involve
a journal of mathematics

The isoperimetric problem in the plane with the sum of two Gaussian densities

John Berry, Matthew Dannenberg, Jason Liang and Yingyi Zeng

msp

2018 vol. 11, no. 4

INVOLVE 11:4 (2018)
dx.doi.org/10.2140/involve.2018.11.549

The isoperimetric problem in the plane with the sum of two Gaussian densities

John Berry, Matthew Dannenberg, Jason Liang and Yingyi Zeng

(Communicated by Gaven Martin)

We consider the isoperimetric problem for the sum of two Gaussian densities in the line and the plane. We prove that the double Gaussian isoperimetric regions in the line are rays and that if the double Gaussian isoperimetric regions in the plane are half-spaces, then they must be bounded by vertical lines.

1. Introduction

Sudakov and Tsirelson, and independently Borell, see [Morgan 2009, 18.2], proved that for $\mathbb{R}^n$ endowed with a Gaussian measure, half-spaces bounded by hyperplanes are isoperimetric, i.e., minimize weighted perimeter for given weighted volume. Cañete et al. [2010, Question 6], in response to a question of Brancolini, conjectured that for $\mathbb{R}^n$ endowed with a finite sum of Gaussian measures centered on the x-axis, half-spaces bounded by vertical hyperplanes are isoperimetric. We consider the case of two such Gaussians in $\mathbb{R}^1$ and $\mathbb{R}^2$. Our Theorem 3.16 proves that on the double Gaussian line, rays are isoperimetric. Section 4 provides evidence that on the double Gaussian plane, half-spaces are isoperimetric.

1.1. ***The double Gaussian line.*** Theorem 3.16 states that the isoperimetric regions in the double Gaussian line are rays. We may assume that the two Gaussians have centers at 1 and -1. For small variances, the theorem follows by comparison with the single Gaussian. For larger variances, additional quantitative and stability arguments are needed to rule out certain nonray cases.

1.2. ***The double Gaussian plane.*** A conjecture of Cañete et al. [2010, Question 6], appearing in this paper as Conjecture 4.1, states that isoperimetric regions in the double Gaussian plane are half-planes bounded by vertical lines. We use variational arguments to show that horizontal and vertical lines are the only lines that are candidates, and that vertical lines always beat horizontal lines.

MSC2010: 49Q10, 53A10.
Keywords: isoperimetric, Gaussian.

2. First and second variations

Formulas 2.3 and 2.6 state standard first and second variation formulas, analogous to the first and second derivative conditions for local minima of twice-differentiable real functions.

Definition 2.1. A *density* e^{ψ} on $\mathbb{R}^n$ is a positive, continuous function used to weight volume and hypersurface area. Given a density e^{ψ}, the (weighted) *volume* of a region R is given by

$$\int_R e^{\psi}\, dV_0.$$

The (weighted) *hypersurface area* of its boundary ∂R is given by

$$\int_{\partial R} e^{\psi}\, dA_0.$$

R is called *isoperimetric* if no other region of the same weighted volume has a boundary with smaller hypersurface area.

We now assume that the density e^{ψ} is smooth. The existence and regularity of isoperimetric regions for densities of finite total volume is standard.

Existence and Regularity 2.2 [Morgan 2009, 5.5, 9.1, 8.5]. *Suppose that e^{ψ} is a density in the line or plane such that the line or plane has finite measure A_0. Then for any $0 < A < A_0$, an isoperimetric region R of weighted volume A exists and is a finite union of intervals bounded by finitely many points in the line or a finite union of regions with smooth boundaries in the plane.*

Let e^{ψ} be a smooth density on $\mathbb{R}^{n+1}$. Let R be a smooth region in $\mathbb{R}^{n+1}$. Let φ_t be a smooth, one-parameter family of deformations on $\mathbb{R}^{n+1}$ such that φ_0 is the identity. For a given $x \in \partial R$, the deformation $\varphi_t(x)$ traces out a small path in $\mathbb{R}^{n+1}$ beginning at x and $\varphi_t(\partial R)$ is a curve for each t. Therefore $\{\varphi_t\}$, where $|t| < \epsilon$, describes a perturbation of ∂R. Define

$$V(t) = \int_{\varphi_t(R)} e^{\psi}\, dV_0 \quad \text{and} \quad P(t) = \int_{\varphi_t(\partial R)} e^{\psi}\, dA_0.$$

First Variation Formulas 2.3 [Rosales et al. 2008, Lemma 3.1]. *Suppose that $\boldsymbol{n}$ and H are the inward unit normal and mean curvature of ∂R. Let X be the vector field $d\varphi_t/dt$ and $u = \langle X, \boldsymbol{n}\rangle$. Then we have that*

$$V'(0) = -\int_{\partial R} e^{\psi} u\, dA_0 \quad \textit{and} \quad P'(0) = -\int_{\partial R} (nH - \langle \nabla\psi, \boldsymbol{n}\rangle) e^{\psi} u\, dA_0.$$

Since any isoperimetric curve is a local minimum among all curves enclosing a certain volume A, it satisfies $P'(0) = 0$ for any φ_t such that $V(t) = A$ for small t.

Corollary 2.4. *If a curve ∂R is isoperimetric, then $(nH - \langle \nabla\psi, \boldsymbol{n}\rangle)$ is constant on ∂R.*

Proof. If a curve ∂R is isoperimetric, then it satisfies $P'(0) = 0$. By Formula 2.3, this occurs if and only if $(nH - \langle \nabla\psi, \boldsymbol{n}\rangle)$ is constant on ∂R. □

Definition 2.5. Let C be a boundary in the line or plane with unit inward normal $\boldsymbol{n}$ and let κ denote the standard curvature. For a density e^{ψ}, we call $\kappa_{\psi} = \kappa - d\psi/d\boldsymbol{n}$ the *generalized curvature* of C.

By Corollary 2.4, all isoperimetric curves have constant generalized curvature. In the real line, $n = 0$, so isoperimetric curves have $\langle \nabla\psi, \boldsymbol{n}\rangle$ constant. For the interval $[a, b]$, the generalized curvature evaluated at b is equal to $\psi'(b)$, while the generalized curvature evaluated at a is equal to $-\psi'(a)$.

