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On G-graphs of certain finite groups

Mohammad Reza Darafsheh and Safoora Madady Moghadam

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The notion of G-graph was introduced by Bretto et al. and has interesting properties. This graph is related to a group G and a set of generators S of G and is denoted by $\Gamma(G, S)$. In this paper, we consider several types of groups G and study the existence of Hamiltonian and Eulerian paths and circuits in $\Gamma(G, S)$.

1. Introduction

Let G be a finitely generated group with a generating set $S = \{s_1, s_2, \ldots, s_n\}$. The left transversal of the left cosets of the subgroup $\langle s_i \rangle$ in G is denoted by $T_{\langle s_i \rangle}$. This means that $\{\langle s_i \rangle x \mid x \in T_{\langle s_i \rangle}\}$ is the set of all the distinct left cosets of $\langle s_i \rangle$ in G. A simple graph $\Gamma(G, S)$ is defined as follows: the vertex set of $\Gamma(G, S)$ is the set $\{\langle s_i \rangle x_j \mid x_j \in T_{\langle s_i \rangle}\}$, and two distinct vertices $\langle s_i \rangle x_j$ and $\langle s_k \rangle x_l$ are joined by an edge if $\langle s_i \rangle x_j \cap \langle s_k \rangle x_l \neq \varnothing$.

The G-graphs were introduced in [Bretto and Faisant 2005] to study the group isomorphism problem. They also defined a similar graph $\bar{\Gamma}(G, S)$, which differs from $\Gamma(G, S)$ by the fact that there are p edges between $\langle s_i \rangle x_j$ and $\langle s_k \rangle x_l$ if $|\langle s_i \rangle x_j \cap \langle s_k \rangle x_l| = p$. In this paper, we are more concerned with the simple graph $\Gamma(G, S)$. For more information on the subject see, for example, [Bretto et al. 2007; Bretto and Gillibert 2005]. By [Bretto et al. 2007], if S is a generating set of G, then $\Gamma(G, S)$ is a connected graph. We always choose S such that $G = \langle S \rangle$.

The existence of Hamiltonian paths and circuits in $\Gamma(G, S)$ was the main interest of [Bretto and Faisant 2011]. In [Bauer et al. 2008] the authors considered various classes of finite groups G and studied the Eulerianness and Hamiltonicity of the graph $\Gamma(G, S)$. For instance, they studied the Hamiltonicity of certain G-graphs on the groups $Z_m \times Z_n$ and D_{2n}, the dihedral group of order $2n$. In this paper we will consider the groups $Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k}$ such that $n_1 \mid n_2 \mid \cdots \mid n_k$, the dicyclic group T_{4n} of order $4n$ with presentation

$$T_{4n} = \langle a, b \mid a^{2n} = e,\ a^n = b^2,\ b^{-1}ab = a^{-1} \rangle,$$

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V_{8n}, a group of order $8n$ with presentation

$$V_{8n} = \langle a, b \mid a^{2n} = b^4 = e,\ ba = a^{-1}b^{-1},\ b^{-1}a = a^{-1}b\rangle,$$

and obtain the conditions under which $\Gamma(G, S)$ is Eulerian or Hamiltonian.

2. Preliminaries

Let $S = \{s_1, s_2, \dots, s_n\}$ be a generating set for the group G. Let

$$V_{s_i} = \{\langle s_i\rangle x_j \mid x_j \in T_{\langle s_i\rangle}\}, \quad 1 \leqslant i \leqslant n,$$

where $T_{\langle s_i\rangle}$ is a complete set of left transversals of $\langle s_i\rangle$ in G. Then by definition the vertex set of $\Gamma(G, S)$ is $V(\Gamma(G, S)) = \bigsqcup_{i=1}^{n} V_{s_i}$. The graph $\Gamma(G, S)$ is connected and n-partite. We recall some results which will be used in this paper.

Result 1 [Bondy and Murty 1976]. Let Γ be a nontrivial connected graph. Then:

(a) Γ has an Eulerian circuit if and only if every vertex of Γ has even degree.

(b) Γ has an Eulerian path if and only if Γ has exactly two vertices of odd degree. Furthermore, the path begins at one of the vertices of odd degree and terminates at the other one.

Result 2 [Bauer et al. 2008]. Let G be a group with a generating set given by $S = \{s_1, s_2, \dots, s_n\}$. Let $S_{ij} = |\langle s_i\rangle \cap \langle s_j\rangle|$. Then the degree of the vertex $\langle s_i\rangle$ in the graph $\Gamma(G, S)$ is equal to $\deg(\langle s_i\rangle) = \sum_{i=1}^{n}(o(s_i)/S_{ij}) - 1$, where $o(s_i)$ denotes the order of the element $s_i \in G$. Note that for all elements $x_j\langle s_i\rangle$ in V_i we have $\deg(x_j\langle s_i\rangle) = \deg(\langle s_i\rangle)$.

Result 3 [Bauer et al. 2008]. Let $G = Z_n \times Z_m$ and $S = \{(1, 0), (0, 1)\}$. Then $\Gamma(G, S)$ has a Hamiltonian path if and only if $|m - n| \leqslant 1$.

In the following we generalize Result 3 to obtain a necessary condition for a Hamiltonian circuit of $\Gamma(G, S)$.

