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A classification of Klein links as torus links

Steven Beres, Vesta Coufal, Kaia Hlavacek, M. Kate Kearney,
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We classify Klein links. In particular, we calculate the number and types of components in a $K_{p,q}$ Klein link. We completely determine which Klein links are equivalent to a torus link, and which are not.

1. Introduction

When we began thinking about Klein knots, we were told that they were uninteresting since all Klein knots are torus knots. We decided to see if we could prove that statement using elementary methods, and to consider whether it was also true about Klein links. In our first paper [Alvarado et al. 2016], we presented our constructions and results leading up to our discovery of a class of Klein links that are not equivalent to any torus links.

In this paper, we show exactly which Klein links are torus links, and which are not. We begin in Section 2 with defining our notation for Klein links, which is based on the standard notation for torus links. In Section 3 we define two functions, the wrapping function and the hitting function, which help us to describe components of our links as they traverse a standard link diagram. We introduce several preliminary results in Section 4. We compute the number of components in a link $K_{p,q}$. Each of these components is itself a Klein knot, and we also describe the knot type of these components. Section 5 includes our main result, Theorem 12, which gives a complete classification of which Klein links are equivalent to torus links and which are *knot*.

Some of our results are identical or similar to results proved by another group using braids. However, our methods are different. Explicitly, our Lemma 2 is [Bush et al. 2014, Proposition 6.1], our Theorem 3 is [Bush et al. 2014, Proposition 6.2], and our Lemma 7 is [Catalano et al. 2010, Theorem 2].

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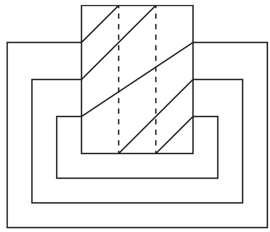


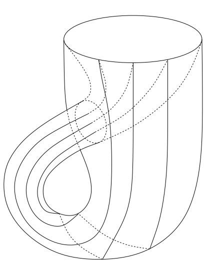
Figure 1. Planar diagram for the torus knot $T_{2,3}$.

2. Constructions

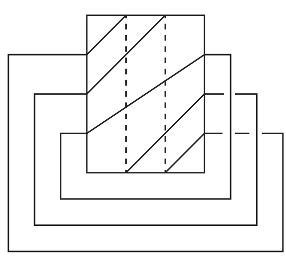
We begin with a brief description of the standard construction of torus links [Adams 1994; Murasugi 1996] and our analogous construction of Klein links. For nonnegative integers p and q, the torus link $T_{p,q}$ is the link on the torus which crosses the longitude p times and crosses the meridian q times, with no crossing on the torus itself. We illustrate the construction of $T_{2,3}$ on a planar diagram in Figure 1. The rectangle in the figure is a planar diagram for the torus, with the gluings (left side to right side, and top to bottom) understood.

We will construct Klein links in a similar way, being careful of certain issues. Klein bottles do not exist in three-dimensional space, and knots are trivial in four-dimensional space. To avoid this, we will work with punctured Klein bottles in three-dimensional space. The puncture occurs where the Klein bottle appears to (but does not) intersect itself. Warning: the notation of the knots and links we work with will be dependent on the relative position of the puncture. Mimicking the construction of $T_{p,q}$, the Klein knot $K_{2,3}$ is illustrated in Figure 2.

The corresponding planar diagram representation of $K_{2,3}$ is modeled after the torus version, except that we need to account for the Möbius-band twist and be mindful of the puncture. We deform the Klein bottle so that the twist produces a pattern of additional crossings as in Figure 2, with the puncture occurring in the lower left corner.



$K_{2,3}$ on a Klein bottle



planar diagram for $K_{2,3}$

Figure 2. Klein link $K_{2,3}$.

Note that $K_{p,0}$ is the p-component unlink.

We emphasize that the class of links that we are denoting by $K_{p,q}$ and the results in this paper are dependent on placing the puncture in the lower left corner. We do not consider Klein links with the puncture placed in different positions in this paper. Furthermore, deformations of our links are as links in space, not on the Klein bottle, and so the puncture does not affect deformations. For this reason, and since our puncture is always in the lower left corner, we do not include it in our illustrations.

It is worth noting that, while the diagrams are configured a bit differently, our $K_{p,q}$ Klein links are the same as the $K(p,q)$ Klein links found in [Bush et al. 2014; Freund and Smith-Polderman 2013; Shepherd et al. 2012]. Additionally, some of the same authors of the previously cited papers have done preliminary work in which they found explicit relationships between Klein links with different choices of puncture. There are certainly more questions to be answered in this regard.

3. The wrapping and hitting functions

We start with some definitions.

Definition 1. A *component* is a maximal connected subset of the link. A *horizontal node* is a position on the top of the planar diagram that a component passes through. A *vertical node* is a position on the left of the planar diagram that a component passes through. A *strand* is a subset of a component that passes exactly once horizontally through the planar diagram. Typically we denote the strand by the vertical node the strand passes through on the left side of the planar diagram.