Second Variation Formula 2.6 [Rosales et al. 2008, Proposition 3.6]. *Let the real line be with smooth density e^{ψ}. If a one-dimensional boundary $l = \partial R$ satisfies $P'(0) = 0$ for any volume-preserving $\{\varphi_t\}$, then*

$$(P - \kappa_{\psi} V)''(0) = \int_l f u^2 \left(\frac{d^2\psi}{dx^2}\right) da.$$

Proof. This formula comes from Proposition 3.6 in [Rosales et al. 2008], where the second variation is stated for arbitrary dimensions. Some terms from the general formula cancel in the one-dimensional case. □

Corollary 2.7. *Let S be a subset of the real line such that $\psi''(x) \leq 0$ for all $x \in S$ with equality holding at no more than one point. If B is an isoperimetric boundary contained in S, then B is connected and thus a single point.*

Proof. If B has at least two connected components, then since by Existence and Regularity 2.2 B consists of a finite union of points, there is a nontrivial volume-preserving flow on B given by moving one component so as to increase the volume and the other so as to decrease it. By Formula 2.6, the second variation satisfies

$$(P - \kappa_{\psi} V)''(0) = \int_B f u^2 (\psi''(x))\, da < 0.$$

This contradicts that B is isoperimetric. □

3. Isoperimetric regions on the double Gaussian line

Theorem 3.16 states that for the real line with density given by the sum of two Gaussians with the same variance a^2, isoperimetric regions are rays bounded by single points. This theorem is a necessary condition for Conjecture 4.1, which states that isoperimetric regions in the double Gaussian plane are half-planes bounded by vertical lines. Propositions 3.4, 3.14, and 3.15 treat the cases $a^2 \geq 1,\ 1 > a^2 > \frac{1}{2}$, and $\frac{1}{2} \geq a^2 > 0$.

Lemma 3.5 shows that if the Gaussians have the same variance, we can reduce the problem to ruling out a few noninterval, but still symmetrical, cases. When the Gaussians have different variances, the problem is harder and not treated by our results.

Let $g_{c,a}$ denote the Gaussian density with mean c and variance a^2, and let

$$f_{c,a}(x) = \frac{1}{2}\left(\frac{e^{-(x-c)^2/(2a^2)} + e^{-(x+c)^2/(2a^2)}}{a\sqrt{2\pi}}\right) = \tfrac{1}{2}(g_{c,a}(x) + g_{-c,a}(x)).$$

Let

$$f(x) = \tfrac{1}{2}(f_1(x) + f_2(x)) = \tfrac{1}{2}(g_{1,a}(x) + g_{-1,a}(x)).$$

In one dimension, the regions are unions of intervals and their boundaries are points. Since the total measure is finite, isoperimetric regions exist by Existence and Regularity 2.2. For a given weighted length A, we seek to find the set of points with the smallest total density which bounds a region of weighted length A. Since the complement of a region of weighted length A has weighted length $1 - A$, we can assume that our regions have weighted length $0 \leq A \leq \frac{1}{2}$.

The following proposition shows that it suffices to consider the density f.

Proposition 3.1. *Suppose B is an isoperimetric boundary enclosing a region L of weighted length A for the density $f_{1,a}(x)$. Then for any $b > 0$, we have bB is an isoperimetric boundary enclosing region bL of weighted length A for the density $f_{b,ab}(x)$.*

Proof. Let g denote the standard Gaussian density.

First, we show that for any boundary P enclosing a region Q, the weighted length of bQ for the density $f_{b,ab}(x)$ is the same as the weighted length of Q for the density $f_{1,a}(x)$. We have that

$$\begin{aligned}|Q| = \int_Q f_{1,a}(x)\,dx &= \frac{1}{2}\int_Q g_{1,a}(x)\,dx + \frac{1}{2}\int_Q g_{-1,a}(x)\,dx \\ &= \frac{1}{2}\int_{(Q-1)/a} g(x)\,dx + \frac{1}{2}\int_{(Q+1)/a} g(x)\,dx \\ &= \frac{1}{2}\int_{(bQ-b)/(ab)} g(x)\,dx + \frac{1}{2}\int_{(bQ+b)/(ab)} g(x)\,dx \\ &= \frac{1}{2}\int_Q g_{b,ab}(x)\,dx + \frac{1}{2}\int_Q g_{-b,ab}(x)\,dx = |bQ|,\end{aligned}$$

where $|\cdot|$ denotes the weighted length in the appropriate densities.

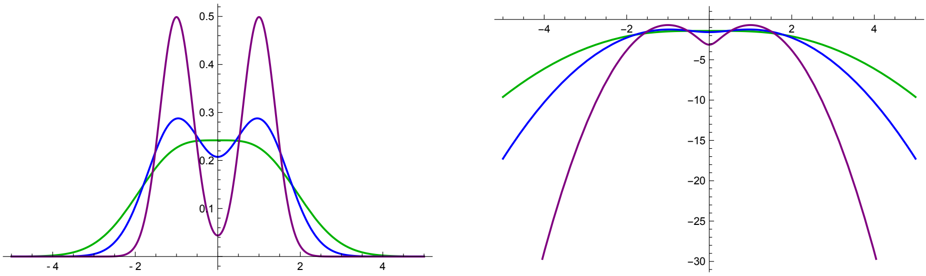


Figure 1. Plots of f (left) and ψ (right). The purple curves are for $a^2 = 0.16$, the blue curves for $a^2 = \frac{1}{2}$, and the green curves for $a^2 = 1$.

Second, for any two boundaries P_1 and P_2, we have $f_{b,ab}(bx) = (1/b) f_{1,a}(x)$ for $x \in P_i$. Thus, $|P_1| \geq |P_2|$ in the density $f_{1,a}(x)$ exactly when $|bP_1| \geq |bP_2|$ in the density $f_{b,ab}(x)$.

Therefore $|bL| = A$ in the density $f_{b,ab}(x)$, and if any other boundary P enclosing region Q satisfies $|Q| = A$ in the density $f_{b,ab}(x)$, then since B is isoperimetric, we have $|B| \leq |P/b|$ in the density $f_{1,a}(x)$. Therefore $|bB| \leq |P|$ in the density $f_{b,ab}(x)$, so bP is isoperimetric. $\square$

As a result of Proposition 3.1, it suffices to consider the density

$$f = \tfrac{1}{2}(f_1 + f_2) = \tfrac{1}{2}(g_{1,a} + g_{-1,a}).$$

Proposition 3.2. *Let X be the disjoint union of two real-lines X_1 and X_2, each with a standard Gaussian density scaled so that it has weighted length $\frac{1}{2}$. For any given length $0 < A < \frac{1}{2}$, the isoperimetric region in X of length A is a ray contained entirely in X_1 or X_2.*

Proof. Let B be an isoperimetric boundary and B_i its intersection with X_i. If B_1 and B_2 are nonempty, then they each must be a single point since the isoperimetric boundaries for the single Gaussian are always single points. Assume, in contradiction to the proposition, that $B_i = \{b_i\}$ for $i = 1, 2$ is the i-th component on the i-th Gaussian bounding a ray L_i of weighted length A_i. Since $A_1 + A_2 < \frac{1}{2}$, it is possible to put a point b_1' on the first Gaussian at the same height as that of b_2 bounding a ray L_1' disjoint from L_1 and with weighted length A_2. Consider the boundary $B' = \{b_1, b_1'\}$, which has the same weighted perimeter as that of B. There exists a single point on B_1 bounding a ray of area A and with weighted density smaller than $|B'| = |B|$. This contradicts the fact that B is isoperimetric. $\square$