Theorem 2.1. *Let $G = \langle a, b\rangle$, $S = \{a, b\}$ and $X = |G|/o(a)$ and $Y = |G|/o(b)$. If $\Gamma(G, S)$ has a Hamiltonian path, then $|X - Y| \leqslant 1$.*

Proof. Let $V_a = \{a_1, a_2 \cdots a_X\}$ and $V_b = \{b_1, b_2 \cdots b_Y\}$.

Case 1: Assume that the Hamiltonian path begins from a vertex in V_a. Call this vertex a_{i_1}. The next vertex can't be from V_a. Thus it is from V_b. Call this vertex b_{i_1}. In this way, the Hamiltonian path can be represented as $a_{i_1}, b_{i_1}, a_{i_2}, b_{i_2}, \dots$.

If this Hamiltonian path ends with a vertex from V_a, it is represented as

$$a_{i_1},\ b_{i_1},\ a_{i_2},\ b_{i_2},\ \dots,\ a_{i_{X-1}},\ b_{i_{X-1}},\ a_{i_X}.$$

Now notice that $b_{i_1}, b_{i_2}, \dots, b_{i_{X-1}}$ should exhaust all the vertices of V_b exactly once. So $\{b_{i_1}, b_{i_2}, \dots, b_{i_{X-1}}\} = \{b_1, b_2, \dots, b_Y\}$; hence $X - 1 = Y$, which implies

$X - Y = 1$. But if this path ends with a vertex of V_b, it is represented as $a_{i_1}, b_{i_1}, a_{i_2}, b_{i_2}, \dots, a_{i_X}, b_{i_X}$. Similarly, $\{b_{i_1}, b_{i_2}, \dots, b_{i_X}\} = \{b_1, b_2, \dots, b_Y\}$, so $X = Y$.

Case 2: Assume that the Hamiltonian path begins with a vertex from V_b. In the same manner as above, this path can be represented as $b_{i_1}, a_{i_1}, b_{i_2}, a_{i_2}, \dots$.

If this path ends with a vertex from V_a, it is represented by $b_{i_1}, a_{i_1}, b_{i_2}, a_{i_2}, \dots, b_{i_Y}, a_{i_Y}$. Notice that $a_{i_1}, a_{i_2}, \dots, a_{i_Y}$ should exhaust all the vertices of V_a exactly once, so $\{a_{i_1}, a_{i_2}, \dots, a_{i_Y}\} = \{a_1, a_2, \dots, a_X\}$; hence $Y = X$. But if this path, ends with a vertex from V_b, it is represented by $b_{i_1}, a_{i_1}, b_{i_2}, a_{i_2}, \dots, b_{i_{Y-1}}, a_{i_{Y-1}}, b_{i_Y}$. Similarly, $\{a_{i_1}, a_{i_2}, \dots, a_{i_{Y-1}}\} = \{a_1, a_2, \dots, a_X\}$, so $Y - 1 = X$, implying $Y - X = 1$.

Thus in the general case the inequality $|X - Y| \leqslant 1$ holds. $\square$

Result 4. Let $G = Z_n \times Z_m$ and $S = \{(1, 0), (0, 1)\}$. Then $\Gamma(G, S)$ has a Hamiltonian circuit if and only if $m = n$.

A generalization of Result 4 for the existence of a Hamiltonian circuit is given in the following theorem.

Theorem 2.2. *Let $G = \langle a, b\rangle$, $S = \{a, b\}$ and $X = |G|/o(a)$ and $|G|/o(b)$. If $\Gamma(G, S)$ has Hamiltonian circuit, then $X = Y$.*

Proof. Let $V_a = \{a_1, a_2, \dots, a_X\}$ and $V_b = \{b_1, b_2, \dots, b_Y\}$, and assume this circuit starts from a vertex in V_a, which is called a_{i_1}. The next vertex can't be from V_a, so it should be from V_b; call this vertex b_{i_1}. Therefore this circuit can be represented by $a_{i_1}, b_{i_1}, a_{i_2}, b_{i_2}, \dots, a_{i_X}, b_{i_X}, a_{i_1}$. Now notice that $b_{i_1}, b_{i_2}, \dots, b_{i_X}$ should exhaust all the vertices of V_b exactly once. So $\{b_{i_1}, b_{i_2}, \dots, b_{i_X}\} = \{b_1, b_2, \dots, b_Y\}$; hence $X = Y$. $\square$

3. Finite abelian groups

From [Rotman 1995] it's well known that every finite abelian group G is isomorphic to a direct product of cycle groups, say $G \cong Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k}$, where $n_1 \mid n_2 \mid \cdots \mid n_k$. We choose

$$S = \{(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, 0, \dots, 1)\}$$

as a generating set of G. The vector $(0, \dots, 1, \dots, 0)$ with 1 in the i-th position is denoted by e_i, and the zero vector is denoted by $0 = (0, 0, \dots, 0)$.

We are going to generalize the results of Section 3 in [Bauer et al. 2008] and obtain necessary and sufficient conditions in order that $\Gamma(G, S)$ contains an Eulerian path or circuit.

Theorem 3.1. *Let G be a finite abelian group which can be represented by $G \cong Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k}$, where $n_1 \mid n_2 \mid \cdots \mid n_k$. Let $S = \{e_1, e_2, \dots, e_k\}$. Then $\Gamma(G, S)$ has an Eulerian circuit if and only if k is odd or n_1 is even. Furthermore $\Gamma(G, S)$ has an Eulerian path if and only if $G \cong Z_1 \times Z_1$ or $G \cong Z_1 \times Z_2$.*

Figure 1. $\Gamma(Z_1 \times Z_1, S)$.