The underlying keys to many of our results are our "wrapping" and "hitting" functions. Given a component entering the left side of the rectangle in the planar diagram construction of $K_{p,q}$ (see Figure 3), the wrapping function describes where that particular component reenters the left side of the rectangle. For $1 \le x \le q$, let x be the node in $K_{p,q}$ as in Figure 3. Then the wrapping function is given by

$$W_{p,q}(x) = 1 - x + p \pmod{q}.$$

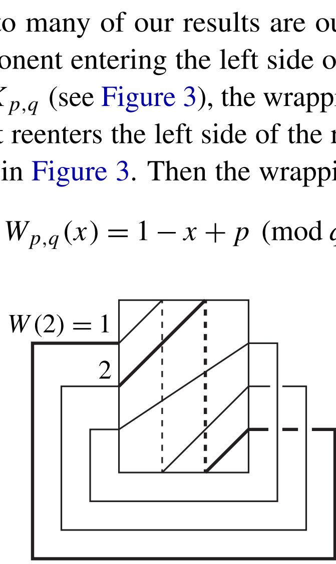


Figure 3. For $K_{2,3}$, $W(2) = 1$.

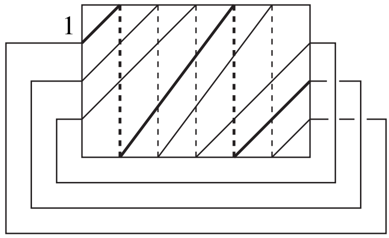


Figure 4. The hitting function $H_{5,3}(1) = 2$.

For an in-depth exploration of the wrapping function, as well as a proof of the next lemma, see [Alvarado et al. 2016].

Lemma 2. *For any $p, q \geq 0$, we have $W^2_{p,q}(x) = x$. Therefore, every component of $K_{p,q}$ wraps at most twice.*

While the wrapping function describes the horizontal movement of a strand, the hitting function addresses the vertical travel. Given a particular strand x starting at node x in a planar diagram of a $K_{p,q}$, we can determine how many times x hits the top of the planar diagram before reaching the right edge of the planar diagram. This is denoted by $H_{p,q}(x)$. Given p, q and x, where $1 \leq x \leq q$, we can use the following formula to find $H_{p,q}(x)$:

$$H_{p,q}(x) = \left\lfloor \frac{p-x}{q} \right\rfloor + 1, \tag{1}$$

where $\lfloor t \rfloor$ is the greatest integer function.

Note that the hitting function depends on the strand. To see how many vertical nodes a component passes through, we apply the hitting function to each strand in the component and add.

Applying the hitting function to the first strand of $K_{5,3}$ gives $H_{5,3}(1) = 2$, which is illustrated in Figure 4.

To see that the hitting function is defined correctly, notice that by construction a strand passes through the $(k+1)$ horizontal nodes $x,\ x+q,\ x+2q, \ldots, x+kq$, where $x+kq \leq p < x+(k+1)q$. So we have

$$\begin{aligned} x+kq \leq p &< x+(k+1)q, \\ kq \leq p-x &< (k+1)q, \\ k \leq \frac{p-x}{q} &< k+1. \end{aligned}$$

It follows that $k+1 = \lfloor (p-x)/q \rfloor + 1$. Thus the hitting function is correctly defined in (1).

4. Preliminary results

Our primary goal in this paper is to describe which Klein links are equivalent to torus links. In the interest of doing so, we will build up several results that break down the link $K_{p,q}$ into components and describe those components. Our first result gives the number of components in the link $K_{p,q}$.

Theorem 3 (number of components). *For a Klein link $K_{tq+n,q}$ with $q > 0$, $t \geq 0$ and $0 \leq n < q$:*

- *For q odd there are $\frac{1}{2}(q+1)$ components.*
- *For q even, n even there are $\frac{1}{2}q$ components.*
- *For q even, n odd there are $\frac{1}{2}q+1$ components.*

Moreover, in the case that $\frac{1}{2}(n+1)$ or $\frac{1}{2}(q+n+1)$ are integers, the strands at these nodes wrap only once. All other strands wrap twice.

Proof. It is enough the count the number of strands that wrap once. Then we divide the number of remaining vertical nodes by 2 to find how many components wrap twice, then add these two values.

To find the single-wrapping components, consider the equation $W_{tq+n,q}(x) = x$ (that is, the strand x wraps to itself). Then,

$$\begin{aligned} x = W_{tq+n,q}(x) &\equiv 1 - x + tq + n \pmod q, \\ x &\equiv 1 - x + n \pmod q, \\ 2x &\equiv n + 1 \pmod q. \end{aligned}$$

We will make use of this last modular equation in the following cases.

Case 1 (q *odd*): Since q is odd, 2 has a multiplicative inverse modulo q, which is $\frac{1}{2}(q+1)$. Solving the modular equation above, we have

$$\begin{aligned} 2x &\equiv n + 1 \pmod q, \\ x &\equiv \tfrac{1}{2}(q+1)(n+1) \pmod q. \end{aligned}$$

Thus $x = \frac{1}{2}(q+1)(n+1) + kq$ for some integer k. Since $1 \leq x \leq q$, we have $1 \leq \frac{1}{2}(q+1)(n+1) + kq \leq q$. The length of this interval is $q-1$; thus there can be at most one k-value solution. Since q is odd, there is at least one strand that wraps only once, which means there is at least one k-value solution. It follows that there is exactly one k-value solution, and thus exactly one component that wraps once and $\frac{1}{2}(q-1)$ components that wrap twice. Therefore, we have $\frac{1}{2}(q-1) + 1 = \frac{1}{2}(q+1)$ components.