Proposition 3.3. *For the double Gaussian density f, the log derivative ψ' is given by*

$$\psi'(x) = a^{-2}\Big(-x + \tanh\frac{x}{a^2}\Big).$$

Proof. We have

$$\psi'(x) = \frac{\dfrac{-e^{-(-1+x)^2/(2a^2)}(-1+x)}{a^2} + \dfrac{-e^{-(1+x)^2/(2a^2)}(1+x)}{a^2}}{e^{-(-1+x)^2/(2a^2)} + e^{-(1+x)^2/(2a^2)}}.$$

By using the substitution

$$\tanh\Big(\frac{x}{a^2}\Big) = \frac{e^{x/a^2} - e^{-x/a^2}}{e^{x/a^2} + e^{-x/a^2}},$$

we get

$$\psi'(x) = a^{-2}\Big(-x + \tanh\frac{x}{a^2}\Big). \qquad \square$$

Proposition 3.4. *For the double Gaussian density f, if $a \geq 1$, isoperimetric boundaries are single points.*

Proof. For any given a, we have

$$\psi'(x) = a^{-2}\Big(-x + \tanh\frac{x}{a^2}\Big),$$

$$\psi''(x) = a^{-4}\Big(-a^2 + \operatorname{sech}^2\frac{x}{a^2}\Big),$$

$$\psi'''(x) = -2a^{-6}\operatorname{sech}^2\frac{x}{a^2}\tanh\frac{x}{a^2}.$$

As shown in Figure 2, $\psi'''(x)$ is positive for any $x < 0$ and negative for $x > 0$, so $\psi''(x)$ achieves its unique maximum at $x = 0$ for any given a. We have $\psi''(0) = (1 - a^2)/a^4$, so $\psi''(0)$ is greater than 0 for $a < 1$, and less than or equal to 0 for $a \geq 1$. If $a \geq 1$, by Corollary 2.7, isoperimetric boundaries are always connected. Since isoperimetric boundaries consist of finite unions of points, they must be single points. $\square$

Lemma 3.5. *Let p and q be two real functions with $p(0) = q(0)$. Suppose p and q satisfy*

(1) $p'(0) = q'(0) \geq 0$,

(2) $q''(0) \geq p''(0)$,

(3) $q''(0) \geq 0$, *and*

(4) $p''' < 0$ *and* $q''' > 0$ *on* $(0, \infty)$.

For any $a, b > 0$, if $p(a) = q(b)$, then $q'(b) > p'(a)$.

Proof. As in Figure 3, for all $x > 0$, by (2) and (4) we have $q''(x) > p''(x)$ and by (3) and (4) we have $q''(x) > 0$. If we choose a' so that $q'(a') = p'(a)$, we will have $a' < a$.

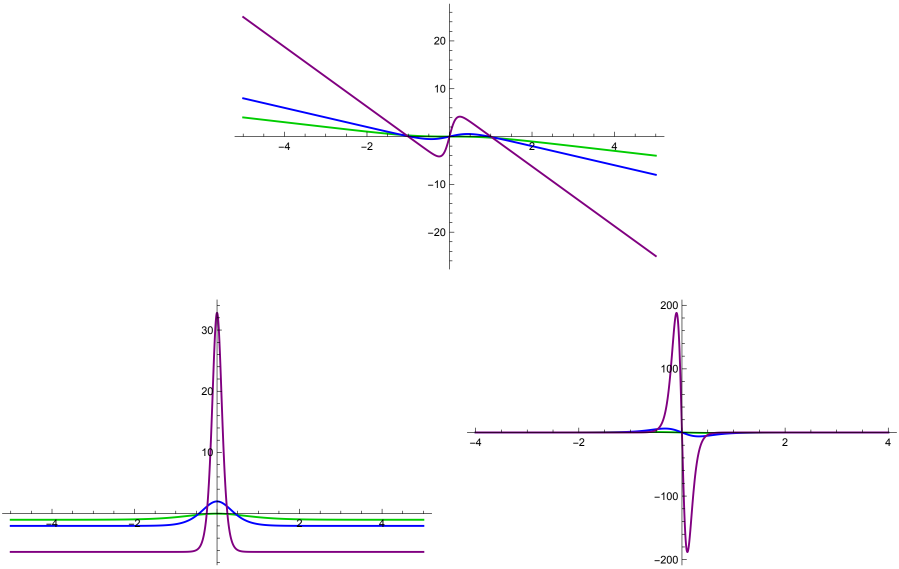


Figure 2. Plots of ψ' (top), ψ'' (bottom left), and ψ''' (bottom right). The purple curves are for $a^2 = 0.16$, the blue curves for $a^2 = \frac{1}{2}$, and the green curves for $a^2 = 1$.

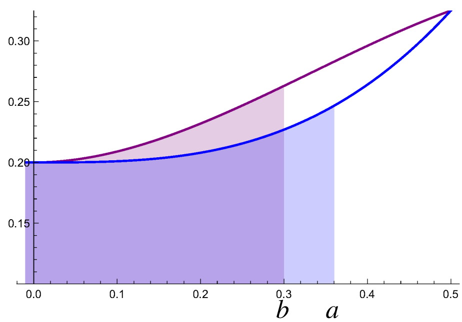


Figure 3. The purple curve is q', and the blue curve p'. When the areas are equal, as in the picture, q' is higher.

Since by (4) p' is concave and q' is convex,

$$q(a') = \int_0^{a'} q'(t)\,dt \le \tfrac{1}{2}(a' * q'(a')) < \tfrac{1}{2}(a * p'(a)) \le \int_0^{a'} q'(t)\,dt = p(a) = q(b).$$

Therefore $b > a'$, so $q'(b) > p'(a)$, as asserted. □

Proposition 3.6. *Suppose* $[a, b]$ *is an interval of* f*-weighted length* $0 < A < \frac{1}{2}$ *with* $-1 < a < b < 1$*. Then there exists a union of rays* $B = (-\infty, c] \cup [d, \infty]$ *of* f_1*-weighted length* A *such that* $f_1(c) < f_1(b) < f(b)$ *and* $f_1(d) < f_2(a) < f(a)$.

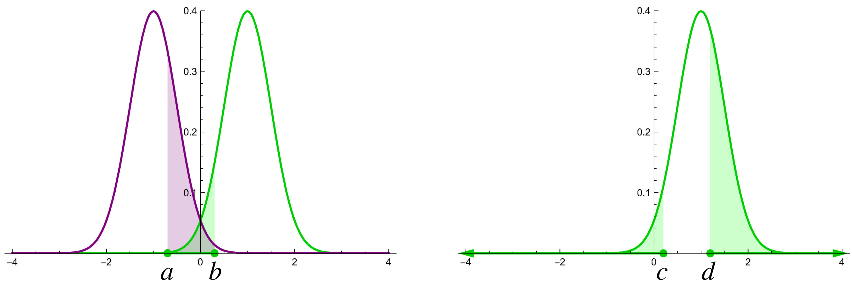


Figure 4. Left: an interval in the double Gaussian. Right: two rays in the single Gaussian. The total areas are the same, but the heights in the right graph are slightly lower.

