of the nine graphs shown in [Figure 2](#).*



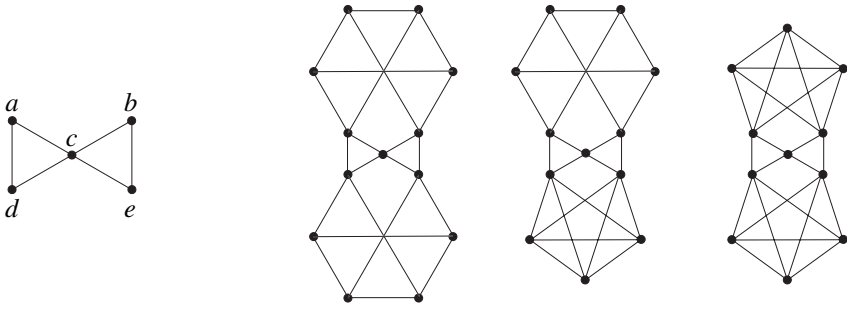
**Figure 2.** The nine MMNA graphs with  $ab \in E(G)$ .

*Proof.* It is straightforward to verify that the nine graphs are MMNA. Let  $G$  be MMNA,  $\kappa(G) = 2$  with cut set  $\{a, b\}$ , and  $ab \in E(G)$ . By Theorems 4.13 and 4.14, there exists a vertex  $c$  such that  $\{a, c\}$  and  $\{b, c\}$  are also 2-cuts for  $G$ . Let  $H_1$  play the role of  $G_1$  for the  $\{a, c\}$  cut set. That is,  $G - a, c = H'_1 \sqcup J'_1$  with  $H'_1 + a$  and  $H'_1 + c$  planar (see Theorem 4.7). Similarly, let  $H_2$  be the  $G_1$  for the  $\{b, c\}$  cut set. By Theorem 4.12,  $G_1 \in \{K_{3,3}, K_5\}$  and by that theorem and Theorem 4.15,  $H_i \in \{K_{3,3}, K_{3,3} - e, K_5, K_5 - e\}$ .

Note that, if  $H_1$  is  $K_{3,3} - e$  or  $K_5 - e$ , then  $G - b$  is planar and similarly for  $H_2$ . Thus,  $H_1, H_2 \in \{K_{3,3}, K_5\}$ . There are three cases depending on whether  $ac, bc \in E(G)$  or not.

First suppose that  $ab$  is the only one of  $ab, bc$ , and  $ac$  present in the graph. As above,  $G_1, H_1$  and  $H_2$  are each either  $K_{3,3}$  or  $K_5$ . However, by Theorem 4.15, this means  $H_1 = H_2 = K_{3,3}$ . So, there are exactly two graphs (graphs (a) and (b) in Figure 2) of this type, depending on whether  $G_1$  is  $K_5$  or  $K_{3,3}$ .





**Figure 3.** Bowtie graphs.

Next suppose that exactly one of  $ac$  and  $bc$ , say  $ac$ , is in the graph. As in the previous case  $H_2$  must be  $K_{3,3}$ . There are three graphs (graphs (c), (d), and (e) of Figure 2) of this type as  $\{G_1, H_1\}$  is either  $\{K_5, K_5\}$ ,  $\{K_5, K_{3,3}\}$ , or  $\{K_{3,3}, K_{3,3}\}$ .

Finally, suppose all three edges  $ab, ac$  and  $bc$  are in the graph. Then, as above,  $G_1, H_1$ , and  $H_2$  are each either  $K_{3,3}$ , or  $K_5$ . There are four graphs of this type, shown as graphs (f) through (i) of Figure 2. For example, such a graph has between zero and three  $K_5$ 's.

This shows that the nine graphs of Figure 2 are the graphs where  $G$  is MMNA,  $\kappa(G) = 2$  with cut set  $\{a, b\}$ , and  $ab \in E(G)$ . □

Henceforth, we can assume  $ab \notin E(G)$ . By Theorem 4.15, this means  $G_1 \in \{K_5 - e, K_{3,3} - e, K_{3,3}\}$ . We will say that  $G$  is a *bowtie* if the neighborhood of  $a, b$  in  $G_2$  is as shown in Figure 3 (left). That is,  $a$  and  $b$  have degree 2 in  $G_2$  and  $c$  has degree 4. Although  $d$  and  $e$  have additional neighbors in  $G_2$  besides  $\{a, c\}$  and  $\{b, c\}$  respectively,  $de \notin E(G_2)$ .

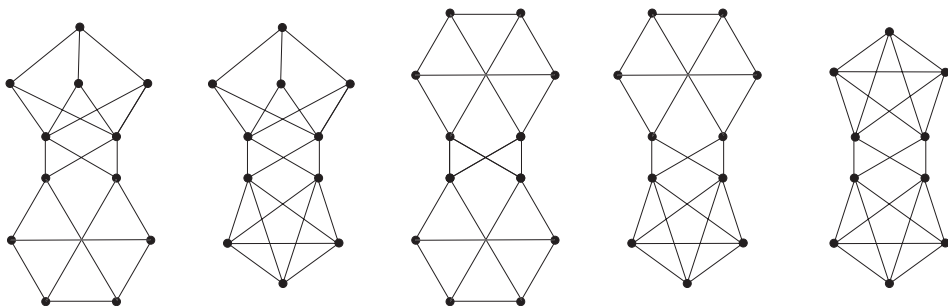
**Theorem 4.17.** *If  $G$  is a bowtie MMNA graph, then  $G$  is one of the three graphs shown in Figure 3 (right).*

*Proof.* It is straightforward to verify that the three graphs in the figure are MMNA. Let  $G$  be a bowtie MMNA graph. Then, referring to Figure 3 (left),  $\{d, e\}$  is a cut set as well. Let  $H_1$  play the role of the  $G_1$  for the  $\{d, e\}$  cut set. By Theorem 4.15,  $G_1$  and  $H_1$  are both drawn from  $\{K_{3,3}, K_{3,3} - e, K_5 - e\}$ .

We will argue that neither  $G_1$  nor  $H_1$  is  $K_{3,3}$ . For the sake of contradiction, assume instead  $G_1 = K_{3,3}$ . Notice  $G_1$  and  $G'_2$  are disjoint, and nonplanar. So,  $G$  has a proper NA minor,  $G_1 \sqcup G'_2$ , which contradicts that  $G$  is to be minor minimal.

So,  $G_1$  and  $H_1$  are both in  $\{K_{3,3} - e, K_5 - e\}$ , where  $ab$  is the missing edge,  $e$ , and the only possibilities are the three graphs shown in Figure 3 (right). □

Let  $G$  be MMNA with cut set  $\{a, b\}$  such that  $ab \notin E(G)$ . We say  $G$  is of  $(2, 2, c)$  type if, in  $G_2$ , the vertices  $a$  and  $b$  are of degree 2 and have  $c$  common neighbors. For example, a bowtie graph is of  $(2, 2, 1)$  type.



**Figure 4.** Graphs of type  $(2, 2, 2)$ .

**Theorem 4.18.** *If  $G$  is MMNA and of  $(2, 2, 2)$  type, then  $G$  is one of the five graphs shown in Figure 4.*

*Proof.* It is straightforward to verify that the five graphs are MMNA. Let  $G$  be MMNA with cut set  $\{a, b\}$  and of  $(2, 2, 2)$  type. Let  $\{c, d\}$  be the common neighbors of  $a$  and  $b$  in  $G_2$ . Note that  $cd \notin E(G)$ , as otherwise  $G$  must be one of the nine graphs of Theorem 4.16 and none of those are  $(2, 2, 2)$  type.

By Theorem 4.15, and using symmetry,  $G_1, G'_2 \in \{K_{3,3}, K_{3,3} - e, K_5 - e\}$ . However, they cannot both be  $K_{3,3}$ , as otherwise  $G_1 \sqcup G'_2$  is a proper NA subgraph, which contradicts that  $G$  is minor minimal. So at most one of the subgraphs can be  $K_{3,3}$ . This leaves the five possibilities shown in Figure 4.  $\square$

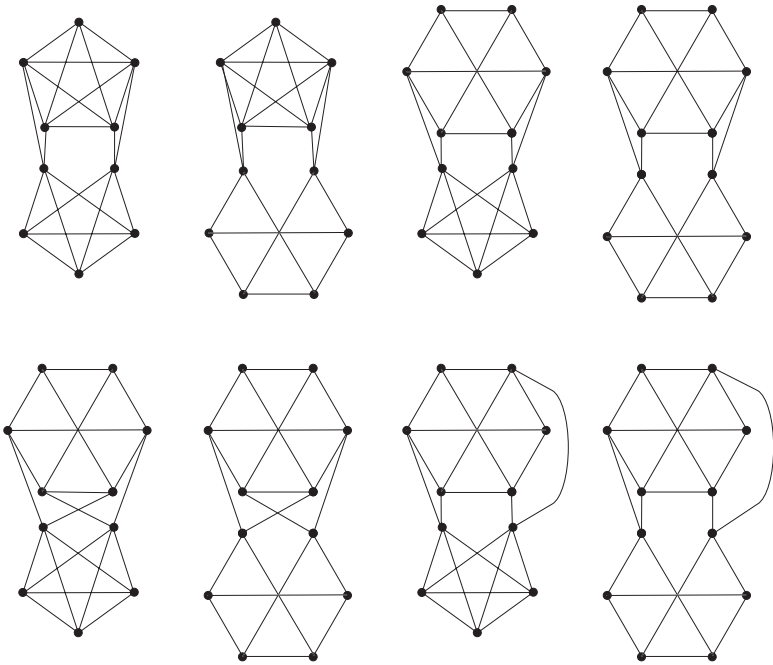
**Theorem 4.19.** *Suppose  $G$  is MMNA and of connectivity 2 with  $G_1 \in \{K_5 - e, K_{3,3} - e\}$ . Then there is no vertex, other than  $a$  and  $b$ , common to all  $a$ - $b$ -paths in  $G_2$ .*

*Proof.* Assume, for the sake of contradiction, that  $G_1 \in \{K_5 - e, K_{3,3} - e\}$  and there is a vertex  $c \in V(G'_2)$  that lies on every  $a$ - $b$ -path in  $G_2$ . Then, as in Theorem 4.14,  $\{a, c\}$  and  $\{b, c\}$  are 2-cuts for  $G$ , and as in the proof of Theorem 4.16, we can let  $H_1$  play the role of the  $G_1$  for the  $\{a, c\}$  cut and similarly  $H_2$  for the  $\{b, c\}$  cut and, by Theorems 4.12 and 4.15, both  $H_1$  and  $H_2$  are drawn from  $\{K_5, K_{3,3}, K_5 - e, K_{3,3} - e\}$ . Then  $G - c$  is planar, contradicting that  $G$  is NA.