Case 2 (q *even, n even*): In this case, rewriting the modular equation we get that $2x = n + 1 + kq$ for some integer k. We have that $2x$ and kq are even integers, and

$n+1$ is an odd integer. Then the equation $2x = n+1+kq$ has no solutions, and thus every component wraps twice. Therefore, there are $\frac{1}{2}q$ components.

Case 3 (*q even, n odd*): In this case, we again have $2x = n+1+kq$. Solving the equation for x gives $x = \frac{1}{2}(n+1) + \frac{1}{2}kq$. Recall that $1 \le x \le q$ and $0 \le n < q$. Thus, we have two solutions: one when $k = 0$ and the other when $k = 1$. Thus there are two components that wrap once and $\frac{1}{2}(q-2)$ components that wrap twice. Therefore, there are $\frac{1}{2}(q-2)+2 = \frac{1}{2}q+1$ components. $\square$

Now that we have determined the number of components in a $K_{tq+n,q}$, we would like to know how many times each of these components wraps around the meridian and longitude of the Klein bottle, as well as their knot type. We will denote by $L = a \cdot P \cup b \cdot Q$ a link which is composed of a copies of a knot (or link) P, and b copies of knot (or link) Q. The copies of P and Q may be linked.

Theorem 4 (types of components). *Consider $K_{tq+n,q}$ with $q > 0$, $t \ge 0$ and $0 \le n < q$. Then*:

(1) *If q even and n odd, then*

$$K_{tq+n,q} \equiv \tfrac{1}{2}(n-1) \cdot K_{2t+2,2} \cup \tfrac{1}{2}(q-n-1) \cdot K_{2t,2} \cup K_{t+1,1} \cup K_{t,1}.$$

(2) *If q, n odd, then*

$$K_{tq+n,q} \equiv \tfrac{1}{2}(n-1) \cdot K_{2t+2,2} \cup \tfrac{1}{2}(q-n) \cdot K_{2t,2} \cup K_{t+1,1}.$$

(3) *If q odd and n even, then*

$$K_{tq+n,q} \equiv \tfrac{1}{2}n \cdot K_{2t+2,2} \cup \tfrac{1}{2}(q-n-1) \cdot K_{2t,2} \cup K_{t,1}.$$

(4) *If q, n even, then*

$$K_{tq+n,q} \equiv \tfrac{1}{2}n \cdot K_{2t+2,2} \cup \tfrac{1}{2}(q-n) \cdot K_{2t,2}.$$

Proof. According to Theorem 3, the only components that wrap once are the components through $x_1^* = \frac{1}{2}(n+1)$ and $x_2^* = \frac{1}{2}(q+n+1)$ when these values are integers (one or both), and all other components wrap twice.

It is advantageous to inspect the wrapping function $W(x)$ for a number of specific values:

$$\begin{aligned} W(1) &= n, & W(n+1) &= q, \\ W(2) &= n-1, & W(n+2) &= q-1, \\ W(3) &= n-2, & W(n+3) &= q-2. \end{aligned}$$

In general, we have

$$\begin{aligned} W(x) &= n+1-x && \text{for } x < x_1^*, \\ W(x) &= q+n+1-x && \text{for } n < x < x_2^*. \end{aligned}$$

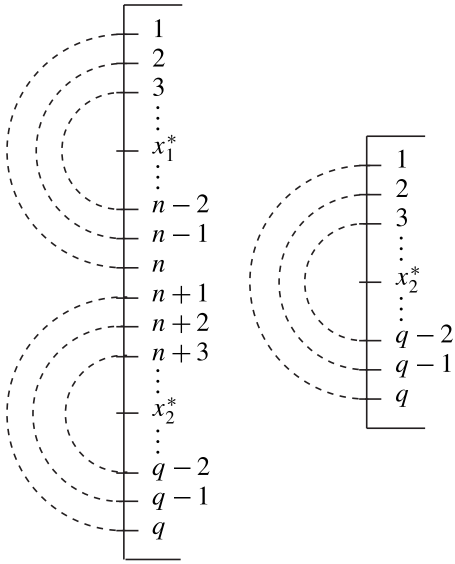


Figure 5. The wrapping of $K_{tq+n,q}$ (on the left), and the wrapping of $K_{tq,q}$ (on the right).

We see that there are now two symmetry points, $x_1^* = \frac{1}{2}(n+1)$ and $x_2^* = \frac{1}{2}(q+n+1)$, regardless of whether these are integers or not, and the wrapping of $K_{tq+n,q}$ can be pictured as in the left side of Figure 5.

If $n = 0$, however, we see that $x_1^* = \frac{1}{2}$ and there are no nodes $x < x_1^*$. In that case, we have only one symmetry point as in the right side of Figure 5.

Next, recalling that $0 \le n < q$ and $1 \le x \le q$, we simplify the hitting function as follows:

$$H(x) = \left\lfloor \frac{tq+n-x}{q} \right\rfloor + 1 = \left\lfloor \frac{n-x}{q} \right\rfloor + t + 1 = \begin{cases} t+1 & \text{if } x \le n, \\ t & \text{if } x > n. \end{cases}$$

Notice that the components symmetric about (but not on) x_1^* wrap twice and hit $t+1$ times on each wrap, so they are all of the form $K_{2(t+1),2} = K_{2t+2,2}$. When n is odd, there is a component passing through x_1^* and it wraps once and hits $t+1$ times, making it a $K_{t+1,1}$. Components symmetric about (but not on) x_2^* wrap twice and hit t times on each wrap, so they are all of the form $K_{2t,2}$. When $q+n$ is odd, there is a component passing through x_2^* and it wraps once and hits t times, making it a $K_{t,1}$.

