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Given real numbers $b, c \in \mathbb{R}$, we form the sequence of polynomials $\{H_m(z)\}_{m=0}^{\infty}$ satisfying the four-term recurrence

$$H_m(z) + cH_{m-1}(z) + bH_{m-2}(z) + zH_{m-3}(z) = 0, \quad m \geq 1,$$

with the initial conditions $H_0(z) = 1$ and $H_{-1}(z) = H_{-2}(z) = 0$. We find necessary and sufficient conditions on b and c under which the zeros of $H_m(z)$ are real for all m, and provide an explicit real interval on which $\bigcup_{m=0}^{\infty} \mathcal{Z}(H_m)$ is dense, where $\mathcal{Z}(H_m)$ is the set of zeros of $H_m(z)$.

1. Introduction

Consider the sequence of polynomials $\{H_m(z)\}_{m=0}^{\infty}$ satisfying the finite recurrence

$$\sum_{k=0}^{n} a_k(z) H_{m-k}(z) = 0, \quad m \geq n, \tag{1-1}$$

where $a_k(z)$, $1 \leq k \leq n$, are complex polynomials. With certain initial conditions, one may ask for the locations of the zeros of $H_m(z)$ on the complex plane. There are two common approaches to answering this question. The first describes the asymptotic location of the zeros of the generated polynomials, while the second provides the exact location of these zeros (or at least for the zeros of $H_m(z)$ for $m \gg 1$). Recent works in the first direction include [Beraha et al. 1975; 1978; Borcea et al. 2006; Boyer and Goh 2007; 2008]. Results using the first approach prove useful when establishing the necessary condition for $H_m(z)$ to be hyperbolic, as we will see in Section 3.

When considering polynomials satisfying a generic recurrence such as (1-1), the task of finding an explicit curve where the zeros of the $H_m(z)$ must lie is difficult. For three-term recurrences with degree two and appropriate initial conditions, the curve containing zeros is given in [Tran 2014]. The corresponding curve for a three-term recurrence with degree n is given in [Tran 2015]. Among all possible

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curves containing the zeros of the $H_m(z)$, the real line plays an important role. We say that a polynomial is hyperbolic if all of its zeros are real. There are a lot of recent works on hyperbolic polynomials and on linear operators preserving hyperbolicity of polynomials; see for example [Bates and Yoshida 2016; Borcea and Brändén 2009; Bunton et al. 2015; Craven and Csordas 2004]. For studies of sequences of hyperbolic polynomials satisfying finite recurrences, see [Eğecioğlu et al. 2001; Forgács and Tran 2016].

The main result of this paper, Theorem 2, is the identification of necessary and sufficient conditions on $b, c \in \mathbb{R}$ under which the zeros of the sequence of polynomials $H_m(z)$ satisfying the recurrence

$$\begin{gathered} H_m(z)+cH_{m-1}(z)+bH_{m-2}(z)+zH_{m-3}(z)=0, \quad m\geq 1,\\ H_0(z)\equiv 1, \\ H_m(z)\equiv 0, \quad m<0, \end{gathered} \tag{1-2}$$

are real. We use the convention that the zeros of the constant zero polynomial are real.

Definition 1. The set of zeros of $H_m(z)$ is denoted by $\mathcal{Z}(H_m)$.

Theorem 2. *Suppose $b, c \in \mathbb{R}$, and let $\{H_m(z)\}_{m=0}^{\infty}$ be defined as in* (1-2)*. The zeros of $H_m(z)$ are real for all m if and only if one of the two conditions below holds*:

(i) $c = 0$ *and* $b \geq 0$.

(ii) $c \neq 0$ *and* $-1 \leq b/c^2 \leq \frac{1}{3}$.

In the first case, if $b > 0$, then $\bigcup_{m=0}^{\infty} \mathcal{Z}(H_m)$ is dense in $(-\infty, \infty)$. In the second case, $\bigcup_{m=0}^{\infty} \mathcal{Z}(H_m)$ is dense in the interval

$$c^3 \cdot \left(-\infty, \tfrac{1}{27}\left(-2+9b/c^2-2\sqrt{(1-3b/c^2)^3}\right)\right].$$

Our paper is organized as follows. In Section 2, we prove a sufficient condition for the zeros of all $H_m(z)$ to be real in the case $c \neq 0$. The case $c = 0$ follows from similar arguments whose key differences will be outlined in Section 3. Finally, in Section 4, we prove the necessary condition for the zeros of $H_m(z)$ to be real.

2. The case $c \neq 0$ and $-1 \leq b/c^2 \leq \frac{1}{3}$

We write the sequence $\{H_m(z)\}_{m=0}^{\infty}$ in (1-2) using its generating function

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1+ct+bt^2+zt^3}. \tag{2-1}$$

Substituting $t \to t/c$, $b/c^2 \to a$, and $z/c^3 \to z$, we will prove the following form of the theorem.

Theorem 3. *Consider the sequence of polynomials* $\{H_m(z)\}_{m=0}^{\infty}$ *generated by*

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1+t+at^2+zt^3}, \tag{2-2}$$

where $a \in \mathbb{R}$. *If* $-1 \le a \le \frac{1}{3}$ *then the zeros of* $H_m(z)$ *lie in the real interval*

$$I_a = \left(-\infty, \tfrac{1}{27}\left(-2+9a-2\sqrt{(1-3a)^3}\right)\right], \tag{2-3}$$

and $\bigcup_{m=0}^{\infty} \mathcal{Z}(H_m)$ *is dense in* I_a.

We will see later that the density of the union of zeros on I_a follows naturally from the fact that $\mathcal{Z}(H_m) \subset I_a$ and thus we focus on proving this claim. We note that each value of $a \in \left[-1, \frac{1}{3}\right]$ generates a sequence $\{H_m(z, a)\}_{m=0}^{\infty}$. The lemma below asserts that it suffices to prove that $\mathcal{Z}(H_m(z, a)) \subset I_a$ for all a in a dense subset of $\left[-1, \frac{1}{3}\right]$.

Lemma 4. *Let S be a dense subset of* $\left[-1, \frac{1}{3}\right]$, *and let* $m \in \mathbb{N}$ *be fixed. If*

$$\mathcal{Z}(H_m(z, a)) \subset I_a$$

for all $a \in S$, *then*

$$\mathcal{Z}(H_m(z, a^*)) \subset I_{a^*}$$

for all $a^* \in \left[-1, \frac{1}{3}\right]$.