Therefore, if  $G$  is MMNA, of connectivity 2 with  $G_1 \in \{K_5 - e, K_{3,3} - e\}$ , then there is no vertex, other than  $a$  and  $b$ , common to all  $a$ - $b$ -paths in  $G_2$ .  $\square$

**Theorem 4.20.** *Let  $G$  be MMNA with  $\kappa(G) = 2$  and  $ab \notin E(G)$ , where  $\{a, b\}$  is a 2-cut. If  $G'_2$  is nonplanar, then there are independent  $a$ - $b$ -paths in  $G_2$ .*

*Proof.* By Theorem 4.15,  $G_1 \in \{K_5 - e, K_{3,3}, K_{3,3} - e\}$ . However, if  $G_1 = K_{3,3}$  then, together with  $G'_2$ , this constitutes a pair of disjoint  $K$ -subgraphs, which would mean  $G$  has a proper disconnected NA minor, a contradiction. So  $G_1 \in \{K_5 - e, K_{3,3} - e\}$  and we can apply Menger's theorem and Theorem 4.19.  $\square$



**Figure 5.** Graphs of type  $(2, 2, 0)$ .

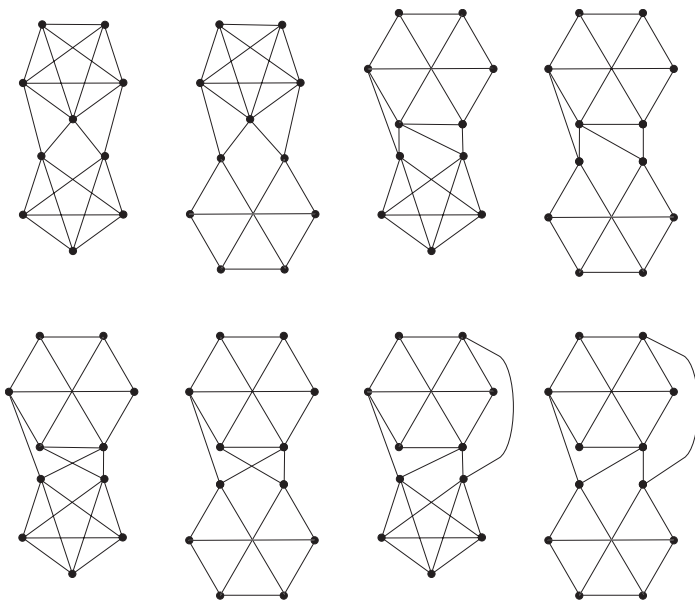
**Theorem 4.21.** *If  $G$  is MMNA of  $(2, 2, 0)$  type and  $G'_2 \in \{K_5, K_{3,3}\}$ , then  $G$  is one of the eight graphs in [Figure 5](#).*

*Proof.* Notice that the eight graphs in the figure are MMNA. Suppose  $G$  is MMNA of  $(2, 2, 0)$  type with  $G'_2$  a Kuratowski graph. By [Theorem 4.15](#),  $G_1 \in \{K_5 - e, K_{3,3}, K_{3,3} - e\}$ . However,  $G_1$  cannot be  $K_{3,3}$  because then, together with  $G'_2$  it forms a disconnected MMNA minor of  $G$ . We continue by examining the ways to construct  $G_2$ .

To construct  $G_2$ , we consider how to add the vertices  $a$  and  $b$  to  $G'_2$ . Let  $a$  have neighbors  $v_1, v_2 \in V(G'_2)$  and let  $v_3, v_4 \in V(G'_2)$  be the neighbors of  $b$ . Since  $G$  is of  $(2, 2, 0)$  type,  $\{v_1, v_2\} \cap \{v_3, v_4\} = \emptyset$ . Up to symmetry, there is only one way to attach  $a$  and  $b$  to  $K_5$ . This gives two of the graphs in the figure, as  $G_1$  is either  $K_5 - e$  or  $K_{3,3} - e$ .

In  $K_{3,3}$ , the vertices are split into two parts  $A$  and  $B$ , each of three vertices. Then the four vertices  $v_i, i = 1, \dots, 4$ , are either divided with two in each part, or else with three in one part and the fourth in the other. In the first case, there are two subcases: either  $\{v_1, v_2\} \subset A$  (and  $\{v_3, v_4\} \subset B$ ) or else  $|\{v_1, v_2\} \cap A| = |\{v_1, v_2\} \cap B| = 1$  (and similarly for  $\{v_3, v_4\}$ ). These three choices for  $G_2$  along with the two choices for  $G_1$ , either  $K_5 - e$  or  $K_{3,3} - e$ , account for the remaining six graphs in [Figure 5](#).  $\square$

**Theorem 4.22.** *If  $G$  is MMNA of  $(2, 2, 1)$  type and  $G'_2 \in \{K_5, K_{3,3}\}$ , then  $G$  is one of the eight graphs of [Figure 6](#).*



**Figure 6.** Graphs of type (2, 2, 1).

*Proof.* The proof is similar to that for (2, 2, 0) type. If  $G'_2$  is a Kuratowski graph, then  $G_1$  cannot be  $K_{3,3}$ , as that would result in a proper NA minor. So  $G_1 \in \{K_5 - e, K_{3,3} - e\}$ . If  $G'_2 = K_5$ , up to symmetry there is only one way to form  $G_2$  and this gives two graphs in the figure, as  $G_1$  is either  $K_5 - e$  or  $K_{3,3} - e$ .

If  $G'_2 = K_{3,3}$ , there are three ways to form  $G_2$ . Together,  $a$  and  $b$  have three neighbors in  $G'_2$ , which can either all lie in one part or else be split with a single vertex in one part and the remaining two in the other. In this second case, there are two further subcases since the vertex that is alone in its part can either be the common neighbor or not. Together with these three choices for  $G_2$ , there are two choices for  $G_1$ , either  $K_5 - e$  or  $K_{3,3} - e$ . This gives the remaining six graphs of **Figure 6**.  $\square$

We conclude this section with a classification of the MMNA graphs of connectivity 2, with 2-cut  $\{a, b\}$  such that  $G - a, b$  has a nonplanar component. By **Theorem 4.11** we must have  $ab \notin E(G)$ , and by **Theorem 4.7**,  $G'_1$  is planar. In other words, if there is a nonplanar component, it must be  $G'_2$ . So far, we have constructed 21 graphs with nonplanar  $G'_2$ , the three bowtie graphs of **Theorem 4.17**, two of the (2, 2, 2) graphs (the two to the left of **Figure 4**), and eight each of (2, 2, 0) type (**Theorem 4.21**) and (2, 2, 1) type (**Theorem 4.22**). This is in fact a complete listing of the graphs with  $G'_2$  nonplanar, as we now show.

**Theorem 4.23.** *Let  $G$  be MMNA with  $\kappa(G) = 2$  and 2-cut  $\{a, b\}$  such that  $G - a, b$  has a nonplanar component. Then  $G$  is of (2, 2,  $c$ ) type with  $c = 0, 1$ , or 2 and appears in one of **Figures 3 (right), 4, 5, or 6**.*

*Proof.* Assume the hypothesis. As remarked above, if  $\{a, b\}$  is a 2-cut, this implies  $ab \notin E(G)$  and  $G'_2$  is nonplanar. Let  $H'_2$  be a K-subgraph of  $G'_2$ . Since  $ab \notin E(G)$ , combining Theorems 4.15 and 4.2, we have  $G_1 \in \{K_5 - e, K_{3,3} - e\}$ . By Theorem 4.20 there are independent  $a$ - $b$ -paths in  $G_2$ , call them  $P_1$  and  $P_2$ . Since, by Theorem 4.15,  $G_1 + ab$  is nonplanar,  $P_1$  and  $P_2$  each have vertices in common with  $H'_2$ . (Otherwise,  $G$  has disjoint nonplanar subgraphs and therefore a disconnected NA minor, by Theorem 4.2, contradicting  $G$  being minor minimal.) By contracting edges if necessary, we have a minor of  $G$  for which the vertices of  $P_i$  are  $a, a_i, \dots, b_i, b$  with  $a_i, b_i \in V(H_2)$ ,  $i = 1, 2$ . Then there are several cases that correspond to  $(2, 2, c)$  type, where  $c = 0, 1, 2$ .