All that is left is to count the number of components of each type, depending on the parity of q and n, using Theorem 3. For example, if q is even and $n > 0$ is even, then there are a total of $\frac{1}{2}q$ components, with $\frac{1}{2}n$ of them symmetric about x_1^* and $\frac{1}{2}q - \frac{1}{2}n = \frac{1}{2}(q-n)$ of them about x_2^*. Thus, in this case, $K_{tq+n,q} \equiv \frac{1}{2}n \cdot K_{2t+2,2} \cup \frac{1}{2}(q-n) \cdot K_{2t,2}$. We leave it to the reader to finish counting for the remaining three cases. □

We now have a complete characterization of the types of components for any Klein link. To establish an equivalence to a torus link, we need to establish an equivalence of the components. We present a collection of lemmas about the components of torus and Klein links that we will use to prove the classification theorem.

In the next lemma, and many of the subsequent results, we make use of the linking number of a pair of components in a link.

Definition 5. To define the *linking number* of two components C_1 and C_2 of a link, we first orient the link (choose a direction of travel for each component). Next, assign $+1$ to a crossing between if the undergoing strand goes from the right side to the left side of the overgoing strand (right-handed crossing). If the undergoing strand moves from left to right (left-handed crossing) it is assigned a -1. Considering all crossings involving a strand from C_1 and a strand from C_2, add all of the signed crossing numbers (the $+1$s and -1s), take the absolute value of this sum, and divide by two. The resulting value is called the linking number of the two components, and is denoted by $\mathrm{lk}(C_1, C_2)$.

Lemma 6. *All components of a torus link have the same knot type. Additionally, every pair of components in a torus link have the same linking number.*

Proof. As discussed in Section 2, the torus link $T_{p,q}$ is given by identifying the top and bottom, and left and right sides of the square together with the knot that hits the top p times and the side q times, that is, the line with slope p/q, and appropriate translation; see [Flapan 2016]. We can identify components by examining each strand along the left-hand side, just as we have for Klein links. In contrast to the picture with Klein links, we can make the observation here that a vertical translation by $1/q$ produces the same link, but with the ordering of strands (and hence components) shifted by 1. Since each component is a translation of the others, all components must have the same knot type.

Next we consider the linking of pairs of components. As we saw above, each component is a translation of the others. Furthermore, considering the strands along the left-hand side, if we have n components they must be represented by the first n strands from the top. If we translate a strand vertically by a/q and find that we have reached another strand of the same component, then every translation by $(c * a)/q$ will also return the same component. Hence, to have n components, we must find our first repeated component in the translation by n/q (so the first strand shifts to the $(n+1)$-st strand), and so the first n strands each represent different components. Now, we see that if we consider components x_i, x_j, and x_k, we know that x_k is a translation of x_j, and in particular it is a translation by less than n/q, and hence does not cross any strand of x_i in the process of translating. This is enough to guarantee that the linking number of x_i with x_j is equal to the linking number of x_i with x_k. Finally, we see that any pair of components in the torus link have the same linking number. □

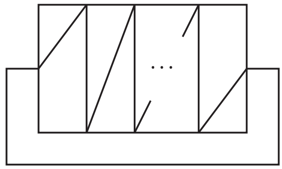


Figure 6. $K_{p,1}$ is an unknot.

The proof of Lemma 7 follows directly from the construction; see Figure 6.

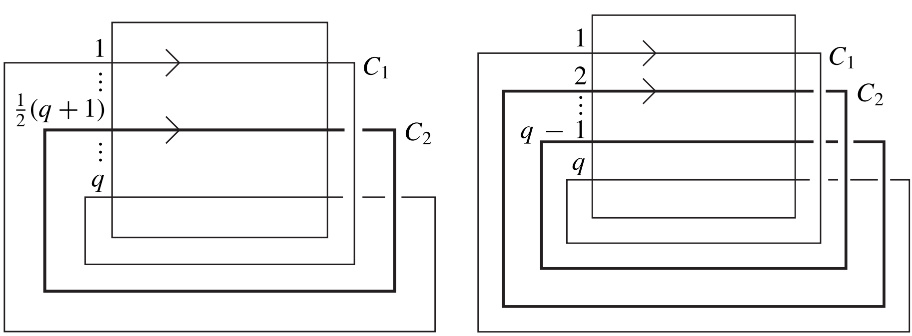
Lemma 7. *For all p, $K_{p,1}$ is an unknot.*

The next two lemmas address the linking numbers of certain components of $K_{p,q}$ in special cases.

Lemma 8. *If $q \geq 3$ is odd, then $K_{0,q}$ contains a pair of components with linking number* 1. *If $q \geq 4$ (even or odd), then $K_{0,q}$ contains a pair of components with linking number* 2.

Proof. First note that $K_{0,q}$ has crossings only outside of the rectangle, and all crossings are of the same type (with all strands oriented to point into the right-hand side of the rectangle, and all crossings are right-hand crossings).