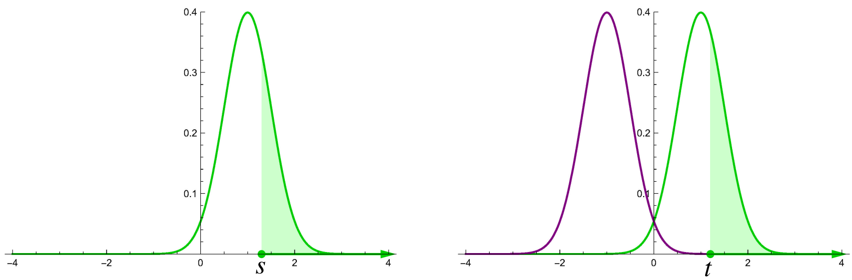


Figure 5. Left: ray in the single Gaussian. Right: ray in the double Gaussian. The total areas are the same, but the height in the right graph is slightly lower.

Proof. Since $2+a = 1+(a-(-1))$, we have $f_1(2+a) = f_2(a)$. The union of rays $(-\infty, b] \cup [2+a, \infty)$ has greater f_1-weighted length than the f-weighted length of $[a, b]$. Therefore there exists $c < t$ and $d > 2+a$ such that $(-\infty, c] \cup [d, \infty)$ has f_1-weighted length A, and

$$f_1(c) + f_1(d) < f_1(b) + f_2(a) < f(b) + f(a).$$

See Figure 4. □

Proposition 3.7. *If* $[s, \infty)$ *has* $\left(\frac{1}{2}\right) f_1$*-weighted length* $0 < A \leq \frac{1}{4}$*, then there exists* $t > s$ *such that* $[t, \infty)$ *has* f*-weighted length* A.

Proof. If $[s, \infty)$ has $\left(\frac{1}{2}\right) f_1$-weighted length $0 < A \leq \frac{1}{4}$, then $s \geq 1$. The interval $[s, \infty)$ has f-weighted length greater than A. Therefore there exists $t > s$ such that $[t, \infty)$ has f-weighted length A. See Figure 5. □

Now we begin analyzing the case where the variance satisfies $0 < a^2 < 1$.

Proposition 3.8. *If* a^2 *satisfies* $0 < a^2 \leq 1$*, then* $\psi''(x) = 0$ *exactly when* $x = \pm a^2 \operatorname{arccosh}(1/a)$.

Proof. This follows from the formula for $\psi''(x)$ given in Proposition 3.4. □

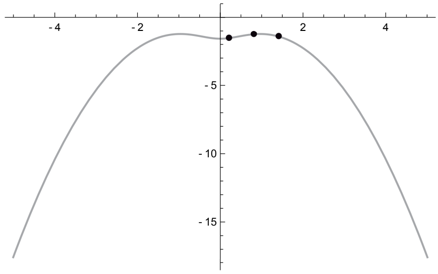


Figure 6. On the graph of $\psi = \log f$, there are at most three points with $x > 0$ with the same value for $|\psi'(x)|$.

Suppose that a^2 is a variance. In the proof of the following proposition, we will use the quantity

$$c_a = a^2 \operatorname{arccosh}(1/a).$$

Proposition 3.9. *Suppose* $0 < a^2 \leq 1$ *and* B *is an isoperimetric boundary with at least one point* s *in* $[0, c]$, *where* $c = c_a$, *enclosing a region of weighted-length* $0 < A < \frac{1}{2}$. *Then the boundary* B *is one of the following*:

(1) *a single point* s *enclosing the ray* $[s, \infty)$,

(2) $\{s, t\}$, *where* $t > s$, *enclosing the interval* $[s, t]$,

(3) $\{s, t\}$, *where* $s > 0 > t$, *enclosing the interval* $[t, s]$,

(4) $\{s, -s, t\}$ *enclosing* $[-s, s] \cup (-\infty, t]$, $[-s, s] \cup [t, \infty)$ *or* $[s, t] \cup (-\infty, -s]$.

The analogous claims apply if $s \in [-c, 0]$.

Proof. Since B is isoperimetric, it can contain at most one point x at which $\psi''(x) < 0$. If it contained two such points, then by slightly shifting the two points we could create a new region with the same weighted length. By Formula 2.6, the boundary of this region would have a smaller total density. Therefore B can contain at most one point outside of $[-c, c]$.

In addition, B has constant curvature, so $|\psi'|$ is constant on B (see Figure 6). Since $\psi''(s)$ is positive on $[0, c)$ and negative on $(c, \infty]$, there exists one point $t > s > 0$ such that $\psi'(t) = \psi'(s)$ and one point $u > t > s > 0$ such that $-\psi'(u) = \psi'(s)$. Therefore B is a subset of $\{s, t, u, -s, -t, -u\}$. Suppose B is not (1). If B contains no points outside of $[-c, c]$, then B is (3). Suppose B contains one point y outside of $[-c, c]$. If $t > 0$, then the only possibilities are (2) or (4). If $t < 0$, then the only possibilities are (3) or (4). The regions enclosed follow from the fact that we assume $0 < A < \frac{1}{2}$. ☐

Proposition 3.10. *Suppose* B *is an isoperimetric boundary with at least one point* $s \in [-c, c]$. *If* B *is of type* (3) *in* Proposition 3.9 *and* $0 < a^2 \leq \frac{1}{2}$, *then the region* R *enclosed by* B *has* f*-weighted length no more than* $\frac{1}{4}$.

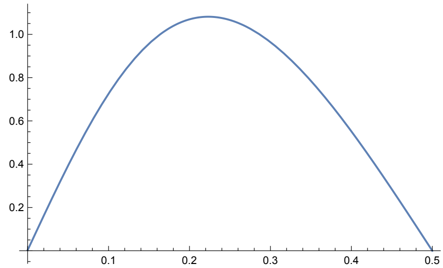


Figure 7. $\psi'(s) - \psi'(1-s)$.

Proof. We have

$$\frac{d}{dx}(x - \operatorname{arccosh}(x)) = 1 - \frac{1}{\sqrt{x-1}\sqrt{1+x}} > 0$$

for $x > \sqrt{2}$, so $x - \operatorname{arccosh}(x)$ is increasing on $(\sqrt{2}, \infty)$. If $y = 1/x$, then the function $y - \operatorname{arccosh}(y) - \frac{1}{2}$ decreases on $(0, 1/\sqrt{2})$. Since $\sqrt{2} - \operatorname{arccosh}(\sqrt{2}) - \frac{1}{2} > 0$, we have $\operatorname{arccosh}(y) < y - \frac{1}{2}$ on $(0, 1/\sqrt{2})$. Therefore

$$c < a - \frac{a^2}{2} \leq \frac{1}{\sqrt{2}} - \frac{1}{4} < \frac{1}{2}.$$

Consider the function

$$I(x) = \int_{x-1}^{x} f_1(x)\,dx + \int_{x-1}^{x} f_2(x)\,dx$$

which sends x to the weighted length of $[x-1, x]$. Then

$$\begin{aligned} I'(x) &= f_1(x) - f_1(x-1) + f_2(x) - f_2(x-1) \\ &= [f_2(x) - f_1(x-1)] + [f_1(x) - f_2(x-1)]. \end{aligned}$$