Proof. Let $a^* \in \left[-1, \frac{1}{3}\right]$ be given. By the density of S in $\left[-1, \frac{1}{3}\right]$, we can find a sequence $\{a_n\}$ in S such that $a_n \to a^*$. For any $z^* \notin I_{a^*}$, we will show that $H_m(z^*, a^*) \neq 0$. We note that the zeros of $H_m(z, a_n)$ lie in the interval I_{a_n} whose right endpoint approaches the right endpoint of I_{a^*} as $n \to \infty$. If we let $z_k^{(n)}$, $1 \le k \le \deg H_m(z, a_n)$, be the zeros of $H_m(z, a_n)$ then

$$|H_m(z^*, a_n)| = \gamma^{(n)} \prod_{k=1}^{\deg H_m(z,a_n)} \left|z^* - z_k^{(n)}\right|,$$

where $\gamma^{(n)}$ is the leading coefficient of $H_m(z, a_n)$. Since $\deg H_m(z, a_n) \le \left\lfloor \frac{1}{3}m \right\rfloor$ (see Lemma 5), using this product representation and the assumption that $z^* \notin I_a$, we conclude that there is a fixed (independent of n) $\delta > 0$ so that $|H_m(z^*, a_n)| > \delta$ for all large n. Since $H_m(z^*, a)$ is a polynomial in a for any fixed z^*, we conclude that

$$H_m(z^*, a^*) = \lim_{n\to\infty} H_m(z^*, a_n) \neq 0$$

and the result follows. □

Lemma 4 allows us to ignore some special values of a. In particular, we may assume $a \neq 0$. In our main argument, we count the number of zeros of $H_m(z)$ on the interval I_a in (2-3) and show that this number is at least as big as the degree of $H_m(z)$. To count the number of zeros of $H_m(z)$ on I_a, we write $z = z(\theta)$ as a strictly increasing function of a variable θ on the interval $\left(\frac{2\pi}{3}, \pi\right)$. Then we construct a function $g_m(\theta)$ on $\left(\frac{2\pi}{3}, \pi\right)$ with the property that θ is a zero of $g_m(\theta)$ on $\left(\frac{2\pi}{3}, \pi\right)$ if and only if $z(\theta)$ is a zero of $H_m(z)$ on I_a. From this construction, we count the number of zeros of $g_m(\theta)$ on $\left(\frac{2\pi}{3}, \pi\right)$, which will be the same as the number of zeros of $H_m(z)$ on I_a by the monotonicity of the function $z(\theta)$.

We next obtain an upper bound for the degree of $H_m(z)$ and provide heuristic arguments for the formulas of $z(\theta)$ and $g_m(\theta)$.

Lemma 5. *The degree of the polynomial $H_m(z)$ defined by* (2-2) *is at most* $\left\lfloor \frac{1}{3}m \right\rfloor$.

Proof. We rewrite (2-2) as

$$(1+t+at^2+zt^3)\sum_{m=0}^{\infty} H_m(z)t^m = 1.$$

By equating the coefficients in t of both sides, we see that the sequence $\{H_m(z)\}_{m=0}^{\infty}$ satisfies the recurrence

$$H_{m+3}(z) + H_{m+2}(z) + aH_{m+1}(z) + zH_m(z) = 0$$

and the initial conditions

$$H_0(z) \equiv 1, \quad H_1(z) \equiv -1, \quad \text{and} \quad H_2(z) \equiv 1-a.$$

The lemma follows by induction. □

2.1. ***Heuristic arguments.*** We now provide heuristic arguments to motivate the formulas for two functions $z(\theta)$ and $g_m(\theta)$ on $\left(\frac{2\pi}{3}, \pi\right)$. Let $t_0 = t_0(z)$, $t_1 = t_1(z)$, and $t_2 = t_2(z)$ be the three zeros of the denominator $1+t+at^2+zt^3$. We will show rigorously in Section 2.2 that t_0, t_1, t_2 are nonzero and distinct with $t_0 = \bar{t}_1$. We let $q = t_1/t_0 = e^{2i\theta}$, $\theta \neq 0, \pi$. We have

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1+t+at^2+zt^3} = \frac{1}{z(t-t_0)(t-t_1)(t-t_2)}.$$

We apply partial fractions to rewrite the generating function given above as

$$\big(z(t-t_0)(t_0-t_1)(t_0-t_2)\big)^{-1} + \big(z(t-t_1)(t_1-t_0)(t_1-t_2)\big)^{-1} + \big(z(t-t_2)(t_2-t_0)(t_2-t_1)\big)^{-1},$$

which can be expanded as a series in t as

$$-\sum_{m=0}^{\infty} \frac{1}{z}\Big(\big((t_0-t_1)(t_0-t_2)t_0^{m+1}\big)^{-1} + \big((t_1-t_0)(t_1-t_2)t_1^{m+1}\big)^{-1} + \big((t_2-t_0)(t_2-t_1)t_2^{m+1}\big)^{-1}\Big)t^m. \tag{2-4}$$

From this expression, we deduce that z is a zero of $H_m(z)$ if and only if

$$\big((t_0-t_1)(t_0-t_2)t_0^{m+1}\big)^{-1}+\big((t_1-t_0)(t_1-t_2)t_1^{m+1}\big)^{-1} + \big((t_2-t_0)(t_2-t_1)t_2^{m+1}\big)^{-1}=0. \quad (2\text{-}5)$$

After multiplying the left side of (2-5) by t_0^{m+3}, we obtain the equality

$$\big((1-t_1/t_0)(1-t_2/t_0)\big)^{-1}+\big((t_1/t_0-1)(t_1/t_0-t_2/t_0)(t_1/t_0)^{m+1}\big)^{-1} + \big((t_2/t_0-1)(t_2/t_0-t_1/t_0)(t_2/t_0)^{m+1}\big)^{-1}=0.$$

Setting $\zeta=t_2/t_0e^{i\theta}$, we rewrite the left side as

$$\big((1-e^{2i\theta})(1-\zeta e^{i\theta})\big)^{-1}+\big((e^{2i\theta}-1)(e^{2i\theta}-\zeta e^{i\theta})(e^{2i\theta})^{m+1}\big)^{-1} + \big((\zeta e^{i\theta}-1)(\zeta e^{i\theta}-e^{2i\theta})(\zeta e^{i\theta})^{m+1}\big)^{-1},$$

or equivalently

$$\big(e^{2i\theta}(-2i\sin\theta)(e^{-i\theta}-\zeta)\big)^{-1}+\big((2i\sin\theta)(e^{i\theta}-\zeta)(e^{2i\theta})^{m+2}\big)^{-1} + \big((\zeta-e^{-i\theta})(\zeta-e^{i\theta})(\zeta)^{m+1}(e^{i\theta})^{m+3}\big)^{-1}.$$