Suppose first that  $a_1 = b_1$  and  $a_2 = b_2$  so that  $G$  is of  $(2, 2, 2)$  type. By contracting edges in  $H'_2$  if needed, we recognize that  $G$  has one of the five graphs of Theorem 4.18 as a minor. Since  $G$  is MMNA,  $G$  is one of these five graphs and since  $G'_2$  is nonplanar,  $G$  must be one of the two graphs with  $G'_2 = K_{3,3}$  (i.e., the two to the left of Figure 4). In other words  $G$  is of  $(2, 2, 2)$  type and appears in one of the figures, as required.

The rest of the argument is a little technical and we introduce some notation to simplify the exposition. The K-subgraph  $H'_2$  is a subdivision of  $K_5$  or  $K_{3,3}$  and, along with vertices of degree 2, has five or six vertices of higher degree that we will call *branch vertices*. Corresponding to the edges of  $K_5$  or  $K_{3,3}$ , the branch vertices are connected by paths that we call *2-paths* whose internal vertices are all of degree 2.

To continue the argument, suppose next that, say,  $a_1 = b_1$ , but  $a_2 \neq b_2$ . By contracting edges in  $H'_2$  if necessary, we can arrange that at least two of the three vertices  $a_1, a_2$ , and  $b_2$  become branch vertices of the K-subgraph. If all three can be made branch vertices, then, by further edge contractions, if necessary, we see that one of the eight  $(2, 2, 1)$  graphs of Theorem 4.22 is a minor of  $G$ . Since  $G$  is MMNA, this means  $G$  is one of the  $(2, 2, 1)$  graphs, with  $G'_2 \in \{K_5, K_{3,3}\}$  appearing in Figure 6, as required. If not, we can assume that it is  $a_1$  that remains as a degree-2 vertex of  $H'_2$ . For, if it is  $a_2$  or  $b_2$  that remains, we can contract edges to make  $a_2 = b_2$  and return to the previous case. With  $a_1$  as a degree-2 vertex in  $G'_2$ , we recognize that, perhaps by further edge contractions,  $G$  has a bowtie graph as a minor. Since  $G$  is MMNA,  $G$  is a bowtie graph. That is  $G$  is of  $(2, 2, 1)$  type and appears in Figure 3 (right), as required.

Finally, suppose  $a_1 \neq b_1$  and  $a_2 \neq b_2$ . If all four can be made distinct branch vertices by edge contractions in  $H'_2$ , then  $G$  has a  $(2, 2, 0)$  minor, so  $G$  is a  $(2, 2, 0)$  graph with  $G'_2 \in \{K_5, K_{3,3}\}$  appearing in Figure 5, as required.

Next, suppose at most three can be made into branch vertices and, without loss of generality, suppose it is  $a_1$  that remains as a degree-2 vertex in  $H'_2$ . This means  $a_1$  lies on a 2-path between two of  $b_1, a_2$ , and  $b_2$ . If the path ends at  $b_1$ , by further

edge contractions in  $H'_2$ , we can realize  $a_1 = b_1$  as a branch vertex and return to an earlier case. So, we can assume that  $a_1$  is on a 2-path between  $a_2$  and  $b_2$ . Use the part of the 2-path between  $a_1$  and  $b_2$  to form a new  $a$ - $b$ -path  $P'_1$  (i.e.,  $a'_1 = a_1$  and  $b'_1 = b_2$ ) and use a path in  $H'_2$  between the branch vertices  $a_2$  and  $b_1$  that avoids the branch vertex  $b_2$  to construct an independent  $a$ - $b$ -path  $P'_2$  (i.e.,  $P'_2$  has  $a'_2 = a_2$  and  $b'_2 = b_1$ ). Now we can contract edges in  $P'_1$  to identify  $a'_1 = a_1$  and  $b'_1 = b_2$  to return to the earlier case where  $a_1 = b_1$ . This completes the argument when at most three of the vertices can be moved to branch vertices.

Finally, suppose that at most two of the vertices can be made into branch vertices of  $H'_2$  by contracting edges, if needed. There are two subcases. If  $a_1$  and  $b_1$  are the branch vertices, then  $a_2$  and  $b_2$  are degree-2 vertices on a 2-path between  $a_1$  and  $b_1$ . Here we can further contract edges in  $H'_2$  to identify  $a_2$  and  $b_2$ , which returns us to an earlier case. In the second subcase, without loss of generality, it is  $a_1$  and  $a_2$  that are the branch vertices of  $H'_2$ . Assuming we cannot easily contract edges to identify  $a_1$  and  $b_1$  or  $a_2$  and  $b_2$ , it must be that the 2-path from  $a_1$  to  $a_2$  passes first through  $b_2$  and then through  $b_1$ . In this case, we replace  $P_1$  and  $P_2$  by the independent paths  $P'_1$ , which uses the 2-path from  $a_1$  to  $b_2$  (so  $a'_1 = a_1$  and  $b'_1 = b_2$ ), and  $P'_2$ , which uses the 2-path from  $a_2$  to  $b_1$  (then  $a'_2 = a_2$  and  $b'_2 = b_1$ ). By further edge contractions, we return to our first case where  $a_1 = b_1$  and  $a_2 = b_2$ .  $\square$

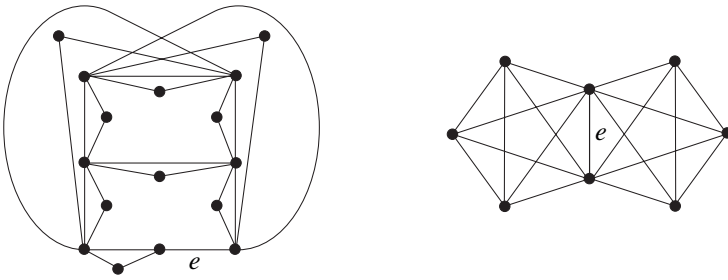
Together, the three bowtie graphs and the eight of [Figure 6](#) give eleven MMNA graphs of  $(2, 2, 1)$  type. In total we have found three disconnected MMNA graphs, nine where  $ab \in E(G)$ , as well as eight, eleven, and five, respectively when  $G$  is of type  $(2, 2, c)$  for  $c = 0, 1, 2$ , respectively. This gives a total of 36 MMNA graphs.

## 5. MMNE and MMNC graphs

In this section we classify MMNE and MMNC graphs of connectivity,  $\kappa(G)$ , at most 1. For MMNE graphs we also show  $\kappa(G) \leq 5$  and determine the graphs with  $\kappa(G) = 2$  and minimum degree at least 3. We conclude the section by describing a computer search that found 55 MMNE and 82 MMNC graphs.

We begin by observing that the MMNE and MMNC graphs are not Kuratowski sets as the opposite properties are not minor closed. Recall that NE is an abbreviation for not edge apex. The opposite property is *edge apex*, meaning there is an  $e \in E(G)$  so that  $G - e$  is planar. We call such an edge an *apex edge*. Similarly, the opposite of NC is *contraction apex*, meaning there is an edge  $e$  such that  $G/e$  is planar. We call  $e$  a *contraction apex*.

**Theorem 5.1.** *Deleting an edge of an edge apex graph results in an edge apex graph. Contracting an edge of an edge apex graph results in an edge apex graph unless the edge that is contracted is the only apex edge.*



**Figure 7.** Examples showing that the sets of MMNE and MMNC graphs are not Kuratowski sets.

*Proof.* Suppose that  $G$  is edge apex, so it contains an edge  $e$  such that  $G - e$  is planar. Let  $G'$  be the result of deleting some edge  $f$  in  $G$ . If  $f \neq e$ , consider  $G' - e$  and note that  $G' - e = G - e, f$ , which is a minor of  $G - e$ . Graph  $G - e$  is planar, so  $G' - e$  is also planar, and  $e$  is an apex edge for  $G'$ , which is therefore edge apex. Otherwise, if  $f = e$ , then  $G'$  would be planar and so would also be edge apex.

Now suppose that  $G$  contains at least two edges  $e_1$  and  $e_2$  ( $e_1 \neq e_2$ ) such that both  $G - e_1$  and  $G - e_2$  are planar. Let  $f$  be an arbitrary edge in  $G$  and let  $G''$  be the result of contracting edge  $f$  in  $G$ . Without loss of generality, suppose that  $f \neq e_1$ . Consider the graph  $G'' - e_1$ , where if  $e_1$  is incident to  $f$  in  $G$  then  $e_1$  is incident to the vertex formed by contracting  $f$  in  $G''$ . Note that this graph  $G'' - e_1$  is a minor of  $G - e_1$ . But  $G - e_1$  is planar, and since planarity is closed under taking minors, the graph  $G'' - e_1$  is planar. So edge  $e_1$  is an apex edge of  $G''$ .  $\square$

**Theorem 5.2.** *The set of graphs that are edge apex is not closed under taking minors.*

*Proof.* Let  $G$  be the graph in Figure 7 (left). This graph can be described as  $K_{3,3}$  with all but one edge replaced by a triangle, and with that one edge subdivided into an edge  $e$  and another edge to be replaced by a triangle. This graph is edge apex with  $e$  as the unique apex edge. However,  $G/e$  is  $K_{3,3}$  with every edge replaced by a triangle, so  $G/e$  is not edge apex.  $\square$

**Theorem 5.3.** *Contracting an edge of a contraction apex graph results in a contraction apex graph. Deleting an edge of a contraction apex graph results in a contraction apex graph unless the edge that is deleted is the only contraction apex.*

*Proof.* Suppose that  $G$  is contraction apex, so it contains an edge  $e$  such that  $G/e$  is planar. Let  $G'$  be the result of contracting some edge  $f$  in  $G$ . If  $f \neq e$ , consider  $G'/e$  and note that  $G'/e = G/e, f$ , which is a minor of  $G/e$ . Graph  $G/e$  is planar, so  $G'/e$  is also planar, and  $e$  is a contraction apex for  $G'$ , which is therefore a contraction apex graph. Otherwise, if  $f = e$ , then  $G'$  would be planar and so would also be contraction apex.