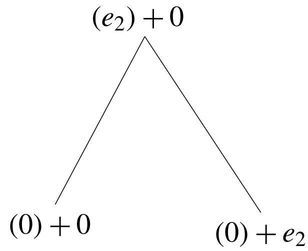

Figure 2. $\Gamma(Z_1 \times Z_2, S)$.

Proof. Let us check the vertices $\langle e_i \rangle + 0$ $(1 \leqslant i \leqslant k)$ of $\Gamma(G, S)$:

$$(e_1)+0 = (0, e_1, 2e_1, \ldots, (n_1-1)e_1),$$
$$(e_2)+0 = (0, e_2, 2e_2, \ldots, (n_2-1)e_2),$$
$$\vdots$$
$$(e_k)+0 = (0, e_k, 2e_k, \ldots, (n_k-1)e_k).$$

For all i, j such that $1 \leqslant i, j \leqslant k,\ \ i \neq j$, we have $((e_i)+0 \cap (e_j)+0) = 0$, so $|(e_i)+0 \cap (e_j)+0| = 1$. Thus for all $(e_i)+x$ and $(e_j)+y$ such that $(e_i)+x \in V_{e_i}$ and $(e_j)+y \in V_{e_j}$, if $|(e_i)+0 \cap (e_j)+0| \neq 0$, then $|(e_i)+0 \cap (e_j)+0| = 1$. So in the simple graph $\Gamma(G, S)$, we have $\deg((e_i)+x) = (k-1)n_i$ for every $(e_i)+x$ from vertices of $\Gamma(G, S)$ (Result 2). Now consider the following cases:

Case 1: If k is odd, then the degree of every vertex of $\Gamma(G, S)$ is even. On the other hand, $G = \langle (1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1) \rangle$. Thus $\Gamma(G, S)$ is connected, so it has an Eulerian circuit but it doesn't have any Eulerian paths (Result 1).

Case 2: Assume that k is even:

Case 2.1: If n_1 is even, then n_i is even for each $1 \leqslant i \leqslant k$, because $n_1 \mid n_2 \mid \cdots \mid n_k$. So the degree of every vertex of $\Gamma(G, S)$ is even; thus it has an Eulerian circuit but it doesn't have any Eulerian paths (Result 1).

Case 2.2: If n_1 is odd and $G \cong Z_1 \times Z_1$, then $\Gamma(G, S)$ is given in Figure 1. It has an Eulerian path, but it doesn't have any Eulerian circuits (Result 1).

Case 2.3: If n_1 is odd and $G \cong Z_1 \times Z_2$, then $\Gamma(G, S)$ is given in Figure 2. It has an Eulerian path, but it doesn't have any Eulerian circuits.

Case 2.4: If n_1 is odd, $n_1 \geqslant 3$ and $G = Z_{n_1} \times Z_{n_2}$, then $n_1 \mid n_2$, so $n_2 \geqslant 3$. On the other hand, the number of vertices of V_{e_1} is $|G|/o(e_1) = n_2$. So $\Gamma(G, S)$ has at least three vertices of odd order. Thus it doesn't have any Eulerian paths or circuits (Result 1).

Case 2.5: If $G = Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k}$ such that n_1 is odd and $k > 2$, then $\Gamma(G, S)$ doesn't have any Eulerian paths or circuits: the number of vertices of V_{e_1} is $|G|/o(e_1) = \prod_{j=2}^{k} n_{i_j}$.

If $\prod_{j=2}^{k} n_{i_j} = 1$, then $G = Z_1 \times \cdots \times Z_1 \times Z_1$, so $\Gamma(G, S)$ has k vertices of odd degree (the degree is $k-1$). Thus $\Gamma(G, S)$ has at least four vertices of odd degree, and hence it doesn't have any Eulerian paths or circuits (Result 1).

If $\prod_{j=2}^{k} n_{i_j} = 2$, then $G = Z_1 \times \cdots \times Z_1 \times Z_2$, so

$$\sum_{r=1}^{k-1} |V_{e_r}| = \sum_{r=1}^{k-1} \frac{|G|}{o(e_r)} = 2(k-1) \geqslant 6.$$

Thus $\Gamma(G, S)$ has at least six vertices of odd degree (the degree is $k-1$), so it doesn't have any Eulerian paths or circuits (Result 1).

If $\prod_{j=2}^{k} n_{i_j} \geqslant 3$, then $\Gamma(G, S)$ has at least three vertices of odd degree (the degree is $n_1(k-1)$), so it doesn't have any Eulerian paths or circuits (Result 1). Therefore the theorem is proved. □

4. Dicyclic group

Let G be the dicyclic group whose presentation is

$$T_{4n} = \langle a, b \mid a^{2n} = e,\ a^n = b^2,\ b^{-1}ab = a^{-1} \rangle, \tag{1}$$

which is a group of order $4n$. We want to check the existence of Eulerian and Hamiltonian circuits and paths in the graph $\Gamma(G, S)$ for a suitable subset S of G.