For $q \geq 3$ and odd, let C_1, C_2 be the components passing through nodes 1 and $\frac{1}{2}(q+1)$, respectively. Using the wrapping function, we have that $W(1) = 1-1+0 \equiv q \pmod{q}$ and $W\big(\frac{1}{2}(q+1)\big) = 1 - \frac{1}{2}(q+1) + 0 \equiv q + \frac{1}{2}(1-q) \equiv \frac{1}{2}(q+1) \pmod{q}$. Thus, component C_1 passes through nodes 1 and q, wrapping twice, and C_2 passes through node $\frac{1}{2}(q+1)$ and wraps only once. See Figure 7(a). Since C_1 wraps twice, while C_2 wraps only once, they cross each other exactly twice. Hence C_1 and C_2 have exactly two crossings, both outside of the rectangle, and the linking number is $\mathrm{lk}(C_1, C_2) = \frac{2}{2} = 1$.



(a) $K_{0,q}$ with q odd (b) $K_{0,q}$ with q even

Figure 7. $K_{0,q}$.

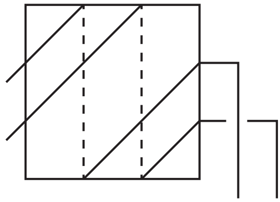


Figure 8. Two components of $K_{n,n}$ on a single wrap.

For $q \geq 4$, let C_1 again be the component passing through nodes 1 and q. Using the wrapping function, we denote by C_2 the component that passes through 2 and $q-1$. See Figure 7(b). In particular, they both wrap twice. It follows that they cross each other exactly four times, and the linking number is $\mathrm{lk}(C_1, C_2) = \frac{4}{2} = 2$. □

Lemma 9. *For $n \geq 3$, $K_{n,n}$ has a pair of components with nonzero linking number.*

Proof. Considering the planar diagram, all crossings inside of the rectangle are left-hand crossings, with our choice of orientation, and every crossing outside of the rectangle is right-handed. Let C_1, C_2 be the components passing through nodes 1 and 2, respectively. We will calculate the linking number for the pair C_1, C_2 by counting the number of crossings inside of the rectangle and the number of crossings outside.

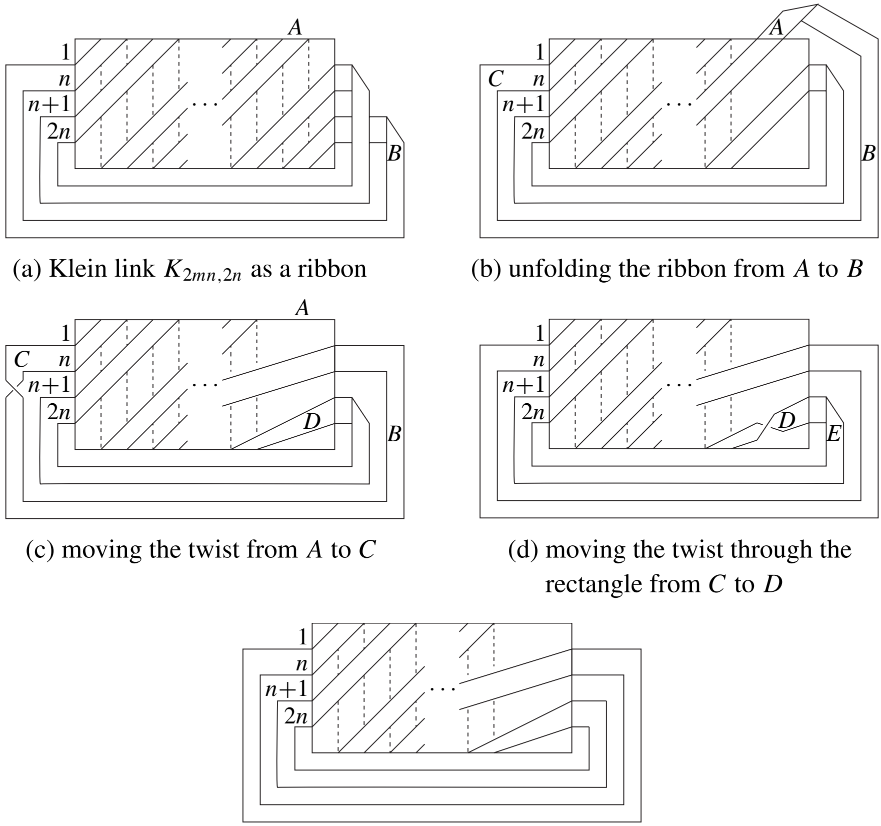
First, we have that $W(1) = 1 - 1 + n = n \neq 1 \pmod n$ and so C_1 wraps twice for $n \geq 3$. If $n = 3$, then $W(2) = 1 - 2 + 3 = 2 \pmod 3$, and thus C_2 wraps once. If $n \geq 4$, then $W(2) = n - 1 \neq 2 \pmod n$ and so C_2 wraps twice. For all $1 \leq x \leq n$, $H(x) = \lfloor (n-x)/n \rfloor + 1 = 1$. Thus each component hits the top of the rectangle exactly once each time it wraps. It follows that, on each wrap, the two components cross twice in the rectangle and once outside of the rectangle, as shown in Figure 8.

For $n = 3$, C_1 wraps twice and C_2 wraps once, so they cross a total of $2(2) = 4$ times inside the rectangle and $2(1) = 2$ times outside the rectangle. The linking number is $|(2-4)/2| = 1$. For $n \geq 3$, both C_1 and C_2 wrap twice, so they cross a total of $4(2) = 8$ times inside the rectangle and $2(2) = 4$ times outside the rectangle, giving a linking number of $\left|\frac{1}{2}(4-8)\right| = 2$. In both cases, the pair has nonzero linking number. □

The next lemma is a generalization of [Alvarado et al. 2016, Theorem 6] and was proved by one of the authors of that paper, Enrique Alvarado.