We multiply this expression by $(\zeta-e^{-i\theta})(\zeta-e^{i\theta})e^{i(m+3)\theta}$ and set the summation equal to zero to arrive at

$$\begin{aligned}
0&=\frac{(\zeta-e^{i\theta})e^{i(m+1)\theta}}{2i\sin\theta}+\frac{e^{-i\theta}-\zeta}{(2i\sin\theta)e^{i(m+1)\theta}}+\frac{1}{\zeta^{m+1}}\\
&=\frac{(\zeta-e^{i\theta})e^{i(m+1)\theta}-(\zeta-e^{-i\theta})e^{-i(m+1)\theta}}{2i\sin\theta}+\frac{1}{\zeta^{m+1}}\\
&=\frac{\zeta(e^{i(m+1)\theta}-e^{-i(m+1)\theta})+e^{-i(m+2)\theta}-e^{i(m+2)\theta}}{2i\sin\theta}+\frac{1}{\zeta^{m+1}}\\
&=\frac{\zeta(2i\sin((m+1)\theta))-2i\sin((m+2)\theta)}{2i\sin\theta}+\frac{1}{\zeta^{m+1}}\\
&=\frac{2i\zeta\sin((m+1)\theta)-2i\sin((m+1)\theta)\cos\theta-2i\cos((m+1)\theta)\sin\theta}{2i\sin\theta}+\frac{1}{\zeta^{m+1}}\\
&=\frac{(\zeta-\cos\theta)\sin((m+1)\theta)}{\sin\theta}-\cos((m+1)\theta)+\frac{1}{\zeta^{m+1}}. \qquad (2\text{-}6)
\end{aligned}$$

The last expression will serve as the definition of $g_m(\theta)$; see (2-15).

We next provide a motivation for the specific form of $z(\theta)$. Since t_0, t_1, and t_2 are the zeros of $D(t,z)=1+t+at^2+zt^3$, they satisfy the three identities

$$t_0+t_1+t_2=-\frac{a}{z},\quad t_0t_1+t_0t_2+t_1t_2=\frac{1}{z},\quad\text{and}\quad t_0t_1t_2=-\frac{1}{z}.$$

If we divide the first equation by t_0, the second by t_0^2, and the third by t_0^3 then these identities become

$$1+e^{2i\theta}+\zeta e^{i\theta}=-\frac{a}{zt_0}, \tag{2-7}$$

$$e^{2i\theta}+\zeta e^{i\theta}+\zeta e^{3i\theta}=\frac{1}{zt_0^2}, \tag{2-8}$$

$$\zeta e^{3i\theta}=-\frac{1}{zt_0^3}. \tag{2-9}$$

We next divide (2-7) by (2-8), and (2-8) by (2-9) to obtain

$$\frac{1+e^{2i\theta}+\zeta e^{i\theta}}{e^{2i\theta}+\zeta e^{i\theta}+\zeta e^{3i\theta}}=-at_0 \quad \text{and} \quad \frac{e^{2i\theta}+\zeta e^{i\theta}+\zeta e^{3i\theta}}{\zeta e^{3i\theta}}=-t_0,$$

from which we deduce that

$$(1+e^{2i\theta}+\zeta e^{i\theta})\zeta e^{3i\theta}=a(e^{2i\theta}+\zeta e^{i\theta}+\zeta e^{3i\theta})^2.$$

This equation is equivalent to

$$(e^{-i\theta}+e^{i\theta}+\zeta)\zeta e^{4i\theta}=ae^{4i\theta}(1+\zeta e^{-i\theta}+\zeta e^{i\theta})^2,$$

or simply

$$(2\cos\theta+\zeta)\zeta=a(1+2\zeta\cos\theta)^2.$$

Lemma 6. *For any $a\in\left[-1,\frac{1}{3}\right]$ and $\theta\in\left(\frac{2\pi}{3},\pi\right)$, the zeros in ζ of the polynomial*

$$(2\cos\theta+\zeta)\zeta-a(1+2\zeta\cos\theta)^2 \tag{2-10}$$

are real and distinct.

Proof. We consider the discriminant of the above polynomial in ζ:

$$\Delta=(1-4a)\cos^2\theta+a.$$

There are three possible cases depending on the value of a. If $\frac{1}{4}\le a\le\frac{1}{3}$, the inequality $\Delta>0$ comes directly from

$$a\ge 4a-1>(1-4a)\cos^2\theta.$$

If $0\le a<\frac{1}{4}$, the claim $\Delta>0$ is trivial since $1-4a>0$. Finally, if $a<0$, we have

$$\Delta\ge\tfrac{1}{4}(1-4a)+a=\tfrac{1}{4}.$$

It follows that the zeros of (2-10) are real and distinct for any $a\in\left[-1,\frac{1}{3}\right]$ and $\theta\in\left(\frac{2\pi}{3},\pi\right)$. □

To obtain a formula for $z(\theta)$, we multiply (2-7) and (2-8) to get

$$(1+e^{2i\theta}+\zeta e^{i\theta})(e^{2i\theta}+\zeta e^{i\theta}+\zeta e^{3i\theta})=-\frac{a}{z^2t_0^3},$$

and divide (2-9) by this equation to arrive at

$$\begin{aligned} z &= \frac{ae^{3i\theta}\zeta}{(1+e^{2i\theta}+\zeta e^{i\theta})(e^{2i\theta}+\zeta e^{i\theta}+\zeta e^{3i\theta})} \\ &= \frac{ae^{3i\theta}\zeta}{e^{3i\theta}(e^{-i\theta}+e^{i\theta}+\zeta)(1+\zeta e^{-i\theta}+\zeta e^{i\theta})} = \frac{a\zeta}{(2\cos\theta+\zeta)(1+2\zeta\cos\theta)}. \end{aligned} \tag{2-11}$$

2.2. *Rigorous proof.* Motivated by Section 2.1, we now rigorously prove Theorem 3. We start by defining the function $\zeta(\theta)$ according to (2-10).

Definition 7. The function $\zeta(\theta)$ is defined on $\left(\frac{2\pi}{3}, \pi\right)$ as

$$\zeta = \zeta(\theta) = \frac{(2a-1)\cos\theta + \sqrt{(1-4a)\cos^2\theta + a}}{1-4a\cos^2\theta}. \tag{2-12}$$

Remark 8. From Lemma 6, $\zeta(\theta)$ is a real function on $\left(\frac{2\pi}{3}, \pi\right)$ with a possible vertical asymptote at

$$\theta = \cos^{-1}\left(-\frac{1}{2\sqrt{a}}\right) \tag{2-13}$$

when $\frac{1}{4} < a \le \frac{1}{3}$. However, we note that the function $1/\zeta(\theta)$ is a real continuous function on $\left(\frac{2\pi}{3}, \pi\right)$.