Theorem 4.1. *Let G be the group* (1) *and $S = \{a, b\}$. If n is even, $\Gamma(G, S)$ has an Eulerian circuit and doesn't have any Eulerian paths. If n is odd, $\Gamma(G, S)$ has an Eulerian path and doesn't have any Eulerian circuits.*

Proof. Clearly $o(b) = 4$. Now we check the vertices $(a)e$ and $(b)e$, where e is the identity element of G:

$$(a)e = (e, a, a^2, \ldots, a^{2n-1}),$$
$$(b)e = (e, b, b^2, b^3) = (e, b, a^n, a^n b).$$

So $(a)e \cap (b)e = \{e, a^n\}$, and thus $|(a)e \cap (b)e| = 2$. Now we know that if $(a)x \cap (b)y \neq \varnothing$, then by [Bauer et al. 2008], $|(a)x \cap (b)y| = 2$. Notice that the number of vertices of V_a is $|G|/o(a) = (4n)/(2n) = 2$. On the other hand $o(b) = 4$, so $\deg((b)y) = 4$ for every $(b)y \in V_b$. Thus every vertex of V_b has exactly

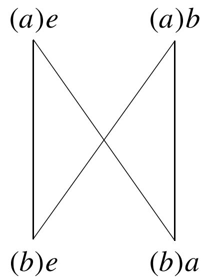

Figure 3. $\Gamma(T_8, \{a, b\})$.

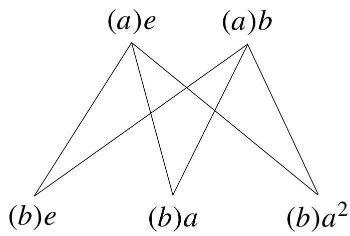

Figure 4. $\Gamma(T_{12}, \{a, b\})$.

two edges to every vertex of V_a. Also we know that the number of vertices of V_b is $|G|/o(b) = 4n/4 = n$; thus $\bar{\Gamma}(G, S)$ is isomorphic to $K^2_{n,2}$, so $\Gamma(G, S) \approxeq K_{n,2}$.

Next if n is even, then $\deg(v)$ is even for every vertex v of $\Gamma(G, S)$; hence $\Gamma(G, S)$ has an Eulerian circuit and it doesn't have any Eulerian paths (Result 1).

But if n is odd, then $\deg(b)y$ is 2 for every $(b)y$ in V_b, and $\deg(a)x$ is n, which is odd for every $(a)x$ in V_a. So $\Gamma(G, S)$ has exactly two vertices of odd order; thus it has an Eulerian path and it doesn't have any Eulerian circuits (Result 1). □

Theorem 4.2. *Let G be the group* (1) *and $S = \{a, b\}$. If $n = 2$, then $\Gamma(G, S)$ has a Hamiltonian path and circuit. If $n = 1$ or 3, then $\Gamma(G, S)$ has Hamiltonian path but it doesn't have any Hamiltonian circuits. If $n \neq 1, 2, 3$, then $\Gamma(G, S)$ doesn't have any Hamiltonian paths or circuits.*

Proof. Assume that $\Gamma(G, S) = K_{n,2}$ has a Hamiltonian path; then $|n - 2| \leqslant 1$ (Theorem 2.1). Therefore just one of the following cases happens:

<u>Case 1</u>: $n = 2$. So $\Gamma(G, S)$ is as in Figure 3. Thus its Hamiltonian path is $(a)e$, $(b)a$, $(a)b$, $(b)e$, and the Hamiltonian circuit is $(a)e$, $(b)a$, $(a)b$, $(b)e$, $(a)e$.

<u>Case 2</u>: $(n-2=1) \Rightarrow (n=3)$. So $\Gamma(G, S)$ is as in Figure 4. Thus its Hamiltonian path is $(b)e$, $(a)e$, $(b)a$, $(a)b$, $(b)a^2$, but it doesn't have any Hamiltonian circuits because $n \neq 2$ (Theorem 2.2).

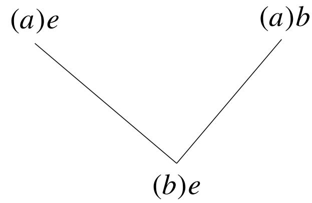

Figure 5. $\Gamma(T_4, \{a, b\})$.

Case 3: $(2-n=1) \Rightarrow (n=1)$. So $\Gamma(G, S)$ is as in Figure 5. Thus its Hamiltonian path is $(a)e$, $(b)e$, $(a)b$, but it doesn't have any Hamiltonian circuits because $n \neq 2$ (Theorem 2.2).

So $\Gamma(G, S)$ has a Hamiltonian circuit if and only if $n=2$, and it has a Hamiltonian path if and only if $n = 1$ or 3. □

Theorem 4.3. *Let G be the group* (1) *and $S = \{ab, b\}$. Then $\Gamma(G, S)$ has Eulerian and Hamiltonian circuits, and the Hamiltonian circuit is just the Eulerian circuit. Also $\Gamma(G, S)$ has a Hamiltonian path, but it doesn't have any Eulerian paths.*

Proof. Clearly $o(ab) = 4$. Now let us check the vertices of V_b:

$$\begin{aligned}(b)e &= (e, b, b^2, b^3),\\ (b)a &= (a, ba, b^2, b^3a),\\ (b)a^2 &= (a^2, ba^2, b^2, b^3a^2),\\ &\vdots\\ (b)a^{n-1} &= (a^{n-1}, ba^{n-1}, b^2, b^3a^{n-1}).\end{aligned}$$