Lemma 10. *For all m and n, we have $K_{2mn,2n} \equiv T_{2mn-n,2n}$.*

Proof. A $K_{2mn,2n}$ has $2n$ strands entering or leaving each side of the rectangle in the planar diagram. We collect together the first n strands (strands 1 through n) to form a single ribbon. Notice that since $W(1) \equiv 2n \pmod{2n}$ and $W(n) \equiv n+1 \pmod{2n}$, the ribbon exits the right side and wraps around to reenter the left side through



(a) Klein link $K_{2mn,2n}$ as a ribbon

(b) unfolding the ribbon from A to B

(c) moving the twist from A to C

(d) moving the twist through the rectangle from C to D

(e) canceling the twists at D and E

Figure 9. Manipulating the ribbon form of $K_{2mn,2n}$ into $T_{2mn-n,2n}$.

strands $n+1$ through $2n$. Thus the entire link $K_{2mn,2n}$ consists of just one ribbon that wraps twice from left to right, as in Figure 9(a).

The transformation to $T_{2mn-n,2n}$ is illustrated in Figure 9. First, unfold the ribbon between the points labeled A and B, as in Figure 9(b), then move the remaining twist at A through B to C, as in Figure 9(c). We also slide the ribbon at point A down from the top of the rectangle to the right side, leaving $2mn-n$ strands through the top and $2n$ strands through the right side of the rectangle. Next, move the twist through the rectangle to point D. To do this, we are doing a series of moves as shown in Figure 10.

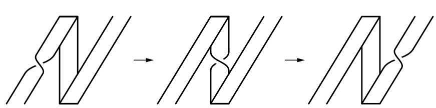


Figure 10. Moving the twist through the rectangle.

We end up with a twist at D, as in Figure 9(d). The twist at D cancels the twist at E, resulting in the ribbon form of the torus link $T_{2mn-n,2n}$, as in Figure 9(e). □

Lemma 11. *For $t \geq 2$, we have $K_{2t+2,2} \not\equiv K_{2t,2}$, and neither are unknots.*

Proof. By Lemma 10, $K_{2t+2,2} \equiv T_{2t+1,2}$ and $K_{2t,2} \equiv T_{2t-1,2}$. The torus links are nontrivial and not equivalent since they have different determinants [Livingston 1993]. Hence, $K_{2t+2,2} \equiv T_{2t+1,2} \not\equiv T_{2t-1,2} \equiv K_{2t,2}$. □

5. The classification theorem

Having built our preliminary results, we are now ready to state and prove our main result, which describes exactly which Klein links are equivalent to torus links, and which are not. Without further ado…

Theorem 12 (the classification theorem). *Let $p = tq + n$ with $t \geq 0$ and $0 \leq n < q$. All Klein links $K_{p,q}$ which are equivalent to torus links are listed in the following table*:

q	0, 1, 2	3	4	*even*	*odd*
p	$0 \leq p$	$0 \leq p \leq 4$	2	$p = tq$	$p = q + 1$

All other Klein links have no torus equivalent.

We present an immediate (but important) corollary before the proof of Theorem 12.

Corollary 13. *Every Klein knot is equivalent to some torus knot.*

Proof. A Klein knot is a Klein link with one component. By Theorem 3, the only possible q values for a Klein knot are 1 and 2. Thus, the only Klein knots are of the forms $K_{p,1}$ and $K_{p,2}$. By Theorem 12, all such knots have a torus equivalent. □

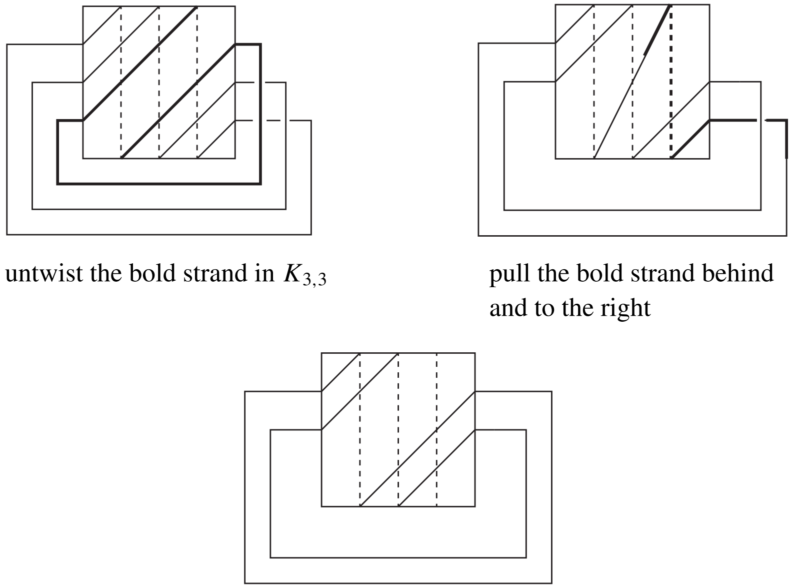
We emphasize that the corollary is a result about knots, not links. It is well-known and can be found in [Alvarado et al. 2016; Catalano et al. 2010; Freund and Smith-Polderman 2013].

Proof of Theorem 12. We first show that the Klein links listed in the table are, indeed, equivalent to some torus link.