Now suppose that  $G$  contains at least two edges  $e_1$  and  $e_2$  ( $e_1 \neq e_2$ ) such that both  $G/e_1$  and  $G/e_2$  are planar. Let  $f$  be an arbitrary edge in  $G$  and let  $G''$  be the result of deleting edge  $f$  in  $G$ . Without loss of generality, suppose that  $f \neq e_1$ . Consider the graph  $G''/e_1$  and note that it is a minor of  $G/e_1$ . But  $G/e_1$  is planar, and since planarity is closed under taking minors, the graph  $G''/e_1$  is planar. So edge  $e_1$  is a contraction apex of  $G''$ .  $\square$

**Theorem 5.4.** *The set of graphs that are contraction apex is not closed under taking minors.*

*Proof.* Define the graph  $G$  as two copies of  $K_5$  that share a common edge  $e$ ; see Figure 7 (right). We show that  $G$  is contraction apex, but has a minor that is NC. Indeed, contracting the common edge,  $G/e = K_4 \dot{\cup} K_4$ , which is planar. Note that this is the unique contraction apex of  $G$ .

Now define the subgraph  $G'$  as  $G - e$ . Label the two subgraphs isomorphic to  $K_5 - e$  as  $G'_1$  and  $G'_2$ . Without loss of generality, suppose we contract an edge  $f$  in  $G'_2$ . Notice that we are left with  $G'_1 = K_5 - e$ , and a path through  $G'_2$  that connects the two degree-3 vertices of  $G'_1$ . Thus,  $G'/f$  has a subgraph homeomorphic to  $K_5$  and is nonplanar. By symmetry, whatever edge  $f \in E(G')$  we choose,  $G'/f$  is nonplanar. Thus  $G'$  is NC.  $\square$

We next classify the disconnected and connectivity-1 MMNE and MMNC graphs, which turn out to be the same sets.

**Theorem 5.5.** *The disconnected MMNE graphs are  $K_5 \sqcup K_5$ ,  $K_5 \sqcup K_{3,3}$ , and  $K_{3,3} \sqcup K_{3,3}$ .*

*Proof.* First observe that these three graphs are MMNE. Let  $G$  be MMNE and disconnected. Suppose one of  $G_1, G_2$  is planar, say  $G_1$ . Then let  $e_1 \in E(G_1)$ , and note that  $G - e_1$  is not NE and nonplanar. Let  $e_2$  be the edge whose removal from  $G - e_1$  gives a planar graph. Since  $G_1$  is planar, it must be that  $e_2$  is in  $E(G_2)$ . But, since  $G_1$  is planar, this means that removing  $e_2$  from  $G$  gives the disconnected union of the planar  $G_1$  and a planar minor of  $G_2$ . So, this graph,  $G - e_2$ , is planar, which is a contradiction since  $G$  is NE. So it must be that  $G_1$  and  $G_2$  are both nonplanar. Thus one of the graphs generated by  $G_1 \sqcup G_2$ , where  $G_1, G_2 \in \{K_5, K_{3,3}\}$ , must be a minor of  $G$ . Since  $G$  is minor minimal,  $G$  must be one of these three graphs.  $\square$

**Theorem 5.6.** *The disconnected MMNC graphs are  $K_5 \sqcup K_5$ ,  $K_5 \sqcup K_{3,3}$ , and  $K_{3,3} \sqcup K_{3,3}$ .*

*Proof.* First observe that these three graphs are MMNC. Let  $G$  be MMNC and disconnected. Suppose one of  $G_1, G_2$  is planar, say  $G_1$ . Then let  $e_1 \in E(G_1)$ , and note that  $G - e_1$  is not NC and nonplanar. Then there is an edge  $e_2 \in E(G - e_1)$  such that  $(G - e_1)/e_2$  is planar. Since  $G_1$  is planar, it must be that  $e_2$  is in  $E(G_2)$ . But, since  $G_1$  is planar, this means that contracting  $e_2$  in  $G$  gives the disconnected



union of the planar  $G_1$  and a planar minor of  $G_2$ . This graph  $G/e_2$  is planar, which is a contradiction since  $G$  is NC. So it must be that  $G_1$  and  $G_2$  are both nonplanar. Then one of the graphs  $G = G_1 \sqcup G_2$ , with  $G_i \in \{K_5, K_{3,3}\}$ , is a minor of  $G$ . Since  $G$  is minor minimal, it is one of those three graphs.  $\square$

**Corollary 5.7.** *Let  $G$  be disconnected. The following are equivalent:  $G$  is MMNA;  $G$  is MMNE;  $G$  is MMNC.*

Recall that  $G_1 \dot{\cup} G_2$  is the union of  $G_1$  and  $G_2$  with one vertex identified.

**Theorem 5.8.** *If  $G$  is MMNE and  $\kappa(G) = 1$  then  $G = G_1 \dot{\cup} G_2$ , where  $G_1, G_2 \in \{K_5, K_{3,3}\}$ , and they share exactly one vertex.*

*Proof.* First observe that these three graphs are MMNE. Let  $G = G_1 \dot{\cup} G_2$  and suppose for the sake of contradiction that one of  $G_1$  and  $G_2$ , say  $G_1$ , is planar. Let  $e$  be an edge of  $G_1$ . Then  $G - e$  is not NE and nonplanar. Let  $f$  be the apex edge of  $G - e$ . Since  $G_1$  is planar,  $f$  must lie in  $E(G_2)$ . Since  $G_2 - f$  is a subgraph of the planar  $G - e, f$ , it must itself be planar. Note that  $G - f = G_1 \cup (G_2 - f)$  is the union of two planar graphs that share at most one vertex, which is clearly planar. This is a contradiction, since  $G$  is NE. So both  $G_1$  and  $G_2$  are nonplanar. So  $G$  has one of the graphs  $G_1 \dot{\cup} G_2$ ,  $G_1, G_2 \in \{K_5, K_{3,3}\}$  as a minor. Since these graphs are NE and  $G$  is minor minimal,  $G$  must be one of these three graphs.  $\square$

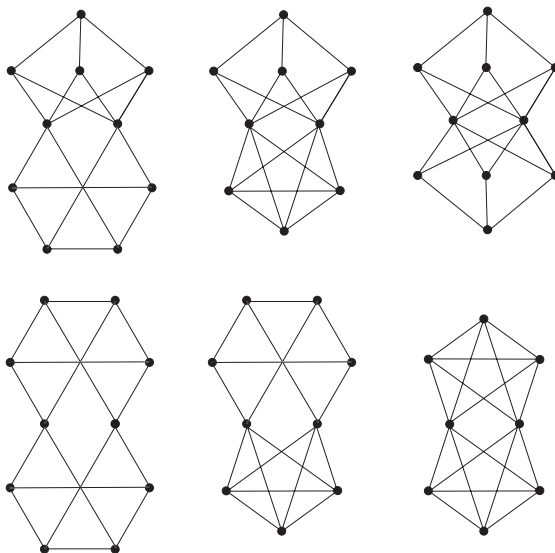
**Theorem 5.9.** *If  $G$  is MMNC and  $\kappa(G) = 1$  then  $G = G_1 \dot{\cup} G_2$ , where  $G_1, G_2 \in \{K_5, K_{3,3}\}$ , and they share exactly one vertex.*

*Proof.* First observe that these three graphs are MMNC. Let  $G = G_1 \dot{\cup} G_2$  and suppose for the sake of contradiction that one of  $G_1$  and  $G_2$ , say  $G_1$ , is planar. Let  $e$  be an edge of  $G_1$ . Then  $G - e$  is not NC and nonplanar. Let  $f \in E(G - e)$  be the contraction apex of  $G - e$ ; that is,  $(G - e)/f$  is planar. Since  $G_1$  is planar,  $f$  must lie in  $G_2$ . Since  $G_2/f$  is a subgraph of the planar  $(G - e)/f$ , it must itself be planar. Note that  $G/f = G_1 \cup (G_2/f)$  is the union of two planar graphs that share at most one vertex, which is clearly planar. This is a contradiction, since  $G$  is NC.