Now notice that $ba^i = a^{2n-i}b$, $(b)^2a^i = a^{n+i}$ and $(b)^3a^i = a^{n-i}b$. So

$$\begin{aligned}(b)e &= (e, b, a^n, (a)^nb),\\ (b)a &= (a, a^{2n-1}b, a^{n+1}, (a)^{n-1}b),\\ (b)a^2 &= (a^2, a^{2n-2}b, a^{n+2}, (a)^{n-2}b),\\ &\vdots\\ (b)a^{n-1} &= (a^{n-1}, a^{n+1}b, a^{2n-1}, ab).\end{aligned}$$

Next let us see the vertices of V_{ab}:

$$\begin{aligned}(ab)e &= (e, ab, (ab)^2, (ab)^3),\\ (ab)a &= (a, aba, (ab)^2a, (ab)^3a),\\ (ab)a^2 &= (a^2, aba^2, (ab)^2a^2, (ab)^3a^2),\\ &\vdots\\ (ab)a^{n-1} &= (a^{n-1}, aba^{n-1}, (ab)^2a^{n-1}, (ab)^3a^{n-1}).\end{aligned}$$

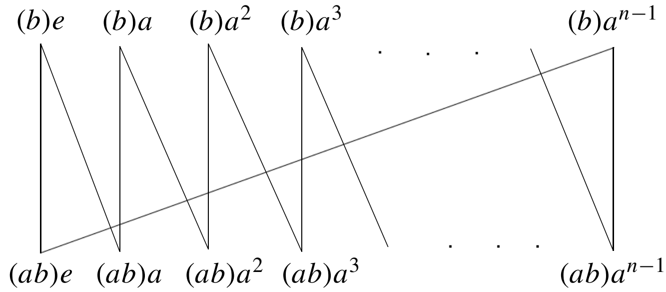

Figure 6. $\Gamma(T_{4n}, \{ab, b\})$.

Since $aba^i = a(ba^i) = a^{2n-1+i}$, we know $(ab)^2 a^i = a_n a^i = a^{n+i}$ and $(ab)^3 a^i = a^{n+1} b a^i = a^{n-i+1}$. So

$$\begin{aligned}(ab)e &= (e, ab, (a)^n, (a)^{n+1}b),\\ (ab)a &= (a, b, (a)^{n+1}, (a)^n b),\\ (ab)a^2 &= (a^2, a^{2n-1}b, (a)^{n+2}, (a)^{n-1}b),\\ &\vdots\\ (ab)a^{n-1} &= (a^{n-1}, a^{n+2}b, (a)^{2n-1}, (a)^2 b).\end{aligned}$$

Thus we have

$$\begin{aligned}(ab)a^i \cap (b)a^i &= \{a^i, a^{n+i}\},\\ (ab)a^{i+1} \cap (b)a^i &= \{a^{2n-i}, a^{n-i}b\},\\ (ab)e \cap (b)a^{n-1} &= \{ab, a^{n+1}b\}.\end{aligned}$$

Therefore $\Gamma(G, S)$ is as shown in Figure 6.

Hence the Eulerian and Hamiltonian circuit is

$$(ab)e,\ (b)e,\ (ab)a,\ (b)a,\ (ab)a^2,\ (b)a^2,\ \dots,\ (ab)a^{n-1},\ (b)a^{n-1},\ (ab)e,$$

the Hamiltonian path is

$$(ab)e,\ (b)e,\ (ab)a,\ (b)a,\ (ab)a^2,\ (b)a^2,\ \dots,\ (ab)a^{n-1},\ (b)a^{n-1}$$

and $\Gamma(G, S)$ doesn't have any Eulerian paths because the degree of every vertex of $\Gamma(G, S)$ is even (Result 1). □

Theorem 4.4. *Let G be the group* (1) *and $S = \{a, ab\}$. If n is even, $\Gamma(G, S)$ has an Eulerian circuit and it doesn't have any Eulerian paths, and if n is odd, $\Gamma(G, S)$ has an Eulerian path and it doesn't have any Eulerian circuits.*

Proof. Let us check the vertices $(a)e$ and $(ab)e$:

$$\begin{aligned}(a)e &= (e, a, a^2, \dots, a^{2n-1}),\\ (b)e &= (e, ab, a^n, a^{n+1}b).\end{aligned}$$

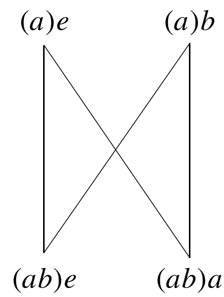

Figure 7. $\Gamma(T_8, \{a, ab\})$.

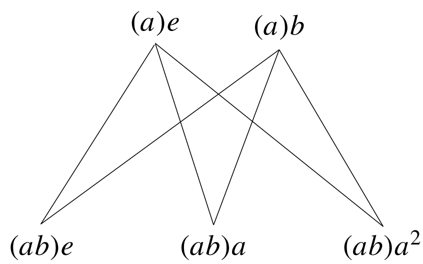

Figure 8. $\Gamma(T_{12}, \{a, ab\})$.

So $(a)e \cap (ab)e = \{e, a^n\}$; thus $|(a)e \cap (ab)e| = 2$. We know that for $(a)x \in V_a$ and $(ab)y \in V_{ab}$, if $(a)x \cap (ab)y \neq \varnothing$, then by [Bauer et al. 2008], $|(a)x \cap (ab)y| = 2$. On the other hand $o(ab) = 4$ so $\deg(ab)x = 4$ for every $(ab)x \in V_{ab}$, and also we know that the number of vertices of V_a is $|G|/o(a) = (4n)/(2n) = 2$. Thus in $\Gamma(G, S)$, every vertex of V_b has an edge to every vertex of V_a, so $\Gamma(G, S)$ is $K_{n,2}$. Now if n is even, the degree of every vertex of $\Gamma(G, S)$ is even, so it has an Eulerian circuit and doesn't have any Eulerian paths (Result 1).