Lemma 9. *Let $\zeta(\theta)$ be defined as in* (2-12). *Then $|\zeta(\theta)| > 1$ for every $a \in \left(-1, \frac{1}{3}\right)$ and every $\theta \in \left(\frac{2\pi}{3}, \pi\right)$ with $1-4a\cos^2\theta \neq 0$.*

Proof. From (2-10), we note that $\zeta_+ := \zeta(\theta)$ and

$$\zeta_- := \frac{(2a-1)\cos\theta - \sqrt{(1-4a)\cos^2\theta + a}}{1-4a\cos^2\theta}$$

are the zeros of

$$f(\zeta) := (2\cos\theta+\zeta)\zeta - a(1+2\zeta\cos\theta)^2.$$

Note that

$$f(-1)f(1) = (-1+2\cos\theta)(1+2\cos\theta)(4a^2\cos^2\theta - (a-1)^2).$$

If $\theta \in \left(\frac{2\pi}{3}, \pi\right)$ and $a \in \left(-1, \frac{1}{3}\right)$, this product is negative since

$$4a^2\cos^2\theta - (a-1)^2 \le 4a^2 - (a-1)^2 = (a+1)(3a-1) < 0.$$

Thus exactly one of the zeros of the quadratic function $f(\zeta)$ lies outside the interval $[-1, 1]$. The claim follows from the fact that $|\zeta_+| > |\zeta_-|$. □

Although one can prove Lemma 9 for the extreme values $a = -1$ or $a = \frac{1}{3}$, that will not be necessary by Lemma 4. Next, motivated by (2-11), we define the real function $z(\theta)$ as follows.

Definition 10. The function $z(\theta)$ is defined on $\left(\frac{2\pi}{3}, \pi\right)$ as

$$z = z(\theta) := \frac{a\zeta}{(2\cos\theta + \zeta)(1 + 2\zeta\cos\theta)}. \tag{2-14}$$

Lemma 9 implies $1+2\zeta\cos\theta \neq 0$, and by (2-10), neither is $2\cos\theta+\zeta$. Dividing the numerator and the denominator of (2-14) by $\zeta^2(\theta)$ and combining with the fact that $1/\zeta(\theta)$ is continuous on $\left(\frac{2\pi}{3}, \pi\right)$, we conclude that the possible discontinuity of $z(\theta)$ in (2-13) is removable. Finally, motivated by (2-6), we define the function $g_m(\theta)$ as follows.

Definition 11. The function $g_m(\theta)$ is defined on $\left(\frac{2\pi}{3}, \pi\right)$ as

$$g_m(\theta) := \frac{(\zeta - \cos\theta)\sin((m+1)\theta)}{\sin\theta} - \cos((m+1)\theta) + \frac{1}{\zeta^{m+1}}. \tag{2-15}$$

We note that $g_m(\theta)$ has the same vertical asymptote as that of $\zeta(\theta)$ in (2-13) when $\frac{1}{4} < a \leq \frac{1}{3}$.

From Lemma 9, we see that the sign of the function $g_m(\theta)$ alternates at values of θ where $\cos(m+1)\theta = \pm 1$. Thus by the intermediate value theorem, the function $g_m(\theta)$ has at least one root on each subinterval whose endpoints are solutions of $\cos(m+1) = \pm 1$. However, in the case $\frac{1}{4} \leq a \leq \frac{1}{3}$, one of the subintervals contains the vertical asymptote given in (2-13). The lemma below counts the number of zeros of $g_m(\theta)$ on such a subinterval.

Lemma 12. *Let $g_m(\theta)$ be defined as in* (2-15). *Suppose $\frac{1}{4} < a \leq \frac{1}{3}$ and $m \geq 6$. Then there exists $h \in \mathbb{N}$ such that*

$$\theta_{h-1} := \frac{h-1}{m+1}\pi < \cos^{-1}\left(-\frac{1}{2\sqrt{a}}\right) \leq \frac{h}{m+1}\pi =: \theta_h,$$

where $\left\lfloor \frac{2}{3}(m+1) \right\rfloor + 1 \leq h - 1 < h \leq m+1$. Furthermore, as long as

$$\cos^{-1}\left(-\frac{1}{2\sqrt{a}}\right) \neq \frac{h}{m+1}\pi, \tag{2-16}$$

the function $g_m(\theta)$ has at least two zeros on the interval

$$\theta \in \left(\frac{h-1}{m+1}\pi, \frac{h}{m+1}\pi\right) := J_h \tag{2-17}$$

whenever h is at most m, and at least one zero when h is $m+1$.

Proof. Suppose $a \in \left(\frac{1}{4}, \frac{1}{3}\right]$. Since the function $\cos^{-1}(-1/(2\sqrt{x}))$ is decreasing on the interval $\left(\frac{1}{4}, \frac{1}{3}\right]$, we conclude that

$$\cos^{-1}\left(-\frac{1}{2\sqrt{a}}\right) \geq \frac{5\pi}{6}.$$

The existence of h now follows directly from

$$\frac{\left\lfloor \frac{2}{3}(m+1)\right\rfloor+1}{m+1}\pi < \frac{5\pi}{6},$$

when $m \geq 6$.

The vertical asymptote of $g_m(\theta)$ at $\cos^{-1}(-1/(2\sqrt{a}))$ divides the interval J_h in (2-17) into two subintervals. We will show that each subinterval contains at least one zero of $g_m(\theta)$ if $h \leq m$. In the case $h = m+1$, only the subinterval on the left contains at least one zero of $g_m(\theta)$. We analyze these two subintervals in the two cases below.

We consider the first case when $\theta \in J_h$ and $\theta < \cos^{-1}(-1/(2\sqrt{a}))$. By Lemma 9 and (2-15) we see that the sign of $g_m(\theta_{h-1})$ is $(-1)^h$. We now show that the sign of $g_m(\theta)$ is $(-1)^{h-1}$ when $\theta \to \cos^{-1}(-1/(2\sqrt{a}))$. From (2-12), we observe that $\zeta(\theta) \to +\infty$ as $\theta \to \cos^{-1}(-1/(2\sqrt{a}))$. Since $\theta \in J_h$, the sign of $\sin((m+1)\theta)$ is $(-1)^{h-1}$ and consequently the sign of $g_m(\theta)$ is $(-1)^{h-1}$ when $\theta \to \cos^{-1}(-1/(2\sqrt{a}))$ by (2-15). By the intermediate value theorem, we obtain at least one zero of $g_m(\theta)$ in this case.

Next we consider the case when $\theta \in J_h$ and $\theta > \cos^{-1}(-1/(2\sqrt{a}))$. In this case the sign of $g_m(\theta_h)$ is $(-1)^{h-1}$ if $h \leq m$ by Lemma 9. Since $\zeta(\theta) \to -\infty$ as $\theta \to \cos^{-1}(-1/(2\sqrt{a}))$ and the sign of $\sin((m+1)\theta)$ is $(-1)^{h-1}$, the sign of $g_m(\theta)$ is $(-1)^h$ as $\theta \to \cos^{-1}(-1/(2\sqrt{a}))$. By the intermediate value theorem, we obtain at least one zero of $g_m(\theta)$ in this case if $h \leq m$. □

Note that as a consequence of Lemma 4, we may assume that none of the partitioning points under consideration are the points $\cos^{-1}(-1/(2\sqrt{a}))$. From the fact that the sign of $g_m(\theta)$ in (2-15) alternates when $\cos((m+1)\theta) = \pm 1$, we can find a lower bound for the number of zeros of $g_m(\theta)$ on $\left(\frac{2\pi}{3}, \pi\right)$ by the intermediate value theorem. We will relate the zeros of $g_m(\theta)$ to the zeros of $H_m(z)$ by (2-6). However to ensure that the partial fractions procedure preceding (2-6) is rigorous, we need the lemma below.