For $|x| < \frac{1}{2}$, both the bracketed quantities are negative, so I is decreasing on $[0, c]$. We have

$$I(0) = \int_{-1}^{0} f_2(x)\,dx + \int_{-1}^{x} f_1(x)\,dx = \int_{-1}^{1} f_2(x)\,dx < \int_{-1}^{\infty} f_2(x)\,dx = \tfrac{1}{4}.$$

Therefore if we can show that $s - t \leq 1$, we will have that the f-weighted length of $[s, t]$ is less than $I(s) \leq \frac{1}{4}$ and be done. This follows immediately when $t = -s$, since $s \leq c < \frac{1}{2}$. When $t \neq -s$, we observe that $s - 1$ is to the left of $-c$, so it suffices to show that $\psi'(s-1) \geq \psi'(t) = -\psi'(s)$. Thus we want to show that

$$\begin{aligned} \psi'(s-1) + \psi'(s) &= \psi'(s) - \psi'(1-s) \\ &= ([(1-s) - s] + [\tanh(s/a^2) - \tanh((1-s)/a^2)])/(a^2) \geq 0 \end{aligned}$$

on $\left[0, \frac{1}{2}\right]$; see Figure 7.

This is equivalent to showing that

$$\gamma(s) := \left([(1-s)-s] + \left[\tanh\left(\frac{s}{a^2}\right) - \tanh\left(\frac{1-s}{a^2}\right) \right] \right) \geq 0$$

on $[0, c]$. Since $|\tanh| < 1$, we have $\gamma(0) > 0$. In addition, $\gamma\left(\frac{1}{2}\right) = 0$. Therefore it suffices to show that γ achieves its minimum value on $\left[0, \frac{1}{2}\right]$ at $s = \frac{1}{2}$. We will do this by using the first derivative test to show that there is only one other local extremum in the interval and further demonstrating that this local extremum is not the minimum point.

We have

$$\gamma'(s) = \frac{\operatorname{sech}^2(s/a^2)}{a^2} + \frac{\operatorname{sech}^2((1-s)/a^2)}{a^2} - 2.$$

Since $1/a^2 \geq 2$, we have $\gamma'(0) > 0$. In addition,

$$\gamma'\left(\tfrac{1}{2}\right) = \frac{2\operatorname{sech}^2(1/(2a^2))}{a^2} - 2.$$

By using the substitution

$$\operatorname{sech}^2(x) = \frac{4}{e^{2x} + e^{-2x} + 2},$$

we get

$$\operatorname{sech}^2\left(\frac{x}{2}\right) = \frac{4}{e^{1/x} + e^{-1/x} + 2} \leq \frac{4}{e^{1/x} + 2}.$$

Therefore

$$\operatorname{sech}^2\left(\frac{x}{2}\right)\left(\frac{1}{x}\right) \leq \frac{4}{xe^{1/x} + 2x}.$$

We have

$$\alpha(x) := (xe^{1/x} + 2x)' = (2 + e^{1/x} - e^{1/x}/x).$$

When $0 < x \leq \frac{1}{2}$, we have

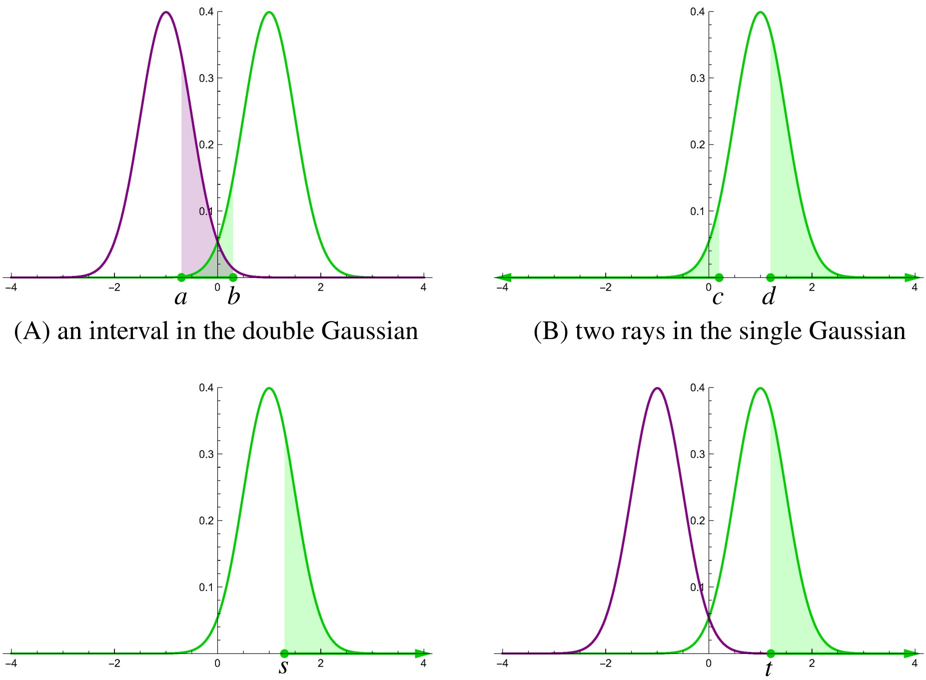
$$\alpha(x) \leq 2 + e^{1/x} - 2e^{1/x} = 2 - e^{1/x} \leq 2 - e^2 < 0.$$

Therefore $\alpha(x)$ attains a minimum value of $\frac{1}{2}e^2 + 1 > 4$ on $\left(0, \frac{1}{2}\right]$. This shows that

$$\operatorname{sech}^2\left(\frac{x}{2}\right)\left(\frac{1}{x}\right) \leq \frac{4}{xe^{1/x} + 2x} < 1$$

on $\left(0, \frac{1}{2}\right]$, so $\gamma'\left(\frac{1}{2}\right) < 0$.

By the intermediate value theorem, there exists $z_1 \in \left(0, \frac{1}{2}\right)$ such that $\gamma'(z_1) = 0$. It follows that $z_2 = 1 - z_1 > \frac{1}{2}$ is also a zero of γ'. Now $\operatorname{sech}^2(x) = \operatorname{sech}^2(-x)$ tends to 0 as x tends to ∞, so $\gamma' < 0$ for some $s \ll 0$. Therefore there exists z_3 in $(-\infty, 0)$ such that $\gamma'(z_3) = 0$, and $z_4 = 1 - z_3 > 1$ is also a zero of γ'.



(A) an interval in the double Gaussian

(B) two rays in the single Gaussian

(C) a ray in the single Gaussian

(D) a ray in the double Gaussian

Figure 8. When all the areas are the same, we have (A) > (B) > (C) and (D) > (C).