But if n is odd, $\Gamma(G, S)$ has exactly two vertices of odd degree ($(a)e$ and $(a)b$), so it has an Eulerian path and doesn't have any Eulerian circuits (Result 1). □

Theorem 4.5. *Let G be the group* (1) *and $S = \{a, ab\}$. If $n = 2$, then $\Gamma(G, S)$ has a Hamiltonian path and circuit, if $n = 1$ or $n = 3$, then $\Gamma(G, S)$ has a Hamiltonian path and it doesn't have any Hamiltonian circuits, and if $n \neq 1, 2, 3$, then $\Gamma(G, S)$ doesn't have any Hamiltonian paths or circuits.*

Proof. The G-graph $\Gamma(G, S)$ is isomorphic to $K_{n,2}$ (as we have already proved). Assume that it has a Hamiltonian path; then $|n - 2| \leqslant 1$ (Theorem 2.1). So just one of the following cases happens:

Case 1: $n = 2$. So $\Gamma(G, S)$ is as in Figure 7. Therefore its Hamiltonian path is $(a)e$, $(ab)e$, $(a)b$, $(ab)a$, and its Hamiltonian circuit is $(a)e$, $(ab)e$, $(a)b$, $(ab)a$, $(a)e$.

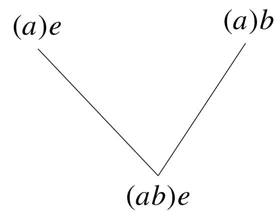

Figure 9. $\Gamma(T_4, \{a, ab\})$.

Case 2: $(n-2 = 1) \Rightarrow (n = 3)$. So $\Gamma(G, S)$ is as in Figure 8. Therefore its Hamiltonian path is $(ab)e$, $(a)e$, $(ab)a$, $(a)b$, $(ab)a^2$. But it doesn't have any Hamiltonian circuits because $n \neq 2$ (Theorem 2.2).

Case 3: $(2-n = 1) \Rightarrow (n = 1)$. So $\Gamma(G, S)$ is as in Figure 9. Therefore its Hamiltonian path is $(a)e$, $(ab)e$, $(a)b$. But it doesn't have any Hamiltonian circuits because $n \neq 2$ (Theorem 2.2). So $\Gamma(G, S)$ has a Hamiltonian circuit if and only if $n = 2$, and it has a Hamiltonian path if and only if $n = 1$ or 3. □

5. The group V_{8n} of order $8n$

The group $G = V_{8n}$ has presentation

$$V_{8n} = \langle a, b \mid a^{2n} = b^4 = e,\ ba = a^{-1}b^{-1},\ b^{-1}a = a^{-1}b\rangle. \tag{2}$$

We want to check the existence of Eulerian and Hamiltonian paths and circuits in $\Gamma(G, S)$.

Theorem 5.1. *Let G be the group* (2) *and $S = \{a, b\}$. Then $\Gamma(G, S)$ always has an Eulerian circuit and never has Eulerian paths.*

Proof. Let us check $(a)e$ and $(b)e$:

$$(a)e = (e, a, a^2, \dots, a^{2n-1}),$$
$$(b)e = (e, b, b^2, b^3).$$

So, $(a)e \cap (b)e = \{e\}$; thus $|(a)e \cap (b)e| = 1$. Hence, for every $(a)x \in V_a$ and $(b)y \in V_b$, if $(a)x \cap (b)y \neq \varnothing$, then $|(a)x \cap (b)y| = 1$ [Bauer et al. 2008]. Now notice that $o(a) = 2n$, so the number of vertices of V_a is $|G|/o(a) = (8n)/(2n) = 4$. Also we know that $o(b) = 4$, so $\deg(b)y = 4$ for every $(b)y \in V_b$. Thus every vertex of V_b has exactly one edge to every vertex of V_a. On the other hand, the number of vertices of V_b is $|G|/o(b) = 8n/4 = 2n$, so $\Gamma(G, S) = K_{2n,4}$.

Hence the degree of every vertex of $\Gamma(G, S)$ is even ($2n$ or 4), so it has an Eulerian circuit but it doesn't have any Eulerian paths (Result 1). □

Theorem 5.2. *Let G be the group* (2) *and $S = \{a, b\}$. Then $\Gamma(G, S)$ has a Hamiltonian circuit if and only if $n = 2$.*

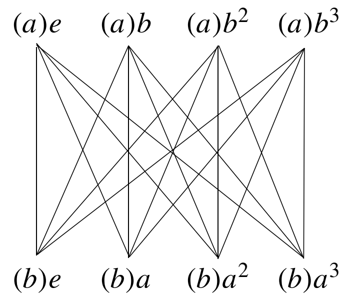

Figure 10. $\Gamma(V_{16}, \{a, b\})$.