Lemma 13. *Let $\theta \in (0, \pi)$ be such that $\theta \neq \cos^{-1}(-1/(2\sqrt{a}))$ whenever $a > \frac{1}{4}$. The zeros in t of $1 + t + at^2 + z(\theta)t^3$ are*

$$t_0 = -\frac{e^{2i\theta} + \zeta e^{i\theta} + \zeta e^{3i\theta}}{\zeta e^{3i\theta}}, \quad t_1 = t_0 e^{2i\theta} \quad \textit{and} \quad t_2/t_0 = \zeta e^{i\theta},$$

where $\zeta := \zeta(\theta)$ is given in (2-12).

Proof. We first note that

$$\begin{aligned} P(t_0) &= 1+t_0+at_0^2+zt_0^3 \\ &= -\frac{1}{\zeta e^{i\theta}} - e^{-2i\theta} + \frac{a}{\zeta^2 e^{2i\theta}}(1+\zeta e^{-i\theta}+\zeta e^{i\theta})^2 - \frac{z}{\zeta^3 e^{3i\theta}}(1+\zeta e^{-i\theta}+\zeta e^{i\theta})^3, \end{aligned}$$

where ζ is a root of the quadratic equation $(2\cos\theta+\zeta)\zeta - a(1+2\zeta\cos\theta)^2 = 0$. We apply the identities

$$(1+\zeta e^{-i\theta}+\zeta e^{i\theta})^2 = (1+2\zeta\cos\theta)^2 = \frac{1}{a}(2\cos\theta+\zeta)\zeta = \frac{1}{a}(e^{-i\theta}+e^{i\theta}+\zeta)\zeta$$

and

$$z = \frac{a\zeta}{(2\cos\theta+\zeta)(1+2\zeta\cos\theta)} = \frac{\zeta^2}{(1+2\zeta\cos\theta)^3} = \frac{\zeta^2}{(1+\zeta e^{-i\theta}+\zeta e^{i\theta})^3}, \tag{2-18}$$

to conclude that $P(t_0) = 0$. Similarly, we have

$$\begin{aligned} P(t_1) &= 1 + t_0 e^{2i\theta} + a t_0^2 e^{4i\theta} + z t_0^3 e^{6i\theta} \\ &= -\frac{e^{i\theta}}{\zeta} - e^{2i\theta} + \frac{a e^{2i\theta}}{\zeta^2}(1+\zeta e^{-i\theta}+\zeta e^{i\theta})^2 - \frac{z e^{3i\theta}}{\zeta^3}(1+\zeta e^{-i\theta}+\zeta e^{i\theta})^3 \\ &= -\frac{e^{i\theta}}{\zeta} - e^{2i\theta} + \frac{a e^{2i\theta}}{\zeta^2}\frac{(e^{-i\theta}+e^{i\theta}+\zeta)\zeta}{a} - \frac{e^{3i\theta}}{\zeta^3}\zeta^2 = 0. \end{aligned}$$

Finally,

$$\begin{aligned} P(t_2) &= P(\zeta t_0 e^{i\theta}) \\ &= -\zeta e^{-i\theta} - \zeta e^{i\theta} + a(1+\zeta e^{-i\theta}+\zeta e^{i\theta})^2 - z(1+\zeta e^{-i\theta}+\zeta e^{i\theta})^3 \\ &= -\zeta e^{-i\theta} - \zeta e^{i\theta} + a\frac{1}{a}(e^{-i\theta}+e^{i\theta}+\zeta)\zeta - \zeta^2 = 0. \end{aligned}$$ □

As a consequence of Lemma 13, if $\theta \in \left(\frac{2\pi}{3}, \pi\right)$, then the zeros of $1 + t + at^2 + z(\theta)t^3$ will be distinct and $t_1 = \bar{t}_0$ since $\zeta \in \mathbb{R}$ by Lemma 6. Thus we can apply the partial fractions given in the beginning of Section 2.1. From this partial fraction decomposition, we conclude that if θ is a zero of $g_m(\theta)$, then $z(\theta)$ will be a zero of $H_m(z)$. In fact, we claim that each distinct zero of $g_m(\theta)$ on $\left(\frac{2\pi}{3}, \pi\right)$ produces a distinct zero of $H_m(z)$ on I_a. This is the content of the following two lemmas.

Lemma 14. *Let $\zeta(\theta)$ be defined as in* (2-12). *The function $z(\theta)$ defined as in* (2-14) *is increasing on $\theta \in \left(\frac{2\pi}{3}, \pi\right)$.*

Proof. Lemma 13 gives

$$-z = \frac{1 + t_0 + a t_0^2}{t_0^3} = \frac{1 + t_1 + a t_1^2}{t_1^3}.$$

We differentiate the three terms and obtain

$$dz = \frac{3 + 2t_0 + a t_0^2}{t_0^4} dt_0 = \frac{3 + 2t_1 + a t_1^2}{t_1^4} dt_1, \tag{2-19}$$

where

$$dt_1 = d(t_0 e^{2i\theta}) = e^{2i\theta} dt_0 + 2i t_0 e^{2i\theta} d\theta.$$

If we set

$$f(t_0)=\frac{3+2t_0+at_0^2}{t_0^4},\qquad f(t_1)=\frac{3+2t_1+at_1^2}{t_1^4},$$

then $f(t_0)=\overline{f(t_1)}$, and consequently $f(t_0)f(t_1)\geq 0$. Thus (2-19) implies

$$f(t_0)dt_0=f(t_1)(e^{2i\theta}dt_0+2it_0e^{2i\theta}d\theta).$$

After solving this equation for dt_0 and substituting it into (2-19), we obtain

$$\frac{dz}{d\theta}=\frac{2if(t_0)f(t_1)t_0e^{2i\theta}}{f(t_0)-f(t_1)e^{2i\theta}}. \tag{2-20}$$

With $t_0=\tau e^{-i\theta},\ \tau\in\mathbb{R}$, we have

$$\frac{f(t_0)-f(t_1)e^{2i\theta}}{2it_0e^{2i\theta}}=\frac{f(t_0)e^{-i\theta}-f(t_1)e^{i\theta}}{2it_0e^{i\theta}}=\frac{\Im(f(t_0)e^{-i\theta})}{\tau}=\frac{1}{\tau}\Im\left(\frac{3+2t_0+at_0^2}{t_0^4}e^{-i\theta}\right).$$