Again using the substitution

$$\operatorname{sech}^2(x) = \frac{4}{e^{2x} + e^{-2x} + 2},$$

we see that $\gamma'(s)$ is a rational function of e^{2s/a^2} whose numerator is quartic. Therefore γ' has at most four zeros, so z_1 is the only zero of γ' in $\left(0, \frac{1}{2}\right)$. Since $\gamma'(0) > 0$,

$$\gamma(z_1) > \gamma(0) > \gamma\left(\tfrac{1}{2}\right),$$

so $\gamma(s) \geq \gamma\left(\frac{1}{2}\right) = 0$ for $s \in \left[0, \frac{1}{2}\right]$. □

Proposition 3.11. *If the variance satisfies $0 < a^2 \leq \frac{1}{2}$, then the isoperimetric boundaries B with one point b in $[0, c]$ cannot be of type (3) in Proposition 3.9.*

Proof. Let A be the weighted length of B. If, in contradiction to the proposition, B is of type (3) in Proposition 3.9, then B is of the form $[a, b]$, where $-1 < a < b < 1$, as shown in Figure 8(A). By Proposition 3.6, there exists a union of rays $(-\infty, c] \cup [d, \infty)$ with f_1-weighted length A such that $f_1(c) + f_1(d) < f(a) + f(b)$. This is shown in Figure 8(B). By the solution to the single Gaussian isoperimetric

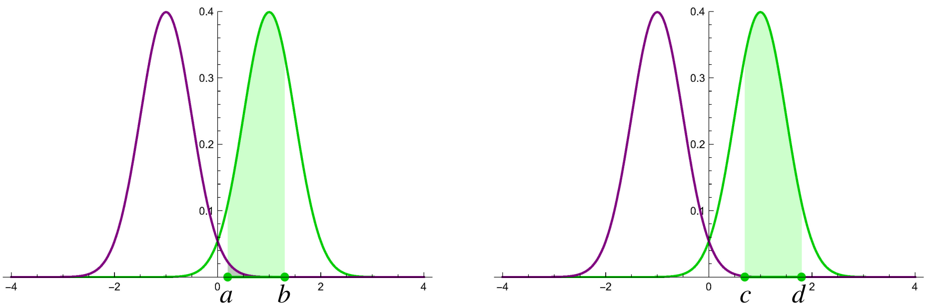


Figure 9. Left: original ray. Right: reflected ray.

problem, there exists a ray $[s, \infty)$, as shown in Figure 8(C), with f_1-weighted length A such that $f_1(t) < f_1(c) + f_1(d)$. By Proposition 3.10, $A \leq \frac{1}{4}$, so $s \geq 1$. By Proposition 3.7, there exists a ray $[t, \infty)$, as shown in Figure 8(D), with f-weighted length A such that $t > s$.

To get a contradiction to the fact that B is isoperimetric, we show $(f(a)+f(b)) - f(t) > 0$. Write

$$(f(a)+f(b)) - f(t) = [(f(a)+f(b)) - (f_1(c)+f_1(d))] \\ + [(f_1(c)+f_1(d)) - f_1(s)] + [f_1(s) - f(t)].$$

Since $[(f_1(c) + f_1(d)) - f_1(s)] > 0$, it suffices to show that $[(f(a) + f(b)) - (f_1(c) + f_1(d))] > [f(t) - f_1(s)]$. Since $f(a) > f_1(d)$, we have

$$[(f(a)+f(b)) - (f_1(c)+f_1(d))] > f(b) - f_1(c) > f(b) - f_1(b) = f_2(b).$$

Since $f(t) < f(s)$, we have

$$[f(t) - f_1(s)] < f(s) - f_1(s) = f_2(s).$$

Since $-1 < b < 1 < s$, we have $f_2(s) < f_2(b)$, and this proves the claim. □

Proposition 3.12. *If the variance satisfies $a^2 \leq \frac{1}{2}$, then the isoperimetric boundaries B with one point $b > 0$ in $[-c, c]$ cannot be of type (2) in Proposition 3.9.*

Proof. We know $f(a) < f(b)$ (recall the concavity/convexity argument), and since $f_2(b) < f_2(a)$, we must have $f_1(a) < f_1(b)$.

Pick $d > c$ such that $f_1(c) = f_1(b)$ and $f_1(d) = f_1(a)$. In other words, we get $[c, d]$ by reflecting $[a, b]$ over the line $x = 1$. See Figure 9. Since $a < 1$, we either have $c < 1 < d$ or $1 < d < c$.

In the first case, we have that $[c, d]$ has the same f_1-length as $[a, b]$, and since $c > a$ and $d > b$, we have $f_2(c) < f_2(a)$ and $f_2(d) < f_2(b)$. Therefore $f(c)+f(d) < f(a) + f(b)$. At the same time, the f_2-length of $[c, d]$ is less than that of $[a, b]$. This difference is at most the f_2-length of $[a, \infty)$. Since $f_1(d) = f_1(a) > f_2(a)$, we can find $e > d$ such that $[c, e]$ has f-length A. In addition, $f(c) + f(e) < f(c) + f(d) < f(a) + f(b)$, so $[a, b]$ is not isoperimetric.

In the second case, we have that $[d, c]$ has the same f_1-length as $[a, b]$, and since $d, c > a, b$, we have $f_2(d) < f_2(a)$ and $f_2(c) < f_2(b)$. Therefore $f(c) + f(d) < f(a) + f(b)$. At the same time, the f_2-length of $[d, c]$ is less than that of $[a, b]$. This difference is at most the f_2-length of $[a, \infty)$. Since $f_1(c) = f_1(a) > f_2(a)$, we can find $e > c$ such that $[d, e]$ has the f-length A. In addition, $f(c) + f(e) < f(c) + f(d) < f(a) + f(b)$, so $[a, b]$ is not isoperimetric. □

Proposition 3.13. *If the variance satisfies $a^2 \leq \frac{1}{2}$, then the isoperimetric boundaries B with one point b in $[-c, c]$ cannot be of type (4) in Proposition 3.9.*

Proof. We may assume without loss of generality that $b \geq 0$. Suppose B is of type (4) in Proposition 3.9. Then the region L enclosed by B consists of the union of an interval of type (2) or (3) in Proposition 3.9 and a ray. Apply Propositions 3.11 and 3.12 to get a new region L' that beats the interval. Since $A < \frac{1}{2}$, L' may be chosen to not intersect the ray. Then the union of L' and the ray beats L. □

Proposition 3.14. *If B is an isoperimetric boundary and the variance satisfies $a^2 \leq \frac{1}{2}$, then B is a single point.*

Proof. If B does not contain a point $s \in [-c, c]$, then by Corollary 2.7, then B is a single point. Otherwise, apply Propositions 3.11–3.13 to complete the proof. □

Proposition 3.15. *For the line endowed with density $f(x)$, if the variance a^2 is such that $\frac{1}{2} \leq a^2 < 1$, then isoperimetric regions R are always rays with boundary B consisting of a single point.*

Proof. By Proposition 3.8, we have that $\psi''(x) = 0$ exactly when x is $c = \pm a^2 \operatorname{arccosh}(1/a)$. Since $\psi'''(x) > 0$ for $x < 0$ and $\psi'''(x) < 0$ for $x > 0$, we have that ψ'' is negative outside of $[-c, c]$ and is positive in $(-c, c)$.