Proof. The G-graph $\Gamma(G, S)$ is isomorphic to $K_{2n,4}$. Assume that it has a Hamiltonian path, so $|2n-4| \leqslant 1$ (Theorem 2.1); hence one of the following cases happens:

Case 1: $(2n=4) \Rightarrow (n=2)$. So $\Gamma(G, S)$ is as in Figure 10. The Hamiltonian path is $(a)e$, $(b)e$, $(a)b$, $(b)a$, $(a)b^2$, $(b)a^2$, $(a)b^3$, $(b)a^3$, and the Hamiltonian circuit is $(a)e$, $(b)e$, $(a)b$, $(b)a$, $(a)b^2$, $(b)a^2$, $(a)b^3$, $(b)a^3$, $(a)e$.

Case 2: $(4-2n=1) \Rightarrow (2n=3)$, which is not possible.

Case 3: $(2n-4=1) \Rightarrow (2n=5)$, which is not possible.

Notice that if $n \neq 2$, then $\Gamma(G, S)$ doesn't have any Hamiltonian circuits (Theorem 2.2). So $\Gamma(G, S)$ has a Hamiltonian path and circuit if and only if $n=2$. $\square$

Theorem 5.3. *Let G be the group* (2) *and $S = \{b, ab\}$. Then $\Gamma(G, S)$ always has an Eulerian circuit and doesn't have any Eulerian paths.*

Proof. Clearly $o(ab) = 2$. Now notice that $aba^i = b^3a^{i-1}$ and $ab^2a^i = b^2a^{i+1}$. Next let us check the vertices of V_{ab}:

$$\begin{aligned}
(ab)e &= (e, ab) = (e, b^3a^{2n-1}),\\
(ab)a &= (a, aba) = (a, b^3),\\
(ab)a^2 &= (a^2, aba) = (a, b^3a),\\
&\vdots\\
(ab)a^{2n-1} &= (a^{2n-1}, aba) = (a, b^3a^{2n-2}),\\
(ab)b &= (b, ab^2) = (b, b^2a),\\
(ab)ba &= (ba, ab^2a) = (ba, b^2a^2),\\
(ab)ba^2 &= (ba^2, ab^2a^2) = (ba^2, b^2a^3),\\
&\vdots\\
(ab)ba^{2n-1} &= (ba^{2n-1}, ab^2a^{2n-1}) = (ba^{2n-1}, b^2).
\end{aligned}$$

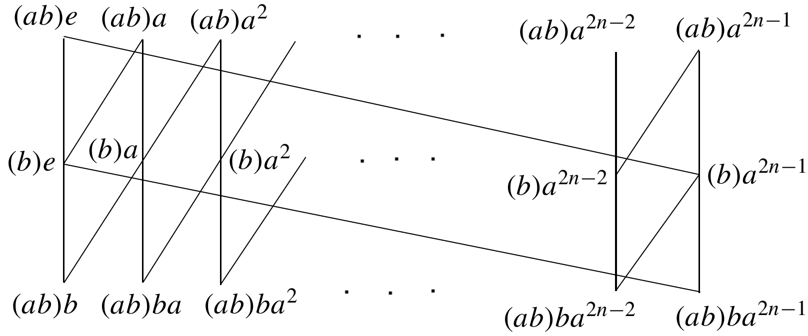

Figure 11. $\Gamma(V_{8n}, \{b, ab\})$.

Let us also check those of V_b:

$$\begin{aligned}
(b)e &= (e, b, b^2, b^3),\\
(b)a &= (a, ba, b^2a, b^3a),\\
(b)a^2 &= (a^2, ba^2, b^2a^2, b^3a^2),\\
&\ \vdots\\
(b)a^{2n-1} &= (a^{2n-1}, ba^{2n-1}, b^2a^{2n-1}, b^3a^{2n-1}).
\end{aligned}$$

So we have $(ab)a^i \cap (b)a^i = \{a^i\}$ and $(ab)a^{i+1} \cap (b)a^i = \{b^3a^i\}$ and $(ab)ba^i \cap (b)a^i = \{ba^i\}$ and $(ab)ba^{i-1} \cap (b)a^i = \{b^2a^i\}$. Hence in $\Gamma(G, S)$, the degree of every vertex of V_{ab} is 2, and the degree of every vertex of V_b is 4. So the degree of every vertex of $\Gamma(G, S)$ is even. On the other hand $G = V_{8n} = \langle ab, b\rangle$, so $\Gamma(G, S)$ is connected [Bretto et al. 2007]. Thus $\Gamma(G, S)$ is a connected graph such that the degree of every vertex is even, so it has an Eulerian circuit and it doesn't have any Eulerian paths (Result 1). The Eulerian circuit in $\Gamma(G, S)$ is

$$\begin{aligned}
&(b)a^{2n-1},\ (ab)e,\ (b)e,\ (ab)a,\ (b)a,\ (ab)a^2,\ (b)a^2,\\
&\quad \dots,\ (ab)a^{2n-2},\ (b)a^{2n-2},\ (ab)a^{2n-1}, (b)a^{2n-1},\ (ab)ba^{2n-1},\\
&\quad\quad (b)e,\ (ab)e,\ (b)a,\ (ab)ba,\ (b)a^2,\ (ab)ba^2,\\
&\quad\quad\quad \dots,\ (b)a^{2n-2},\ (ab)ba^{2n-2},\ (b)a^{2n-1}.
\end{aligned}$$ □

Theorem 5.4. *Let G be the group* (2) *and $S = \{b, ab\}$. Then $\Gamma(G, S)$ doesn't have any Hamiltonian paths or circuits.*