We now substitute $3=-3t_0-3at_0^2-3zt_0^3$ and have

$$\begin{aligned}\frac{f(t_0)-f(t_1)e^{2i\theta}}{2it_0e^{2i\theta}}&=\frac{1}{\tau}\Im\left(\frac{-t_0-2at_0^2-3zt_0^3}{t_0^4}e^{-i\theta}\right)=\frac{1}{\tau^4}\Im(-e^{2i\theta}-2a\tau e^{i\theta}-3z\tau^2)\\&=\frac{1}{\tau^4}(-\sin 2\theta-2a\tau\sin\theta)=\frac{2\sin\theta}{\tau^4}(-\cos\theta-a\tau).\end{aligned}$$

In the formula for t_0 in Lemma 13, we substitute $\tau=-1/\zeta-2\cos\theta$ and obtain

$$\frac{f(t_0)-f(t_1)e^{2i\theta}}{2it_0e^{2i\theta}}=\frac{2\sin\theta}{\tau^4}(-\cos\theta+a/\zeta+2a\cos\theta). \tag{2-21}$$

We finish this lemma by showing that $-\cos\theta+a/\zeta+2a\cos\theta>0$. This strict inequality implies that we cannot have $f(t_0)=f(t_1)=0$ by (2-21), and the lemma follows from (2-20). To prove the inequality, we expand and divide both sides of (2-10) by ζ to get

$$\zeta(1-4a\cos^2\theta)+2\cos\theta(1-2a)-a/\zeta=0,$$

or equivalently,

$$\zeta(1-4a\cos^2\theta)+\cos\theta(1-2a)=-\cos\theta+2a\cos\theta+a/\zeta.$$

Finally, using the definition of ζ in (2-12) and Lemma 6, we calculate

$$\zeta(1-4a\cos^2\theta)+\cos\theta(1-2a)=\sqrt{(1-4a)\cos^2\theta+a}>0. \qquad\square$$

Lemma 15. *The function $z(\theta)$ as defined in* (2-14) *maps the interval $\left(\frac{2\pi}{3},\pi\right)$ onto the interior of I_a.*

Proof. Since $z(\theta)$ is a continuous increasing function on $\left(\frac{2\pi}{3}, \pi\right)$, we only need to evaluate the limits of $z(\theta)$ at the endpoints. Since $|\zeta| > 1$ by Lemma 9, the formula of $\zeta(\theta)$ in (2-12) implies $\zeta(\theta) \to 1^+$ as $\theta \to \left(\frac{2\pi}{3}\right)^+$. Consequently, (2-18) gives

$$\lim_{\theta \to (2\pi/3)^+} z(\theta) = -\infty.$$

Finally, the fact that

$$\lim_{\theta \to \pi} \zeta(\theta) = \frac{1 - 2a + \sqrt{1-3a}}{1-4a},$$

together with (2-14), implies

$$\begin{aligned}\lim_{\theta \to \pi} z(\theta) &= \frac{a(1-2a+\sqrt{1-3a})(1-4a)}{(-1+6a+\sqrt{1-3a})(-1-2\sqrt{1-3a})} \\ &= \frac{a(-1+4a)^2(-2+9a)+2a(-1+3a)(-1+4a)^2\sqrt{1-3a}}{27(1-4a)^2 a} \\ &= \frac{-2+9a-2\sqrt{(1-3a)^3}}{27}, \qquad (2\text{-}22)\end{aligned}$$

where we obtain (2-22) after multiplication and division by $(-1+6a-\sqrt{1-3a})$ $(-1+2\sqrt{1-3a})$. □

Before making the final arguments connecting the results of this section, we check the sign of $g_m(\theta)$ at one of the endpoints.

Lemma 16. *If $-1 \le a < \frac{1}{4}$, then the sign of $g_m(\pi^-)$ is $(-1)^m$.*

Proof. Since $-1 \le a < \frac{1}{4}$, one can check that

$$\lim_{\theta \to \pi^-} \zeta(\theta) = \frac{1-2a+\sqrt{1-3a}}{1-4a} \ge 1.$$

The result follows directly from (2-15) and the fact that

$$\lim_{\theta \to \pi^-} \frac{\sin((m+1)\theta)}{\sin(\theta)} = (m+1)(-1)^m. \qquad □$$

With all the lemmas at our disposal, we produce the final arguments to finish the proof of Theorem 3. We consider the function $g_m(\theta)$ at the points

$$\theta_h = \frac{h\pi}{m+1} \in \left(\tfrac{2\pi}{3}, \pi\right), \quad \left\lfloor \tfrac{2}{3}(m+1) \right\rfloor + 1 \le h \le m.$$

We note that the number of such values of h is

$$m - \left\lfloor \tfrac{2}{3}(m+1) \right\rfloor = \left\lfloor \tfrac{1}{3} m \right\rfloor,$$

where the equality can be checked by considering the residue classes of m modulo 3. From the formula of $g_m(\theta)$ in (2-15) and Lemma 9, the sign of $g_m(\theta_h)$ is $(-1)^{h-1}$.

By the intermediate value theorem and Lemma 12, there are at least $\lfloor \frac{1}{3}m \rfloor - 1$ zeros of $g_m(\theta)$ on $\left(\frac{2\pi}{3}, \pi\right)$. In fact, we claim that there are at least $\lfloor \frac{1}{3}m \rfloor$ zeros of $g_m(\theta)$ on $\left(\frac{2\pi}{3}, \pi\right)$. In the case $-1 \le a < \frac{1}{4}$, we obtain one more zero of $g_m(\theta)$ from Lemma 16. On the other hand, if $\frac{1}{4} < a \le \frac{1}{3}$, then we obtain another zero of $g_m(\theta)$ by Lemma 12. Using Lemmas 14 and 15, we obtain at least $\lfloor \frac{1}{3}m \rfloor$ zeros of $H_m(z)$ on I_a. Since the degree of $H_m(z)$ is at most $\lfloor \frac{1}{3}m \rfloor$ by Lemma 5, all the zeros of $H_m(z)$ lie in I_a. Recall that we can ignore the case $a = \frac{1}{4}$ by Lemma 4. The density of $\bigcup_{m=0}^{\infty} \mathcal{Z}(H_m(z))$ in I_a comes from the density of $\bigcup_{m=0}^{\infty} \mathcal{Z}(g_m(\theta))$ in $\left(\frac{2\pi}{3}, \pi\right)$ and from $z(\theta)$ being a continuous map.

3. The case $c = 0$ and $b \ge 0$

It is trivial that if $c = 0$ and $b = 0$, then the zeros of $H_m(z)$ are real under the convention that the constant zero polynomial is hyperbolic. When $b > 0$, we make the substitution $t \to t/\sqrt{b}$ and reformulate the claim as follows.

Theorem 17. *The zeros of the sequence of polynomials $\{H_m(z)\}_{m=0}^{\infty}$ generated by*

$$\sum_{m=0}^{\infty} H_m(z) t^m = \frac{1}{1 + t^2 + z t^3} \tag{3-1}$$

are real, and the set $\bigcup_{m=0}^{\infty} \mathcal{Z}(H_m)$ is dense in $(-\infty, \infty)$.