Suppose that B is an isoperimetric boundary containing more than two points. By Corollary 2.7, B does not lie entirely outside $[-c, c]$. Since $\psi''(x) > 0$ on $(-c, c)$ and $\psi''(\pm c) = 0$, the maximum and minimum of $\psi'(x)$ on $[-c, c]$ are achieved at c and $-c$ with $\psi'(-c)$ negative and $\psi'(c)$ positive. Since $\psi'(x)$ tends to $-\infty$ as x approaches ∞, there exists a unique point $b > c$ such that $f(\pm b) = f(\pm c)$. Since $b > c$, we have $\psi''(x) < 0$ outside of $[-b, b]$.

We claim that B must lie in $[-b, b]$. Since $|\psi'(x)|$ is constant on B, to show that $B \subset [-b, b]$ it suffices to show that the maximum and minimum of $\psi'(x)$ on $[-b, b]$ are achieved at $-b$ and b. Since 0 is a local minimum for $f(x)$, it suffices to show that $|\psi'(b)| > |\psi'(c)|$. Since $\psi'(c)$ is postive and $\psi''(x) < 0$ for $x > c$, there exists a unique point $d > c$ where $\psi'(d) = 0$ and ψ' changes from positive to negative at d. To apply Lemma 3.5, consider functions p and q denoting the

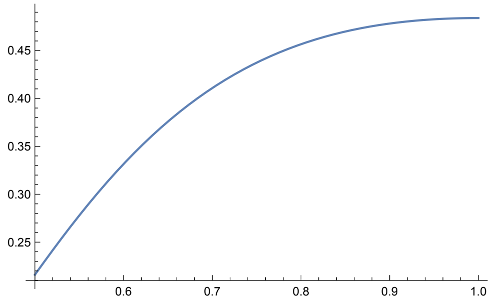


Figure 10. $2f(0, a)$ for various values of a.

increase in ψ moving left of d and the decrease in ψ moving right of d:

$$p(x) = \psi(d) - \psi(d - x),$$
$$q(x) = g(x) = \psi(d) - \psi(d + x),$$

which satisfy the hypotheses of Lemma 3.5. Since $\psi(c) = \psi(b)$, we have

$$|\psi'(c)| = \psi'(c) = p'(d - c) < g'(b - d) - \psi'(b) = |\psi'(b)|.$$

There are five candidates for the minimum points of $f(x)$ on $[-b, b]$: $\pm b$, 0, and $\pm d$. Since $d > c$, we have $\psi''(d) < 0$, so $\pm d$ is not a candidate. Since, also by the preceding paragraph, $\psi'(x)$ is positive between 0 and c, we have $f(b) = f(c) > f(0)$. Therefore the minimum on this interval is $f(0)$. We have

$$\frac{d}{da}(f(0, a)) = -\frac{\sqrt{1/a^2}(-1 + a^2)e^{-1/(2a^2)}}{a^3\sqrt{2\pi}} > 0$$

for all $a \in [-1/\sqrt{2}, 1)$. Therefore we have

$$2f(0, a) \geq 2f(0, 1/\sqrt{2}) \approx 0.415107\ldots.$$

See Figure 10.

To finish the proof, we must show that $f(x, a) < 0.415107\ldots$ for all x and all $a \in [1/\sqrt{2}, 1)$. Consider the numerator n of f given by

$$n(x) = e^{-(x-1)^2/(2a^2)} + e^{-(x+1)^2/(2a^2)}.$$

For a given x, we have n increases when a increases, so

$$n(x) \leq m(x) = e^{-(x-1)^2/2} + e^{-(x+1)^2/2}.$$

Since

$$\frac{d}{dx}(\log m(x)) = \tanh(x) - x,$$

which has the same sign as $-x$, we see that $m(x)$ is maximized at 0. Therefore $n(x) \leq m(0) < 1.22$, so

$$f(x) < \frac{m(0)}{2\sqrt{2\pi}a} \leq \frac{1.22}{2\sqrt{\pi}} \approx 0.345.$$

This means that there is a ray which beats B, contradicting the fact that B is isoperimetric. □

Theorem 3.16. *The isoperimetric boundaries for the double Gaussian density f are always single points enclosing rays.*

Proof. To cover the three cases, apply Propositions 3.4, 3.14, and 3.15. □

4. Isoperimetric regions on the double Gaussian plane

This section describes evidence for the conjecture of Cañete et al., given here as Conjecture 4.1, which states that double Gaussian isoperimetric boundaries in the plane are vertical lines. Proposition 4.4 proves that horizontal and vertical lines are the only stationary lines. Proposition 4.5 proves that vertical lines are better than horizontal lines. First we prove some incidental symmetry results (Propositions 4.2 and 4.3).

Conjecture 4.1 [Cañete et al. 2010, Question 6]. *Let $f(x, y) = e^{\psi(x,y)}$ be the normalized sum of two Gaussian densities with the same variance and different centers. Isoperimetric regions are half-planes enclosed by lines perpendicular to the line connecting the two centers.*

By the planar analogue of Proposition 3.1, it suffices to prove this conjecture in the case where the centers are $c_1 = (1, 0)$ and $c_2 = (-1, 0)$.

Then we have

$$f(x, y) = e^{\psi(x,y)} = \frac{1}{4\pi a^2} e^{-y^2/(2a^2)} (e^{-(x-1)^2/(2a^2)} + e^{-(x+1)^2/(2a^2)}).$$

The next two propositions describe some symmetry properties of isoperimetric curves. For a curve C, let A_C denote the weighted area enclosed by C.

Proposition 4.2. *Consider a density g symmetric about the x-axis. If a closed, embedded curve C encloses the same weighted area above and below the x-axis, then there is a curve C' which is symmetric about the x-axis, encloses the same weighted area, and has weighted perimeter no greater than that of C.*

Proof. Let C_1 and C_2 be the parts of C in the open upper and lower half-planes chosen so that the weighted perimeter of C_1 is no bigger than that of C_2. Consider the curve C' formed by joining C_1 with its reflection over the x-axis and taking the closure. Let w denote the part of C on the x-axis and w_1 denote the part of C' on the x-axis. Since g is symmetric about the x-axis, $A_C = A_{C'}$. In addition,

$$|C'| - |C| = (2|C_1| + |w_1|) - (|C_1| + |C_2| + |w|) = (|C_1| - |C_2|) + (|w_1| - |w|).$$

We have $|C_1| - |C_2| \leq 0$ by assumption, and since the part of C which intersects the x-axis must include w_1, we know $|w_1| - |w| < 0$. Therefore $|C'| - |C| \leq 0$. □

Proposition 4.3. *Consider a density symmetric about the x-axis. If C is a closed embedded planar curve symmetric about the x-axis, then the part C' of C in the open upper half-plane encloses half as much weighted area with half the weighted length.*