Proof. The number of vertices of V_b is $|G|/o(b) = 8n/4 = 2n$, and the number of vertices of V_{ab} is $|G|/o(a) = 8n/2 = 4n$. Now assume that $\Gamma(G, S)$ has a Hamiltonian path, so $|4n - 2n| \leqslant 1$ (Theorem 2.1). Hence one of the following cases will happen:

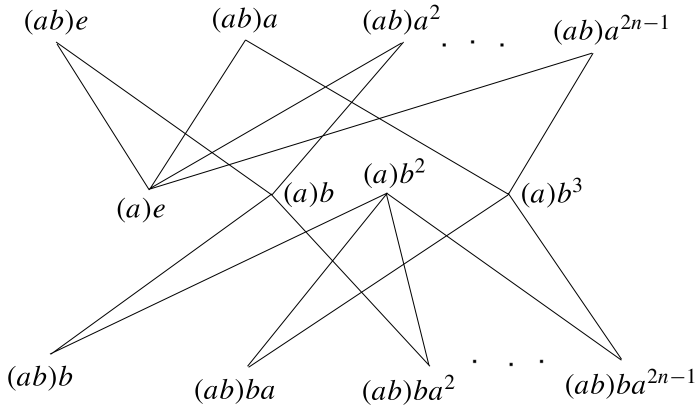

Figure 12. $\Gamma(V_{8n}, \{a, ab\})$.

Case 1: $(4n = 2n) \Rightarrow (n = 0)$.

Case 2: $(4n - 2n = 1) \Rightarrow (2n = 1) \Rightarrow \left(n = \frac{1}{2}\right)$.

Case 3: $(2n - 4n = 1) \Rightarrow (2n = -1) \Rightarrow \left(n = -\frac{1}{2}\right)$.

Obviously none of these cases can happen, so $\Gamma(G, S)$ doesn't have any Hamiltonian paths, and thus it doesn't have any Hamiltonian circuits. $\square$

Theorem 5.5. *Let G be the group* (2) *and $S = \{a, ab\}$. Then $\Gamma(G, S)$ has an Eulerian circuit and doesn't have any Eulerian paths.*

Proof. Notice that $o(a) = 2n$ and $o(ab) = 2$. Also notice that $(ab)e = (e, ab)$ and $(a)e = (e, a, a^2, \cdots, a_{2n-1})$, so $(ab)e \cap (a)e = \{e\}$. Thus, for every $(a)x \in V_a$ and $(ab)y \in V_{ab}$, if $(a)x \cap (ab)y \neq \varnothing$, then $|(a)x \cap (ab)y| = 1$ [Bauer et al. 2008]. So the degree of every vertex of V_a is $2n$, and the degree of every vertex of V_{ab} is 2.

On the other hand $G = \langle a, ab \rangle$, so $\Gamma(G, S)$ is connected [Bretto et al. 2007]. Thus, $\Gamma(G, S)$ is a connected graph such that the degree of every vertex is even. So it has an Eulerian circuit and doesn't have any Eulerian paths (Result 1). $\square$

Theorem 5.6. *Let G be the group* (2) *and $S = \{a, ab\}$. Then $\Gamma(G, S)$ has a Hamiltonian path and circuit if and only if $n = 1$.*

Proof. The number of vertices of V_a is $|G|/o(a) = (8n)/(2n) = 4$, and the number of vertices of V_{ab} is $|G|/o(ab) = 8n/2 = 4n$. Now assume that $\Gamma(G, S)$ has a Hamiltonian path, so $|4n - 4| \leqslant 1$ (Theorem 2.1). Hence one of the following cases happens:

Case 1: $(4n - 4 = 1) \Rightarrow (4n = 5)$, which is impossible.

Case 2: $(4 - 4n = 1) \Rightarrow (4n = 3)$, which is impossible.

Case 3: $(4n - 4 = 0) \Rightarrow (4n = 4) \Rightarrow (n = 1)$. In this case, the image of $\Gamma(G, S)$ is shown in Figure 13. Its Hamiltonian path is $(ab)e$, $(a)b^3$, $(ab)ba$, $(a)b^2$, $(ab)b$,

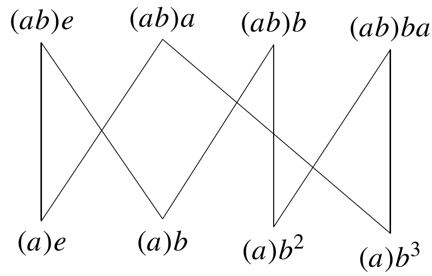

Figure 13. $\Gamma(V_8, \{a, ab\})$.

$(a)b$, $(ab)a$, $(a)e$, and its Hamiltonian circuit is $(ab)e$, $(a)b^3$, $(ab)ba$, $(a)b^2$, $(ab)b$, $(a)b$, $(ab)a$, $(a)e$, $(ab)e$. If $\Gamma(G, S)$ doesn't have any Hamiltonian paths, then it doesn't have any Hamiltonian circuits; thus $\Gamma(G, S)$ has a Hamiltonian path and circuit if and only if $n = 1$. □

6. Conclusion

In this paper we investigated the existence of Eulerian circuits and paths in the G-graphs of finite abelian groups. Also we checked the existence of Hamiltonian and Eulerian circuits and paths in the G-graphs of some nonabelian finite groups. Our method can be applied to other finite groups as well.

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darafsheh@ut.ac.ir *School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran*

madadi.safoora@ut.ac.ir *School of Mathematics, Statistics and Computer Science, University of Tehran, Tehran, Iran*

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