The proof of Theorem 17 follows from a similar procedure as that seen in Section 2. We will point out some key differences. The following lemma comes directly from the recurrence relation

$$H_m(z) + H_{m-2}(z) + z H_{m-3}(z) = 0$$

and induction.

Lemma 18. *The degree of the polynomial $H_m(z)$ generated by* (3-1) *is at most*

$$\begin{cases} \frac{1}{3}m & \text{if } m \equiv 0 \pmod 3, \\ \frac{1}{3}(m-4) & \text{if } m \equiv 1 \pmod 3, \\ \frac{1}{3}(m-2) & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

We define the following three functions on the interval $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$:

$$\begin{aligned} \zeta(\theta) &= -\frac{1}{2\cos\theta}, \\ g_m(\theta) &= \frac{-\sin((m+1)\theta)}{2\cos\theta \sin\theta}(2 + \cos 2\theta) - \cos((m+1)\theta) + (-2\cos\theta)^{m+1}, \quad (3\text{-}2) \\ z(\theta) &= \frac{2\cos\theta}{\sqrt{(1 - 4\cos^2\theta)^3}}. \end{aligned}$$

The proof of the lemma below is similar to that of Lemma 13. We leave the detailed computations to the reader.

Lemma 19. *Suppose* $\theta \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, $\zeta = \zeta(\theta)$, *and* $z = z(\theta)$ *are defined by* (3-2). *The three zeros of* $1 + t^2 + z(\theta)t^3$ *are*

$$t_0 = -\frac{e^{-i\theta}}{z(2\cos\theta + \zeta)}, \quad t_1 = t_0 e^{2i\theta}, \quad t_2/t_0 = \zeta e^{i\theta}.$$

Looking at $z'(\theta)$, one can check that $z(\theta)$ is strictly decreasing on the interval $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$. Using the partial fraction decomposition

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1+t^2+zt^3} = \frac{1}{z(t-t_0)(t-t_1)(t-t_2)},$$

we conclude that for each zero of $g_m(\theta)$ on the interval $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ we obtain two zeros $\pm z(\theta)$ of $H_m(z)$. We can also check by induction that $z = 0$ is a simple zero of $H_m(z)$ if m is odd, and $z = 0$ is not a zero of $H_m(z)$ when m is even. The formula of $g_m(\theta)$ implies that the sign of this function alternates when $\cos((m+1)\theta) = \pm 1$, that is, when,

$$(m+1)\theta = k\pi, \quad \tfrac{1}{3}(m+1) < k < \tfrac{1}{2}(m+1).$$

Since $g_m(\theta)$ is continuous on $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, we may apply the intermediate value theorem to compute the number of zeros of $g_m(\theta)$ on $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ and the corresponding number of zeros of $H_m(z)$ on $(-\infty, \infty)$. We will see that this number is equal to the degree of $H_m(z)$, thereby proving Theorem 17. We summarize the six arising cases, where θ^* denotes the smallest solution $(m+1)\theta = k\pi$ on the interval $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

Case 1: $m \equiv 1 \pmod 3$ and m is even. There are

$$\tfrac{1}{2}m - \tfrac{1}{3}(m+2) = \tfrac{1}{6}(m-4)$$

zeros of $g_m(\theta)$ on $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, which give $\frac{1}{3}(m-4)$ zeros of $H_m(z)$ on $(-\infty, \infty)$.

Case 2: $m \equiv 1 \pmod 3$ and m is odd. There are

$$\tfrac{1}{2}(m-1) - \tfrac{1}{3}(m+2) = \tfrac{1}{6}(m-7)$$

zeros of $g_m(\theta)$ on $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, which give $\frac{1}{3}(m-7)$ nonzero zeros of $H_m(z)$. We add a simple zero $z = 0$ and obtain $\frac{1}{3}(m-4)$ zeros of $H_m(z)$ on $(-\infty, \infty)$.

Case 3: $m \equiv 0 \pmod 3$ and m is even. With the observation that $\lim_{\theta \to \pi/3} g_m(\theta) = -3 < 0$ and $g_m(\theta^*) > 0$, we obtain

$$\tfrac{1}{2}m - \left(\tfrac{1}{3}m + 1\right) + 1 = \tfrac{1}{6}m$$

zeros of $g_m(\theta)$ on $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, which give $\frac{1}{3}m$ zeros of $H_m(z)$ on $(-\infty, \infty)$.

Case 4: $m \equiv 0 \pmod 3$ and m is odd. With the observation that $\lim_{\theta \to \pi/3} g_m(\theta) = 3 > 0$ and $g_m(\theta^*) < 0$, we obtain

$$\tfrac{1}{2}(m-1) - \big(\tfrac{1}{3}(m) + 1\big) + 1 = \tfrac{1}{6}(m-3)$$

zeros of $g_m(\theta)$ on $\big(\frac{\pi}{3}, \frac{\pi}{2}\big)$, which give $\frac{1}{3}(m-3)$ nonzero zeros of $H_m(z)$. We add a simple zero $z = 0$ and obtain $\frac{1}{3}m$ zeros of $H_m(z)$ on $(-\infty, \infty)$.

Case 5: $m \equiv 2 \pmod 3$ and m is even. With the observation that $g_m\big(\frac{\pi}{3}\big) = 0$, $g_m'\big(\frac{\pi}{3}\big) > 0$, and $g_m(\theta^*) < 0$, we obtain

$$\tfrac{1}{2}m - \big(\tfrac{1}{3}(m+1) + 1\big) + 1 = \tfrac{1}{6}(m-2)$$

zeros of $g_m(\theta)$ on $\big(\frac{\pi}{3}, \frac{\pi}{2}\big)$, which give $\frac{1}{3}(m-2)$ zeros of $H_m(z)$ on $(-\infty, \infty)$.

Case 6: $m \equiv 2 \pmod 3$ and m is odd. With the observation that $g_m\big(\frac{\pi}{3}\big) = 0$, $g_m'\big(\frac{\pi}{3}\big) < 0$, and $g_m(\theta^*) > 0$, we obtain

$$\tfrac{1}{2}(m-1) - \big(\tfrac{1}{3}(m+1) + 1\big) + 1 = \tfrac{1}{6}(m-5)$$

zeros of $g_m(\theta)$ on $\big(\frac{\pi}{3}, \frac{\pi}{2}\big)$, which give $\frac{1}{3}(m-5)$ nonzero roots of $H_m(z)$. We add a simple zero $z = 0$ and obtain $\frac{1}{3}(m-2)$ zeros of $H_m(z)$ on $(-\infty, \infty)$.

In all cases above the number of zeros of $H_m(z)$ on $(-\infty, \infty)$ corresponds to the degree of $H_m(z)$ and Theorem 17 follows.

4. Necessary condition for the reality of zeros

To prove the necessary condition of Theorem 2, we first show that if $c = 0$ and $b < 0$ then not all polynomials $H_m(z)$ are hyperbolic. In fact, with the substitution $t \to it$, we conclude that all the zeros of $H_m(z)$ will be purely imaginary by Theorem 17.