Case 1 ($q = 0$): For each $p \geq 0$, by the way we construct Klein links, $K_{p,0}$ is a p-component unlink, hence equivalent to a torus link.

Case 2 ($q = 1$): For each $p \geq 0$, $K_{p,1}$ is an unknot by Lemma 7, hence equivalent to a torus link.

Case 3 ($q = 2$): One can see that $K_{0,2}$ is an unknot, and by [Alvarado et al. 2016, Theorem 5], for each $p \geq 1$, we know $K_{p,2} \equiv T_{p-1,2}$.



untwist the bold strand in $K_{3,3}$

pull the bold strand behind and to the right

$T_{2,2}$

Figure 11. $K_{3,3} \equiv T_{2,2}$.

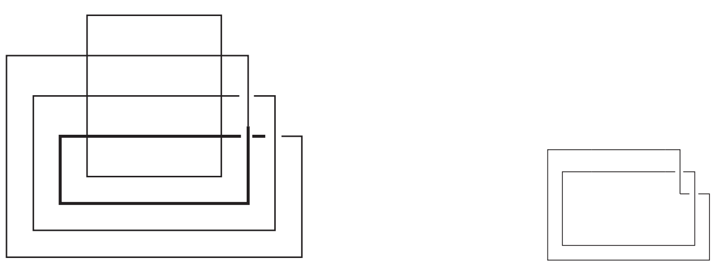
Case 4 ($q = 3$): Both $K_{1,3}$ and $K_{2,3}$ are 2-component unlinks, and thus are torus links. For $p = 3$, we can see in Figure 11 that $K_{3,3}$ is equivalent to $T_{2,2}$, which is a Hopf link.

With a little untwisting as shown in Figure 12 we see that $K_{0,3}$ is also equivalent to a Hopf link, and hence $T_{2,2}$.

Case 5 ($q = 4,\ p = 2$): By inspection, $K_{2,4}$ is a 2-component unlink.

Case 6 (q *even*, $p = tq,\ t \neq 0$): By Lemma 10, $K_{tq,q} \equiv T_{tq-q/2,q}$.

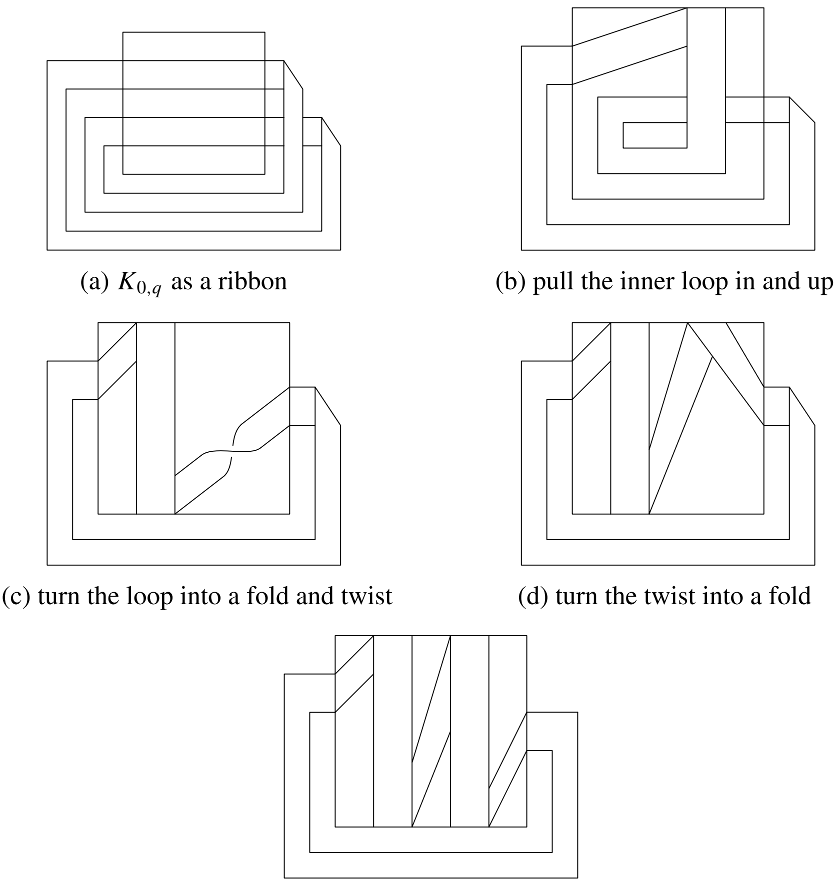
Case 7 ($t = n = 0,\ q \geq 4$ *and even*): Similar to the proof of Lemma 10, we collect together the strands through the first $\frac{1}{2}q$ nodes to form a ribbon. Using the wrapping



untwist the bold strand in $K_{0,3}$

Hopf link

Figure 12. $K_{0,3}$ is equivalent to a Hopf link.



(a) $K_{0,q}$ as a ribbon

(b) pull the inner loop in and up

(c) turn the loop into a fold and twist

(d) turn the twist into a fold

(e) reposition the right fold

Figure 13. $K_{0,q}$ with $q \geq 4$ even is a torus link.

function, these $\frac{1}{2}q$ strands wrap to the strands through nodes $\frac{1}{2}q+1$ through q so that we have a single ribbon that wraps twice. See Figure 13(a). Manipulate the ribbon as in Figure 13(b)–(e). Notice that the resulting ribbon in Figure 13(e) represents a torus link, though with the vertical wrapping opposite to the way we usually wrap.