Thus, both  $G_1$  and  $G_2$  are nonplanar. So  $G$  has one of the graphs  $G_1 \dot{\cup} G_2$  with  $G_1, G_2 \in \{K_5, K_{3,3}\}$  as a minor. Since these graphs are NC and  $G$  is minor minimal,  $G$  must be one of these three graphs.  $\square$

**Corollary 5.10.** *Let  $G$  have connectivity 1. Then  $G$  is MMNE if and only if it is MMNC.*

Recall that there are no MMNA graphs of connectivity 1. In particular, for each of  $K_5 \dot{\cup} K_5$ ,  $K_5 \dot{\cup} K_{3,3}$ , and  $K_{3,3} \dot{\cup} K_{3,3}$ , the cut vertex is an apex. We next classify the MMNE graphs of connectivity 2 under the assumption that the minimum degree,  $\delta(G)$ , is at least 3. We will argue that there are exactly six such graphs and we begin with the observation that those graphs are indeed MMNE. As discussed at the end



**Figure 8.** The six MMNE graphs of connectivity 2 with  $\delta(G) \geq 3$ .

of this section, based on a computer search, these again coincide with the MMNC examples of connectivity 2 with  $\delta(G) \geq 3$ . In addition to being both MMNE and MMNC, these 12 graphs with  $\kappa(G) \leq 2$  are exactly the obstructions, of connectivity at most 2, to embedding a graph in the projective plane; see [Mohar and Thomassen 2001, Section 6.5].

**Theorem 5.11.** *The six graphs of Figure 8 are MMNE.*

Note that these graphs are of the form  $G_1 \dot{\cup} G_2$  with  $G_i \in \{K_5 - e, K_{3,3}, K_{3,3} - e\}$ , i.e., the union of  $G_1$  and  $G_2$  identified on two vertices.

*Proof.* Let  $G$  be one of the six graphs and  $e$  denote an arbitrary edge of  $G$ . It is easy to verify that each  $G - e$  is nonplanar, so  $G$  is NE. We must also show that no minor of  $G$  is NE. We first observe that for each choice of  $e$ , there is another edge  $f$  such that  $G - e, f$  is planar. That is,  $G - e$  is not NE. Also, there is an edge  $g$  such that  $(G/e) - g$  is planar, which shows  $G/e$  is not NE.

By Theorem 5.1, deleting or contracting further edges continues to give minors of  $G$  that are not NE, so long as we do not contract the unique apex edge in a graph. Working around this obstacle is not difficult as we very quickly come to planar minors. Planarity is closed under taking minors and a planar graph is not NE.  $\square$

A key step in the classification is the observation that  $ab$  is not an edge of  $G$ .

**Lemma 5.12.** *If  $G$  is MMNE,  $\kappa(G) = 2$  with cut set  $\{a, b\}$ , and  $\delta(G) \geq 3$ , then  $ab$  is not an edge in  $G$ .*

*Proof.* Let  $G$  be as described. Let  $G - a, b = G'_1 \sqcup G'_2$  and let  $G_i$  be the induced subgraph of  $G$  on the vertices  $V(G'_i) \cup \{a, b\}$ . For a contradiction, suppose that  $ab$  is an edge in  $G$ . There are three cases to consider depending on which of  $G_1$  and  $G_2$  is planar. If both are planar, then  $G$  is the union of two planar graphs that share an edge and therefore is planar. This contradicts  $G$  being MMNE.

Next suppose exactly one of  $G_1$  and  $G_2$  is planar, say  $G_1$ . If  $e \in E(G_2)$  is an edge other than  $ab$ , then  $G_2 - e$  must be nonplanar. For otherwise,  $G - e$ , the union of two planar graphs,  $G_1$  and  $G_2 - e$  along  $ab$ , is planar contradicting  $G$  being NE. If  $G_2 - ab$  is also nonplanar, then  $G_2$  is a proper subgraph that is NE, which contradicts  $G$  being minor minimal. So,  $G_2 - ab$  is planar.

This means that  $G - ab$  is the union of the planar  $G_1 - ab$  and the planar  $G_2 - ab$ , joined at two vertices. However, since  $G$  is NE,  $G - ab$  is nonplanar, so it has a subgraph homeomorphic to  $K_5$  or  $K_{3,3}$ . Using [Lemma 1.8](#), we know that the subgraph must use only a path through one of  $G_1, G_2$ , and nothing else in that component. This means that one of  $G_i^*$  is an edge away from containing a K-subgraph, where  $G_i^*$  denotes  $G_i - ab$ . Since  $G_1$  is planar, it must be  $G_2^*$  that contains a subdivision of  $K_5$  or  $K_{3,3}$  with an edge removed. Thus,  $G_2$  has a subgraph homeomorphic to  $K_5$  or  $K_{3,3}$  that uses the edge  $ab$ .

Replace  $G_1^*$  by the path of [Lemma 1.8](#) to form a subgraph  $H$  of  $G$ . We claim that  $H$  is NE. Indeed, deleting  $e \in E(G_2^*)$  leaves  $H - e$  with the nonplanar subgraph  $G_2 - e$ . Deleting  $ab$  or an edge in the  $G_1^*$  path leaves an  $a$ - $b$ -path that completes a K-subgraph in  $G_2^*$ . Since  $G$  is minor minimal,  $G$  must be  $H$ . However,  $H$  has at least one degree-2 vertex, contradicting  $\delta(G) \geq 3$ .

Finally, we have the case where  $G_1$  and  $G_2$  are both nonplanar. Here there are three subcases to consider depending on which of  $G_1^* = G_1 - ab$  and  $G_2^* = G_2 - ab$  is planar.

Suppose first that both  $G_1^*$  and  $G_2^*$  are planar. In this case, each of  $G_1$  and  $G_2$  has a K-subgraph that contains  $ab$ . It follows that one of the graphs of [Theorem 5.11](#) is a proper minor of  $G$ , contradicting the minor minimality of  $G$ .

In the subcase where both  $G_1^*$  and  $G_2^*$  are nonplanar, let  $e$  be the apex edge of  $G - ab$ . Since the only edge common to  $G_1^*$  and  $G_2^*$  is  $ab$ , the edge  $e$  is in exactly one of  $G_1^*$  and  $G_2^*$ . Whichever it is not in will constitute a nonplanar subgraph of  $G - ab, e$ , which is a contradiction.

Finally, assume exactly one of  $G_1^*$  and  $G_2^*$  is planar, say  $G_1^*$ . As above,  $G_1^*$  planar and  $G_1$  not implies  $G_1$  contains a K-subgraph including  $ab$  as an edge. On the other hand, since  $G_2^*$  is nonplanar, it has a K-subgraph  $H$ . Let  $M = G_1 \cup H$  and, for a contradiction, suppose that  $M$  is a proper minor. Then  $M$  must have an apex edge. However, if we remove an edge  $e$  from  $G_1$ , then  $H$  remains, meaning  $M - e$  is nonplanar. If we remove  $e$  from  $H$  (which shares no edges with  $G_1$  since it is a subgraph of  $G_2^*$ ), then  $G_1$  remains, meaning  $M - e$  is still nonplanar. Therefore,

no matter what edge we remove from  $M$ , we cannot make it planar and  $M$  is NE. However,  $M$  is a minor of  $G$ , so this contradicts  $G$  being MMNE. Therefore,  $H$  is not a proper minor of  $G_2^*$ , so  $G_2^*$  is a subdivision of  $K_5$  or  $K_{3,3}$ . A similar argument (replacing  $H$  by  $K_5$  or  $K_{3,3}$ ) shows, in fact,  $G_2^*$  is  $K_5$  or  $K_{3,3}$  and not just a subdivision. However, since  $ab$  is not an edge of  $G_2^*$ , then  $G_2^*$  must be  $K_{3,3}$ .

Thus  $G_2^* = K_{3,3}$  and  $G_1$  contains a subdivision of  $K_{3,3}$  or  $K_5$  that includes  $ab$  as an edge. This means  $G$  includes one of the graphs of [Theorem 5.11](#) as a proper minor and is not minor minimal.

This completes the last subcase of the last case and shows that  $ab$  is not an edge of  $G$ .  $\square$

For  $G$  of connectivity 2 with cut set  $\{a, b\}$ , we have  $G - a, b = G'_1 \sqcup G'_2$ . We will use  $G_i$  to denote the induced subgraph on  $V(G'_i) \cup \{a, b\}$ .

**Lemma 5.13.** *If  $G$  is MMNE,  $\kappa(G) = 2$ , and  $G_1$  and  $G_2$  are both nonplanar, then  $G_1 = G_2 = K_{3,3}$ .*

*Proof.* Let  $G$  be as described. First suppose for the sake of contradiction that  $G_1$  is nonplanar but not  $K_{3,3}$ . Note that  $G_1$  cannot be  $K_5$  because  $ab \notin E(G)$  by [Lemma 5.12](#). So  $G_1$  has some nonplanar proper minor  $H$ , and  $H \cup G_2$  is a proper minor of  $G$ . Since there are no edges between  $H$  and  $G_2$ , the apex edge of  $H \cup G_2$  must be in exactly one of  $H$  or  $G_2$ . Whichever one does not contain the apex edge will be a nonplanar subgraph even when the edge is removed, contradicting the fact that  $G$  is MMNE. Therefore  $G_1 = K_{3,3}$ . A symmetrical argument can be made for  $G_2$ .  $\square$

**Lemma 5.14.** *If  $G$  is MMNE,  $\kappa(G) = 2$ , with cut set  $\{a, b\}$ ,  $\delta(G) \geq 3$ , and both  $G_1$  and  $G_2$  are planar, then  $G_i \in \{K_5 - e, K_{3,3} - e\}$  with  $ab$  as the missing edge.*

*Proof.* Let  $G$  be as described. For a contradiction, assume that  $G_1 + ab$  is planar. Since  $G$  is NE, for every  $e \in E(G)$ , the graph  $G - e$  is nonplanar and, therefore, has a K-subgraph,  $H$ . By [Lemma 1.8](#) and our assumption that  $G_1 + ab$  is planar,  $H \cap G_1$  is an  $a$ - $b$ -path. In particular  $G_2 + ab$  is nonplanar.