It remains to consider the sequence $H_m(z)\}_{m=0}^{\infty}$ generated by

$$\sum_{m=0}^{\infty} H_m(z) t^m = \frac{1}{1 + t + at^2 + zt^3},$$

and to show that if $a \notin \big[-1, \frac{1}{3}\big]$ then there is an m such that not all the zeros of $H_m(z)$ are real. In fact, we will show if $a \notin \big[-1, \frac{1}{3}\big]$, then $H_m(z)$ is not hyperbolic for all large m. To prove this, let us introduce some definitions, discussed in [Sokal 2004], related to the root distribution of a sequence of functions

$$f_m(z) = \sum_{k=1}^{n} \alpha_k(z) \beta_k(z)^m,$$

where $\alpha_k(z)$ and $\beta_k(z)$ are analytic in a domain D. We say that an index k is dominant at z if $|\beta_k(z)| \ge |\beta_l(z)|$ for all l $(1 \le l \le n)$. Let

$$D_k = \{z \in D : k \text{ is dominant at } z\}.$$

Let $\liminf \mathcal{Z}(f_m)$ be the set of all $z \in D$ such that every neighborhood U of z has a nonempty intersection with all but finitely many of the sets $\mathcal{Z}(f_m)$. Let $\limsup \mathcal{Z}(f_m)$ be the set of all $z \in D$ such that every neighborhood U of z has a nonempty intersection with all but infinitely many of the sets $\mathcal{Z}(f_m)$. We will need the following theorem from Sokal.

Theorem 20 [Sokal 2004, Theorem 1.5]. *Let D be a domain in $\mathbb{C}$, and let $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n$ $(n \geq 2)$ be analytic functions on D, none of which is identically zero. Let us further assume a "no-degenerate-dominance" condition: there do not exist indices $k \neq k'$ such that $\beta_k \equiv \omega\beta_{k'}$ for some constant ω with $|\omega| = 1$ and such that D_k $(= D_{k'})$ has nonempty interior. For each integer $m \geq 0$, define f_m by*

$$f_m(z) = \sum_{k=1}^{n} \alpha_k(z)\beta_k(z)^m.$$

Then $\liminf Z(f_m) = \limsup Z(f_m)$, *and a point z lies in this set if and only if either*

(i) *there is a unique dominant index k at z, and $\alpha_k(z) = 0$, or*

(ii) *there are two or more dominant indices at z.*

If $z^* \in \mathbb{C}$ such that the zeros in t of $1+t+at^2+z^*t^3$ are distinct then by the partial fractions given in (2-4) and Theorem 20, z^* will belong to $\liminf \mathcal{Z}(H_m)$ when the two smallest (in modulus) zeros of $1+t+at^2+z^*t^3$ have the same modulus. We also note that $t_0(z)$, $t_1(z)$, and $t_2(z)$ are analytic in a neighborhood of z^* by the implicit function theorem. If we let $\omega = e^{2i\theta}$, then the no-degenerate-dominance condition in Theorem 20 comes directly from equations (2-14) and (2-12) since θ is a fixed constant (and thus z is a fixed point which has empty interior).

Suppose $a \notin \left[-1, \frac{1}{3}\right]$. With the setup in the previous paragraph, our main goal is to find a $z^* \notin \mathbb{R}$ so that the zeros of $1+t+at^2+z^*t^3$ are distinct and the two smallest (in modulus) zeros of this polynomial have the same modulus. If we can find such a point, then $z^* \in \liminf \mathcal{Z}(H_m) = \limsup \mathcal{Z}(H_m)$. This implies that on a small neighborhood of z^* which does not intersect the real line, there is a nonreal zero of $H_m(z)$ for all large m by the definition of $\liminf \mathcal{Z}(H_m)$. Our choice of $z^* = z(\theta^*)$ comes from (2-14) for a special θ^*. Unlike in Section 2, θ^* will not belong to $\left(\frac{2\pi}{3}, \pi\right)$ to ensure that $z^* \notin \mathbb{R}$. In particular, we consider the two cases $a < -1$ and $a > 3$.

The case $a < -1$. We select $0 \ll \theta^* < \frac{\pi}{2}$. Since

$$\lim_{\theta \to \frac{\pi}{2}} \zeta(\theta) = i\sqrt{|a|},$$

see (2-12), we can pick $0 < \theta^* < \frac{\pi}{2}$ sufficiently close to $\frac{\pi}{2}$ so that $\zeta := \zeta(\theta^*) \in \mathbb{C}\backslash\mathbb{R}$ and $|\zeta(\theta^*)| > 1$. By Lemma 13, we have $t_2 = \zeta t_0 e^{i\theta^*}$ and $t_1 = t_0 e^{2i\theta^*}$. The fact that

$|\zeta| > 1$ and $\theta^* \neq 0, \frac{\pi}{2}$ implies that the polynomial $1+t+at^2+z(\theta^*)t^3$ has distinct zeros and not all its zeros are real. We will show that $z(\theta^*) \notin \mathbb{R}$, by contradiction. Indeed, if $z(\theta^*) \in \mathbb{R}$ then the zeros of the polynomial $1+t+at^2+z(\theta^*)t^3 \in \mathbb{R}[t]$ satisfy $t_0 = \bar{t}_1$ and

$$t_2 = t_0 \zeta e^{i\theta^*} \in \mathbb{R}.$$

This gives a contradiction because the first equation implies $t_0 e^{i\theta^*} \in \mathbb{R}$, while the second equation implies $t_0 e^{i\theta^*} \notin \mathbb{R}$ since $\zeta \notin \mathbb{R}$.

The case $a > \frac{1}{3}$. We select $\beta < \cos\theta^* \ll 1$, where $\beta = \sqrt{a/(4a-1)} < 1$. Once more,

$$\Big|\lim_{\cos\theta \to \beta} \zeta(\theta)\Big| = \begin{cases} \big|\sqrt{4a^2-a}/(1-2a)\big| & \text{if } a \neq \frac{1}{2}, \\ \infty & \text{if } a = \frac{1}{2}, \end{cases}$$

where we can easily check that

$$\left|\frac{\sqrt{4a^2-a}}{1-2a}\right| > 1, \quad a > \tfrac{1}{3}.$$

Thus if $\cos\theta^*$ is sufficiently close to β, then $0 < \theta^* < \frac{\pi}{2}$, $|\zeta(\theta^*| > 1$, and $|\zeta(\theta^*)| \notin \mathbb{R}$, where the last statement comes from (2-12) and the inequality

$$(1-4a)\cos^2\theta^* + a < 0.$$

With $0 < \theta^* < \frac{\pi}{2}$, $|\zeta(\theta^*| > 1$, and $|\zeta(\theta^*)| \notin \mathbb{R}$, we apply the same arguments given in the previous case to complete the proof.

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