Case 8 (*q odd, $p=q+1$*): Using [Alvarado et al. 2016, Theorem 4] and Lemma 10, we have $K_{q+1,q} \equiv K_{q+1,q+1} \equiv T_{q+1,(q+1)/2}$.

Our next step is to show that all other Klein links, those not listed in the table in Theorem 12, have no torus equivalence.

Case 9 (*$t=n=0$, $q \geq 5$ and odd*): By Lemma 8, $K_{0,q}$ with $q \geq 5$ odd has pairs of components with different linking numbers, one pair with linking number 1 and another pair with linking number 2. Thus it cannot be equivalent to a torus link by Lemma 6.

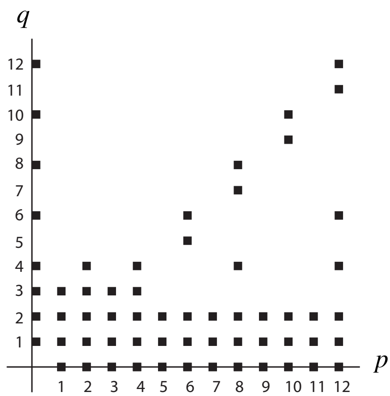


Figure 14. Graph showing which $K_{p,q}$ are torus links.

Case 10 ($t = 0$ *and either* $n = 1,\ q \geq 4$*; or* $n = 2,\ q \geq 5$*; or* $n \geq 3$): Since $t = 0$ and $n < q$, we can use [Alvarado et al. 2016, Theorem 3] to write $K_{p,q} = K_{n,q} \equiv K_{n,n} \cup K_{0,q-n}$, where $K_{n,n}$ and $K_{0,q-n}$ are unlinked. It follows that $K_{n,q}$ must have two components, one from $K_{n,n}$ and one from $K_{0,q-n}$, whose linking number is zero. Now, if $n = 1,\ q \geq 4$ or $n = 2,\ q \geq 5$, then $K_{0,q-n}$ has components with nonzero linking number by Lemma 8. On the other hand, if $n \geq 3$, then $K_{n,n}$ has components with nonzero linking number by Lemma 9. In both cases, $K_{n,q}$ must have components with nonzero linking number. Since it also has components with linking number zero, $K_{n,q}$ cannot be equivalent to a torus link by Lemma 6.

Case 11 ($t = 1$ *and either* $n = 0,\ q \geq 5$ *and odd; or* $n = 1,\ q \geq 4$ *and even*): We are looking at either $K_{q,q}$ with $q \geq 5$ and odd, or $K_{q+1,q} \equiv K_{q+1,q+1}$ with $q + 1 \geq 5$ and odd by [Alvarado et al. 2016, Theorem 4]. Thus, by [Alvarado et al. 2016, Theorem 7], neither can be equivalent to a torus link.

Case 12 (*either* $t = 1,\ n \geq 2$*; or* $t \geq 2,\ n = 0,\ q \geq 3$ *and odd; or* $t \geq 2,\ n = 1,\ q \geq 3$*; or* $t \geq 2,\ n \geq 2$*, n and q not both even; or* $t \geq 2,\ n \geq 2$*, n and q both even*): In each of these cases, by Theorem 4, $K_{p,q}$ contains either:

(1) $K_{2t+2,2}$ and at least one of $K_{t+1,1}$ or $K_{t,1}$ (with $t \geq 1$), or

(2) $K_{2t,2}$ and at least one of $K_{t+1,1}$ or $K_{t,1}$ (with $t \geq 2$), or

(3) $K_{2t+2,2}$ and $K_{2t,2}$ (with $t \geq 2$).

For each situation, $K_{p,q}$ contains components that are nonequivalent knots by Lemmas 7 and 11. Thus, $K_{p,q}$ has no torus equivalence by Lemma 6.

We leave it to the reader to check that all possible cases have been addressed. Figure 14, showing which Klein links have a torus equivalence, might help. □

Recall that every Klein knot is equivalent to a torus knot. From the sparseness of the graph in Figure 14, it is interesting to note that relatively few Klein links are equivalent to torus links. Thus, they warrant further study. For example, we plan to finish calculating the linking numbers for all Klein links (some further work has been done in [Bush et al. 2014]). Other link invariants could also be calculated. As noted in our construction, our Klein links are dependent on the relative position of the puncture on the Klein bottle. We need to investigate the effects on our results if we choose a different position for the puncture. On a more ambitious scale, we would like to determine a complete classification of Klein links, not just in terms of their relation to torus links.

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sberes@zagmail.gonzaga.edu *Gonzaga University, Spokane, WA, United States*

coufal@gonzaga.edu *Department of Mathematics, Gonzaga University, Spokane, WA, United States*

khlavacek@zagmail.gonzaga.edu *Department of Mathematics, Gonzaga University, Spokane, WA, United States*

kearney@gonzaga.edu *Department of Mathematics, Gonzaga University, Spokane, WA, United States*

rlattanzi@zagmail.gonzaga.edu *Gonzaga University, Spokane, WA, United States*

holson3@zagmail.gonzaga.edu *Gonzaga University, Spokane, WA, United States*

jp3465@drexel.edu *Department of Mathematics, Drexel University, Philadelphia, PA, United States*

bstrub@zagmail.gonzaga.edu *Gonzaga University, Spokane, WA, United States*

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