Note that there are edge-disjoint  $a$ - $b$ -paths  $P_1$  and  $P_2$  in  $G_1$ . If not, say every  $a$ - $b$ -path goes through the edge  $e'$ . Then  $G - e'$  must be planar as, by [Lemma 1.8](#), a K-subgraph of  $G - e'$  would either use a path in  $G_1$ , which is not possible as all such paths pass through  $e'$ , or else use a path in  $G_2$ , which is not possible since  $G_1 + ab$  is planar. The contradiction shows there are edge-disjoint paths  $P_1$  and  $P_2$ .

This means we can construct a proper minor  $M$  of  $G$  by adding a triangle on  $ab$ . That is,  $V(M) = V(G_2) \cup \{c\}$  and  $E(M) = E(G_2) \cup \{ab, bc, ac\}$ . Since  $G$  is NE, for any  $e \in E(G_2)$ , the graph  $G - e$  is nonplanar with a K-subgraph that uses only a path in  $G_1$ . So,  $M - e$  is also nonplanar. On the other hand, if we delete any  $e$  in  $\{ab, ac, bc\}$ , we are left with a subgraph of  $M - e$  homeomorphic to  $G_2 + ab$ . So  $M - e$  is again nonplanar. Then  $M$  is a proper NE minor of  $G$  contradicting  $G$  being minor minimal.

We conclude  $G_1 + ab$  is nonplanar. A similar argument shows  $G_2 + ab$  is nonplanar as well. Then  $G$  must have one of the NE graphs  $G_1 \ddot{\cup} G_2$  with  $G_i \in \{K_5 - e, K_{3,3} - e\}$  as a minor. Since  $G$  is minor minimal,  $G$  is a graph of this form.  $\square$

**Lemma 5.15.** *If  $G$  is MMNE,  $\kappa(G) = 2$ ,  $\delta(G) \geq 3$ ,  $G_1$  is planar, and  $G_2$  is nonplanar, then  $G_1 \in \{K_5 - e, K_{3,3} - e\}$ , sharing two vertices and no edges with  $G_2 = K_{3,3}$ .*

*Proof.* Let  $G$  be as described. For a contradiction, suppose  $G_1 + ab$  is planar. Then  $G_2 + ab$  must be NE. Indeed, if we delete  $ab$ , we are left with the nonplanar  $G_2$ . Let  $e \in E(G_2)$ . Since  $G$  is NE,  $G - e$  is nonplanar and has a K-subgraph  $K$ . If  $K$  uses at most one of  $\{a, b\}$ , then  $K$  lies entirely in  $G_2$  and avoids  $e$ . So,  $(G_2 + ab) - e$  is nonplanar in this case. On the other hand, if  $\{a, b\} \subset V(K)$ , then, by Lemma 1.8 and since  $G_1 + ab$  is planar, the part of  $K$  in  $G_1$  is an  $a$ - $b$ -path. So using edge  $ab$  instead,  $K$  remains as a K-subgraph of  $(G_2 + ab) - e$ , which is again nonplanar. However,  $G_2 + ab$  being NE contradicts  $G$  being minor minimal. We conclude  $G_1 + ab$  is nonplanar.

This means  $G_1$  has one of  $K_5 - e$  and  $K_{3,3} - e$  as a minor with the missing edge corresponding to  $ab$ . Replace  $G_1$  by its minor  $K_5 - e$  or  $K_{3,3} - e$ , call it  $H$ , to form  $M = H \cup G_2$ , a minor of  $G$ . We claim  $M$  is again NE. Indeed, if we delete  $e \in E(H)$ , the graph  $G_2$  shows  $M - e$  is nonplanar. For  $e \in E(G_2)$ , we know  $G - e$  has a K-subgraph  $K$ . If  $K$  sees at most one of  $a$  and  $b$ , it must lie entirely in  $G_2$  (since  $H$  is planar) and  $M - e$  is nonplanar. If  $\{a, b\} \subset V(K)$ , then, by Lemma 1.8,  $K$  is simply a path on one side of the 2-cut. If  $K$  is a path in  $G_1$ , then replace that by a path in  $H$  to recognize  $K$  as a subgraph of  $M - e$ , which is therefore nonplanar. On the other hand, if  $K$  is a path in  $G_2$ , this path avoids  $e$ . So, we can use  $H$  along with that path to again find a nonplanar subgraph of  $M - e$ . Since  $G$  is minor minimal,  $G = M$  and  $G_1 \in \{K_5 - e, K_{3,3} - e\}$  as required.

Now,  $G_2$  being nonplanar has a K-subgraph  $K$ . Also, there must be an  $a$ - $b$ -path  $P$  in  $G_2$ , as otherwise  $G$  has connectivity 1. Moreover, both  $K$  and  $G_1 \cup P$  are nonplanar, and so they must overlap, as otherwise  $G$  has a proper disconnected MMNE minor. This means  $P$  passes through  $K$  and, by contracting edges in  $P$  if necessary, we can assume  $G$  has a minor with  $\{a, b\} \subset V(K)$ . From this, form the minor  $M = G_1 \cup K$ . If  $K$  is a subdivision of  $K_5$ , Then  $M$  and hence  $G$  has the MMNA graph  $G_1 \ddot{\cup} (K_5 - e)$  as a proper minor, which is a contradiction. So,  $K$  is a subdivision of  $K_{3,3}$ . After contracting edges,  $G$  either has the MMNA  $G_1 \ddot{\cup} (K_{3,3} - e)$  as a proper minor, which is a contradiction, or else  $G$  has  $G_1 \ddot{\cup} K_{3,3}$  as a minor, where  $a$  and  $b$  are in the same part of  $K_{3,3}$ . Since  $G$  was minor minimal, we conclude  $G = G_1 \ddot{\cup} K_{3,3}$ . In other words, as required,  $G_2 = K_{3,3}$ , sharing two vertices and no edge with  $G_1 \in \{K_5 - e, K_{3,3} - e\}$ .  $\square$

**Theorem 5.16.** *If  $G$  is MMNE,  $\kappa(G) = 2$ , and  $\delta(G) \geq 3$ , then  $G$  is one of the six graphs of Figure 8.*

*Proof.* We showed that these six graphs are MMNE in Theorem 5.11. Lemma 5.13 immediately gives that if  $G_1$  and  $G_2$  are both nonplanar, then they are both  $K_{3,3}$ . Lemmas 5.14 and 5.15 complete the other parts of the proof. In total, these account for six graphs: one from Lemma 5.13, three from Lemma 5.14, and two from Lemma 5.15.  $\square$

The restriction on the minimum degree in the last theorem is necessary. Indeed, there are many MMNE graphs with  $\delta(G) = 2$  (meaning  $\kappa(G) \leq 2$ ). For example, contracting edge  $e$  of Figure 7 (left) results in an MMNE graph that is formed by replacing each edge of  $K_{3,3}$  with a triangle. Similarly, replacing each edge of  $K_5$  with a triangle also yields an MMNE graph. Further examples of MMNE graphs with a degree-2 vertex are the first seven listed in Section A.1 of the Appendix.

We remark that these examples arise in part due to our insistence that edge contraction lead to a simple graph. Contracting an edge of a degree-2 vertex in a triangle gives a (multi)graph with a doubled edge. Our convention is to delete one of the doubled edges to return to a simple graph.

We next show that  $\delta(G) = 2$  is the minimum for MMNE graphs.

**Theorem 5.17.** *The minimum vertex degree in an MMNE graph is at least 2.*

*Proof.* The addition or deletion of an isolated vertex or vertex of degree 1 in a planar graph will again result in a planar graph. So if  $G$  is NE with  $\delta(G) < 2$ , then removing a vertex of degree 0 or 1 will result in a NE graph; hence  $G$  is not MMNE.  $\square$

Although we cannot completely classify the  $\delta(G) = 2$  MMNE graphs, we show that degree-2 vertices must occur as part of a triangle.

**Theorem 5.18.** *In an MMNE graph, the neighbors of a degree-2 vertex are themselves neighbors.*

*Proof.* Let  $G$  be an NE graph with a degree-2 vertex  $v$  with neighbors  $a$  and  $b$ . For a contradiction, suppose  $ab$  is not an edge of  $G$ . Perhaps  $G$  is MMNE so that every proper minor of  $G$  is not NE. Let  $H = G/av$  be the graph that results from contracting edge  $av$  in  $G$ . Since  $G$  is MMNE, there must be some edge  $e$  in  $H$  such that  $H - e$  is planar. Note that  $e$  cannot be the newly formed edge  $ab$  in  $H$ , else, since degree-1 vertices have no impact on the planarity of a graph,  $G - av$  would also be planar, contradicting  $G$  being MMNE. Consider the graph  $G - e$ . Note that  $G - e$  and  $H - e$  are homeomorphic, so since  $H - e$  is planar,  $G - e$  is also planar. But this contradicts  $G$  being MMNE.  $\square$

If graph  $G$  has a triangle  $abc$ , a  $\nabla Y$  move on  $G$  means forming a new graph  $G'$  with one additional vertex  $v$  (i.e.,  $V(G') = V(G) \cup \{v\}$ ) and replacing the edges

$ab$ ,  $ac$ , and  $bc$  with  $va$ ,  $vb$ ,  $vc$ . So,  $G'$  has the same number of edges as  $G$  and one additional vertex. Pierce [2014] shows that  $\nabla Y$  often preserves NA, as was originally observed by Barsotti in unpublished work. (The bowtie graphs of Figure 3 are examples where  $\nabla Y$  does not preserve NA.) Here we give a similar result for NE graphs.