Proof. Suppose that C is a curve that is symmetric about the x-axis and encloses area A. Since C is symmetric about the x-axis, C cannot have nonzero perimeter on the x-axis. Then C' encloses area $\frac{1}{2}A_C$ in the upper half-plane and has weighted perimeter $\frac{1}{2}|C|$. □

Proposition 4.4. *If the plane is endowed with density f, then horizontal and vertical lines have generalized curvature* 0 *and are the only lines which have constant generalized curvature.*

Proof. Let $\psi = \ln f$. Then

$$\nabla\psi(x, y) = \left(\frac{-x + \tanh(x/a^2)}{a^2}, \frac{-y}{a^2}\right).$$

In addition, the normal to the line $y = cx + b$ is $(-c, 1)/\sqrt{c^2+1}$ at all points of the line. Therefore the generalized curvature of such a line evaluated at $(0, b)$ is

$$0 - \nabla\psi(0, b) \cdot \frac{(-c, 1)}{\sqrt{c^2+1}} = \frac{b}{a^2\sqrt{1+c^2}},$$

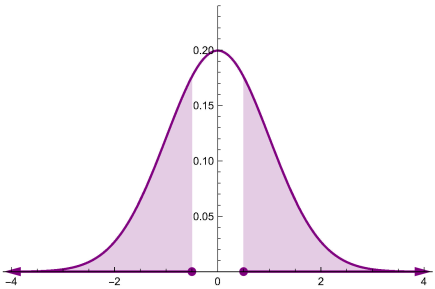
and by an analogous computation the generalized curvature evaluated at $(1, c+b)$ is

$$\frac{c+b}{a^2\sqrt{1+c^2}} + \frac{c(-1+\tanh(1/a^2))}{a^2\sqrt{1+c^2}}.$$

Thus the generalized curvatures at $(0, b)$ and $(1, c+b)$ are equal exactly when $c = 0$. This shows that only nonvertical lines that could possibly have constant curvature are the horizontal lines $y = b$. Such lines have normal $(0, 1)$, and this, combined with our formula with the gradient, shows that horizontal lines have constant curvature b/a^2.

An explicit computation of the same variety shows that the vertical line $x = b$ has constant curvature

$$\frac{b - \tanh(b/a^2)}{a^2}.$$ □



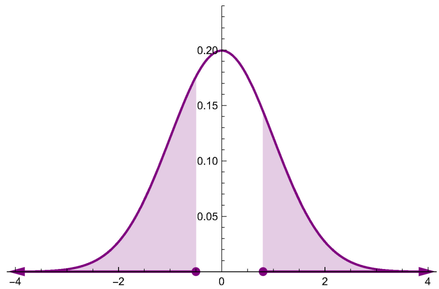


Figure 11. Left: symmetric rays. Right: nonsymmetric rays. When the purple areas are equal, the two nonsymmetric rays are more efficient than the two symmetric rays. The efficiency increases as the disparity between the rays increases, and the limiting case is a single ray, which is the isoperimetric region.

Proposition 4.5. *In the plane with double Gaussian density f, vertical lines enclose a given area with less perimeter than horizontal lines.*

Proof. We now compare the perimeters of and areas enclosed by the horizontal line $x = b$ and the vertical line $y = c$. By symmetry and the fact that we may assume the areas are less than $\frac{1}{2}$, we can assume that b and c are positive and consider the areas of the regions $x > b$ and $y > c$.

The area enclosed by the vertical line is

$$\int_b^\infty \int_{-\infty}^\infty f(x, y)\, dy\, dx = \int_b^\infty \frac{e^{-(x-1)^2/(2a^2)} + e^{-(x+1)^2/(2a^2)}}{2a\sqrt{2\pi}},$$

which is the same as the weighted length of the ray $R_b = [b, \infty)$ on the double Gaussian line. The perimeter of the vertical line is

$$\int_{-\infty}^\infty f(b, y)\, dy = \frac{e^{-(b-1)^2/(2a^2)} + e^{-(b+1)^2/(2a^2)}}{2a\sqrt{2\pi}},$$

which is exactly the cost of R_b on the double Gaussian line.

The area enclosed by the horizontal line is

$$\int_c^\infty \int_{-\infty}^\infty f(x, y)\, dx\, dy = \int_c^\infty \frac{e^{-y^2/(2a^2)}}{2a\sqrt{2\pi}}\, dy,$$

which is the same as the weighted length of the ray $R_c = [c, \infty)$ on the single Gaussian (of total weighted-length 1) line. The perimeter of the horizontal line is

$$\int_{-\infty}^\infty f(x, c)\, dx = \frac{e^{-c^2/(2a^2)}}{2a\sqrt{2\pi}},$$

which is exactly the cost of R_c on the single Gaussian line.

Therefore it suffices to show that a ray on the double Gaussian line of length A costs less than a ray on the single Gaussian line of the same weighted length. Consider the line with density g given by a single Gaussian of total length $\frac{1}{2}$. The ray on the single Gaussian is equivalent to the union of two disjoint, symmetric rays on the g-line. The ray on the double Gaussian is equivalent to the union of two disjoint, nonsymmetric rays on the g-line. By applying the first and second variation arguments to a single Gaussian density, we see that two nonsymmetric rays are always better than two symmetric rays of the same total weighted-length. See Figure 11. □

Therefore if the isoperimetric curve corresponding to area A is a line, then it is a vertical line.

Acknowledgements

This paper is the work of the Williams College NSF "SMALL" 2015 Geometry Group. We thank our advisor Professor Morgan for his support. We would like to thank the National Science Foundation, Williams College, the University of Chicago, and the MAA for supporting the "SMALL" REU and our travel to MathFest 2015.

References

[Cañete et al. 2010] A. Cañete, M. Miranda, Jr., and D. Vittone, "Some isoperimetric problems in planes with density", *J. Geom. Anal.* **20**:2 (2010), 243–290. MR Zbl

[Morgan 2009] F. Morgan, *Geometric measure theory: a beginner's guide*, 4th ed., Academic Press, 2009. MR Zbl

[Rosales et al. 2008] C. Rosales, A. Cañete, V. Bayle, and F. Morgan, "On the isoperimetric problem in Euclidean space with density", *Calc. Var. Partial Differential Equations* **31**:1 (2008), 27–46. MR Zbl

Received: 2016-08-01 Accepted: 2016-08-21

jtb1@williams.edu *Department of Mathematics and Statistics, Williams College, Williamstown, MA, United States*

mdannenberg@g.hmc.edu *Department of Mathematics, Harvey Mudd College, Claremont, CA, United States*

liangj@uchicago.edu *Department of Mathematics, University of Chicago, Chicago, IL, United States*

yzeng@smcm.edu *Department of Mathematics and Computer Science, St. Mary's College of Maryland, St. Mary's City, MD, United States*

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2018 is US $190/year for the electronic version, and $250/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2018 vol. 11 no. 4