**Theorem 5.19.** *Given an NE graph  $G$  with triangle  $t$ , let  $G'$  be the result of performing a  $\nabla Y$  move on triangle  $t$  in  $G$ , and let  $v$  be the vertex added in  $G'$ . Graph  $G'$  is NE if and only if  $G' - e_i$  is nonplanar for each  $e_i$  incident to  $v$ .*

*Proof.* If  $G'$  is NE, then  $G' - e_i$  is nonplanar by definition. Conversely suppose that  $G' - e_i$  is nonplanar for each  $e_i$  incident to  $v$ . Perhaps  $G'$  is not NE, so there is  $e \in E(G')$  such that  $G' - e$  is planar. Note that  $e$  cannot be incident to  $v$ . Since  $e$  is not part of triangle  $t$ , performing a  $\nabla Y$  move on  $G - e$  will result in  $G' - e$ , so  $\nabla Y$  on  $G - e$  is also planar. Note that undoing the  $\nabla Y$  transform on this graph will preserve its planarity. However, graph  $G - e$  being planar contradicts  $G$  being NE.  $\square$

We next give an upper bound on the connectivity of MMNE graphs. We first observe that the minimum degree  $\delta(G)$  is bounded by 5.

**Theorem 5.20.** *If  $G$  is MMNE, then  $\delta(G) \leq 5$ .*

*Proof.* Suppose  $G$  is MMNE with  $\delta(G) \geq 6$  and let  $n = |V(G)|$ . We can assume  $n \geq 6$ , as  $G$  must be nonplanar and the only nonplanar graph with five or fewer vertices is  $K_5$ , which is not MMNE. Since  $\delta(G) \geq 6$ , a lower bound on  $|E(G)|$  is  $6n/2 = 3n$ . Now since  $G$  is MMNE, there exist two edges  $e$  and  $f$  such that  $G - e, f$  is a planar graph with at least  $3n - 2$  edges. However, a planar graph on  $n$  vertices can have no more than  $3n - 6$  edges, the number of edges in a planar triangulation. The contradiction shows there is no MMNE graph with  $\delta(G) \geq 6$ .  $\square$

As  $\kappa(G) \leq \delta(G)$ , we have a bound on the connectivity as an immediate corollary.

**Corollary 5.21.** *If  $G$  is MMNE, then  $\kappa(G) \leq 5$ .*

Finally, we observe a nice connection between MMNE and MMNA graphs.

**Theorem 5.22.** *If  $G$  is MMNE, then  $G$  is MMNA or apex.*

*Proof.* Suppose  $G$  is MMNE and NA. We will argue that  $G$  is in fact MMNA. For this, let  $H$  be a proper minor. Since  $G$  is MMNE,  $H$  is edge apex. This means either  $H$  is already planar, or else there is an edge  $e$  such that  $H - e$  is planar. In the latter case, if  $v$  is a vertex of  $e$ , then  $H - v$  is again planar. This shows that  $H$  is apex, as required.  $\square$

**Results of computer searches.** In addition to the results above, we have found other examples of MMNE and MMNC graphs through brute-force computer searches. Our code is available at <https://github.com/mikepierce/MMGraphFunctions/tree/master/brute-force-search>. See the file `Brute-Force-Search.nb` for documentation.

The algorithms underlying the searches are fairly straightforward. First we generate a list of all the graphs that we are going to search using the tools that are available with the `nauty` and `Traces` graph theory software [McKay and Piperno 2014]. Specifically, we use the tools `geng` and `planarg` to produce all connected, nonplanar graphs of minimum vertex degree at least 2 that either have fewer than 20 edges or that have fewer than 10 vertices. The commands used to generate these graphs in `bash` are the following:

```
$ for i in {6..9}; do
geng -c -d2 ${i} | planarg -v > ${i}v.txt
done
$ for i in {10..16}; do
geng -c -d2 ${i} 0:17 | planarg -v > ${i}v,(0-17)e.txt
geng -c -d2 ${i} 18 | planarg -v > ${i}v,(18)e.txt
geng -c -d2 ${i} 19 | planarg -v > ${i}v,(19)e.txt
done
```

This brute force search was carried out on a standard laptop computer with 4 GB of memory and an Intel Core i3-350M 2.266 GHz processor. The graphs to be searched were split among many different files so that the search could be run in more manageable segments and so that we did not overflow the laptop's memory. We chose to limit our search to graphs with fewer than 20 edges or fewer than 10 vertices due to time constraints. There are a total of 158 505 connected, nonplanar graphs that have 9 vertices and a minimum vertex degree of at least 2. Searching these graphs took about five hours. Since there are 9 229 423 such graphs on 10 vertices, searching these would take more than ten days. Similarly it took about three days to search all 7 753 990 connected, nonplanar graphs that have 19 edges and a minimum vertex degree of at least 2, so searching all 44 858 715 similar graphs on 20 edges is not feasible.

Next we reformat these graphs in each file produced to be read into Wolfram Mathematica. Then we use Mathematica functions to iterate over this list of graphs one file at a time and pull out any that are found to be either MMNE or MMNC. The code in Mathematica was run on a single Mathematica kernel (no attempt was made to parallelize the search in Mathematica). An overview of the method of testing if a graph  $G$  is MMNE is as follows, and an analogous method is used to test if a graph is MMNC:

- (1) For each  $e \in E(G)$ , if  $G - e$  is planar return false.
- (2) Build all the simple minors of  $G$  (the graphs in  $\{G - e, G/e \mid e \in E(G)\}$ ) and remove any duplicates (under isomorphism). If for any of these graphs there is no edge  $f$  such that  $G - f$  is planar, return false.
- (3) Take  $S = \{G\} \cup \{G - e \mid e \in E(G)\}$ . While  $S \neq \emptyset$ :



- (a) Reset  $S$  to the result of contracting each edge of each graph in  $S$ .
- (b) Remove all planar graphs and duplicate graphs from  $S$ .
- (c) If there exists  $G \in S$  such that  $G - e$  is nonplanar for each  $e \in E(G)$  then return false.

(4) Return true.

We need step (3) explicitly because both of the properties edge apex and contraction apex are *not* closed under taking graph minors as shown in Theorems 5.2 and 5.4.

In addition to the 12 MMNE graphs that have been considered in this section, the brute-force search has found 15 more examples of MMNE graphs (listed in Section A.1 of the Appendix). Notable graphs in this list are  $K_{4,3}$ ,  $K_6 - e$ , the rook’s graph on 9 vertices, and some examples of MMNE graphs with degree-2 vertices. The brute-force search also found new examples of MMNC graphs in addition to the six graphs considered in this section. In particular, the computer demonstrated that the six MMNE graphs of connectivity 2 in Figure 8 are also MMNC. Along with these graphs there are 69 other MMNC graphs on 19 or fewer edges or 9 or fewer vertices. Section A.2 of the Appendix is an abridged list of these graphs (those on 17 or fewer edges or 9 or fewer vertices).

Beyond a simple brute-force search, we also conducted a more intelligent graph search using the knowledge that performing  $\nabla Y$  and  $Y \nabla$  moves on a graph has the potential to preserve the NE or NC property of that graph; see Theorem 5.19. The idea is that the  $\nabla Y$  or  $Y \nabla$  families of an MMNE or MMNC graph may contain new MMNE or MMNC graphs. The details of the methodology of this search, as well as the Mathematica code, can be found in [Pierce 2014]. In total, we have found 55 MMNE graphs and 82 MMNC graphs, and we suspect that there are many more of each. Tables 3 and 4 below give a classification of the MMNE and MMNC graphs we have found organized by graph size.

graph size ( $ E(G) $ )	$\leq 11$	12	13	14	15	16	17	18	19	20
number of MMNE graphs	0	1	0	2	0	2	3	11	6	$\geq 2$

graph size ( $ E(G) $ )	21	22	23	24	25	26	27	28	29	30
number of MMNE graphs	$\geq 13$	$\geq 7$	$\geq 4$	$\geq 2$	$\geq 0$	$\geq 0$	$\geq 1$	$\geq 0$	$\geq 0$	$\geq 1$

**Table 3.** The number of MMNE graphs we have found grouped by size. Note that this is a complete classification based on graph size up to and including size 19.

graph size ( $ E(G) $ )	$\leq 11$	12	13	14	15	16	17	18	19	20
number of MMNC graphs	0	1	0	0	1	6	14	32	25	$\geq 3$

**Table 4.** The number of MMNC graphs we have found grouped by size. Note that this is a complete classification based on graph size with the exception of size 20.

### Appendix: Edge lists of graphs found through computer searches

**A.1. MMNE graphs.** The following 15 MMNE graphs are the result of a computer search conducted on the set of graphs that have 19 or fewer edges or 9 or fewer vertices, and that all have a minimum vertex degree of at least 2. These graphs, together with eleven other graphs considered explicitly in the paper (i.e., all but  $K_5 \sqcup K_5$ , which has order 10 and size 20) make up all 26 MMNE graphs on 19 or fewer edges or on 9 or fewer vertices. (Note that Table 3 gives 25 graphs of size 19 or less. Adding the graph  $K_5 \dot{\cup} K_5$ , of order 9 and size 20, is what brings the total to 26.)

$\{(1, 8), (1, 9), (2, 4), (2, 7), (2, 8), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 8), (5, 6), (5, 7), (5, 9), (6, 7), (6, 9), (7, 9), (8, 9)\}$

$\{(1, 6), (1, 7), (2, 5), (2, 7), (3, 7), (3, 8), (3, 9), (4, 5), (4, 6), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (8, 9)\}$

$\{(1, 8), (1, 9), (2, 6), (2, 7), (2, 9), (3, 5), (3, 7), (3, 9), (4, 5), (4, 6), (4, 9), (5, 6), (5, 7), (5, 8), (6, 7), (6, 8), (7, 8), (8, 9)\}$

$\{(1, 8), (1, 9), (2, 7), (2, 10), (3, 6), (3, 8), (3, 10), (4, 6), (4, 7), (4, 9), (5, 6), (5, 7), (5, 8), (6, 9), (6, 10), (7, 8), (7, 10), (8, 9), (9, 10)\}$

$\{(1, 9), (1, 10), (2, 7), (2, 8), (2, 10), (3, 7), (3, 8), (3, 9), (4, 6), (4, 8), (4, 10), (5, 6), (5, 7), (5, 9), (6, 7), (6, 8), (7, 10), (8, 9), (9, 10)\}$

$\{(1, 6), (1, 9), (2, 7), (2, 8), (3, 6), (3, 7), (3, 10), (4, 5), (4, 6), (4, 7), (4, 10), (5, 8), (5, 9), (5, 10), (6, 9), (7, 8), (8, 9), (8, 10), (9, 10)\}$

$\{(1, 8), (1, 10), (2, 4), (2, 8), (2, 9), (3, 4), (3, 5), (3, 9), (4, 5), (4, 6), (5, 7), (5, 10), (6, 7), (6, 8), (6, 9), (7, 9), (7, 10), (8, 10), (9, 10)\}$

$\{(1, 6), (1, 7), (1, 9), (2, 7), (2, 8), (2, 9), (3, 6), (3, 8), (3, 9), (4, 5), (4, 8), (4, 9), (5, 6), (5, 7), (5, 9), (6, 8), (7, 8)\}$

$\{(1, 7), (1, 8), (1, 9), (2, 6), (2, 8), (2, 9), (3, 6), (3, 7), (3, 9), (4, 6), (4, 7), (4, 8), (5, 6), (5, 7), (5, 8), (5, 9)\}$

{(1, 6), (1, 7), (1, 8), (2, 5), (2, 7), (2, 8), (3, 4), (3, 7), (3, 8), (4, 5),  
(4, 6), (4, 7), (4, 8), (5, 6), (5, 7), (5, 8), (6, 7), (6, 8)}

{(1, 6), (1, 7), (1, 9), (2, 5), (2, 7), (2, 8), (3, 7), (3, 8), (3, 9), (4, 5),  
(4, 6), (4, 8), (4, 9), (5, 7), (5, 9), (6, 7), (6, 8), (8, 9)}

{(1, 4), (1, 7), (1, 8), (2, 3), (2, 7), (2, 8), (3, 5), (3, 6), (4, 5), (4, 6),  
(5, 7), (5, 8), (6, 7), (6, 8)}

{(1, 5), (1, 6), (1, 7), (2, 5), (2, 6), (2, 7), (3, 5), (3, 6), (3, 7), (4, 5),  
(4, 6), (4, 7)}

{(1, 6), (1, 7), (1, 8), (1, 9), (2, 4), (2, 5), (2, 8), (2, 9), (3, 4), (3, 5),  
(3, 6), (3, 7), (4, 7), (4, 9), (5, 6), (5, 8), (6, 9), (7, 8)}

{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5),  
(3, 6), (4, 5), (4, 6), (5, 6)}

**A.2. MMNC graphs.** The following 22 MMNC graphs are the result of a computer search conducted on the set of graphs that have 17 or fewer edges or 9 or fewer vertices, and that all have a minimum vertex degree of at least 2.

{(1, 9), (1, 12), (2, 8), (2, 11), (3, 6), (3, 7), (4, 5), (4, 10), (5, 11), (5, 12),  
(6, 9), (6, 11), (7, 8), (7, 12), (8, 10), (9, 10)}

{(1, 6), (1, 10), (2, 5), (2, 9), (3, 4), (3, 6), (3, 8), (4, 5), (4, 7), (5, 10),  
(6, 9), (7, 9), (7, 11), (8, 10), (8, 11), (9, 11), (10, 11)}

{(1, 6), (1, 10), (2, 7), (2, 8), (2, 9), (3, 6), (3, 8), (3, 9), (4, 7), (4, 9),  
(4, 10), (5, 7), (5, 8), (5, 10), (6, 7), (8, 10), (9, 10)}

{(1, 9), (1, 10), (2, 3), (2, 6), (2, 7), (3, 4), (3, 5), (4, 7), (4, 10), (5, 6),  
(5, 9), (6, 8), (6, 10), (7, 8), (7, 9), (8, 9), (8, 10)}

{(1, 9), (1, 11), (2, 9), (2, 10), (3, 4), (3, 6), (3, 11), (4, 5), (4, 10), (5, 8),  
(5, 9), (6, 7), (6, 9), (7, 10), (7, 11), (8, 10), (8, 11)}

{(1, 9), (1, 11), (2, 9), (2, 10), (3, 5), (3, 6), (3, 7), (4, 5), (4, 6), (4, 9),  
(5, 11), (6, 10), (7, 8), (7, 9), (8, 10), (8, 11), (10, 11)}

{(1, 4), (1, 11), (2, 6), (2, 9), (3, 5), (3, 6), (3, 7), (4, 5), (4, 9), (5, 10),  
(6, 11), (7, 9), (7, 10), (8, 9), (8, 10), (8, 11), (10, 11)}

{(1, 9), (1, 11), (2, 4), (2, 5), (2, 6), (3, 5), (3, 6), (3, 7), (4, 8), (4, 9),  
(5, 11), (6, 10), (7, 9), (7, 10), (8, 10), (8, 11), (10, 11)}

{(1, 10), (1, 11), (2, 3), (2, 7), (2, 9), (3, 6), (3, 8), (4, 5), (4, 9), (4, 10),  
(5, 8), (5, 11), (6, 7), (6, 11), (7, 10), (8, 10), (9, 11)}

{(1, 8), (1, 9), (2, 6), (2, 12), (3, 5), (3, 11), (4, 11), (4, 12), (5, 7), (5, 9), (6, 7), (6, 8), (7, 10), (8, 11), (9, 12), (10, 11), (10, 12)}

{(1, 9), (1, 11), (2, 5), (2, 12), (3, 4), (3, 12), (4, 8), (4, 9), (5, 7), (5, 9), (6, 7), (6, 8), (6, 11), (7, 10), (8, 10), (10, 12), (11, 12)}

{(1, 4), (1, 8), (1, 9), (2, 3), (2, 8), (2, 9), (3, 4), (3, 6), (3, 9), (4, 5), (4, 8), (5, 6), (5, 7), (5, 9), (6, 7), (6, 8), (7, 8), (7, 9)}

{(1, 4), (1, 8), (1, 9), (2, 4), (2, 7), (2, 9), (3, 4), (3, 6), (3, 9), (5, 6), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (7, 8)}

{(1, 5), (1, 6), (1, 8), (2, 3), (2, 4), (2, 7), (3, 6), (3, 10), (4, 5), (4, 10), (5, 9), (6, 9), (7, 9), (7, 10), (8, 9), (8, 10)}

{(1, 5), (1, 6), (1, 8), (2, 3), (2, 4), (2, 7), (3, 6), (3, 10), (4, 5), (4, 9), (5, 10), (6, 9), (7, 9), (7, 10), (8, 9), (8, 10)}

{(1, 2), (1, 9), (1, 10), (2, 7), (2, 8), (3, 8), (3, 9), (3, 10), (4, 7), (4, 9), (4, 10), (5, 7), (5, 8), (5, 10), (6, 7), (6, 8), (6, 9)}

{(1, 2), (1, 4), (1, 10), (2, 3), (2, 9), (3, 4), (3, 7), (4, 8), (5, 7), (5, 8), (5, 10), (6, 7), (6, 8), (6, 9), (7, 10), (8, 9), (9, 10)}

{(1, 5), (1, 6), (1, 7), (2, 5), (2, 6), (2, 7), (3, 5), (3, 6), (3, 7), (4, 5), (4, 6), (4, 7)}

{(1, 2), (1, 4), (1, 7), (1, 9), (2, 3), (2, 6), (2, 8), (3, 5), (3, 6), (3, 9), (4, 5), (4, 7), (4, 8), (5, 8), (5, 9), (6, 8), (6, 9), (7, 8), (7, 9)}

{(1, 6), (1, 7), (1, 8), (1, 9), (2, 4), (2, 5), (2, 8), (2, 9), (3, 4), (3, 5), (3, 6), (3, 7), (4, 7), (4, 9), (5, 6), (5, 8), (6, 9), (7, 8)}

{(1, 5), (1, 6), (1, 7), (1, 8), (2, 3), (2, 4), (2, 7), (2, 8), (3, 4), (3, 6), (3, 8), (4, 5), (4, 8), (5, 6), (5, 7), (6, 7)}

{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)}

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