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Interpolation on Gauss hypergeometric functions with an application

Hina Manoj Arora and Swadesh Kumar Sahoo

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We use some standard numerical techniques to approximate the hypergeometric function

$$ {}_2F_1[a,b;c;x] = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots $$

for a range of parameter triples (a,b,c) on the interval $0 < x < 1$. Some of the familiar hypergeometric functional identities and asymptotic behavior of the hypergeometric function at $x = 1$ play crucial roles in deriving the formula for such approximations. We also focus on error analysis of the numerical approximations leading to monotone properties of quotients of gamma functions in parameter triples (a,b,c). Finally, an application to continued fractions of Gauss is discussed followed by concluding remarks consisting of recent works on related problems.

1. Introduction and preliminaries

For a complex number z and $c \neq 0, -1, -2, -3, \ldots$, the *hypergeometric series* is defined by

$$ 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n. $$

Here $(a)_n$ denotes the shifted factorial notation defined, in terms of the gamma function, by

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\cdots(a+n-1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0,\ a \neq 0. \end{cases} $$

Note that the hypergeometric series defines an analytic function, denoted by the symbol ${}_2F_1[a,b;c;z]$, in $|z| < 1$. As quoted in the historical remarks in [Anderson et al. 1997, 1.55, p. 24], the concept of hypergeometric series was first introduced

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by J. Wallis in 1656 to refer to a generalization of the geometric series. Less than a century later, Euler extensively studied the analytic properties of the hypergeometric function and found, for instance, its integral representation; see [Anderson et al. 1997, Theorem 1.19(2)]. Gauss made his first contribution to the subject in 1812. Due to the outstanding contribution made by Gauss to the field, the hypergeometric function is also sometimes known as the *Gauss hypergeometric function*. Most elementary functions which are solutions to certain differential equations can be written in terms of the Gauss hypergeometric functions. One can easily verify by using the Frobenius technique that the function ${}_2F_1[a, b; c; z]$ is one of the solutions of the *hypergeometric differential equation* [Andrews et al. 1999; Beals and Wong 2010; Rainville 1960]

$$z(1-z)w'' + (c-(a+b+1)z)w' - abw = 0.$$

We refer to [Rainville 1943; 1960] for Kummer's 24 solutions to the hypergeometric differential equation, and to [Beals and Wong 2010] for related applications. The asymptotic behavior of ${}_2F_1[a, b; c; z]$ near $z = 1$ reveals that

$${}_2F_1[a, b; c; 1] = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} < \infty, \quad \text{valid for } \operatorname{Re}(c-a-b) > 0. \tag{1-1}$$

Interpolating polynomials for elementary real functions such as trigonometric functions, logarithmic functions, exponential functions, etc. have already been derived in undergraduate texts in numerical analysis; see for instance [Atkinson 1978]. These elementary functions are in fact hypergeometric functions with specific parameters a, b, c; see for instance [Andrews et al. 1999; Rainville 1960]. Most of such polynomial approximations are computed when the functional values at the given boundary points are possible. Hence the asymptotic behavior (1-1) of the hypergeometric function near $z = 1$ motivates us to construct interpolating polynomials for real hypergeometric functions ${}_2F_1[a, b; c; x]$, $a, b, c \in \mathbb{R}$, $c \notin \{0, -1, -2, -3, \dots\}$, of a real variable x using several numerical techniques in the interval $[0, 1]$; however, the interval may be extended to $[-1, 1]$ as the hypergeometric series in x is convergent for $|x| < 1$ and it has a certain asymptotic behavior near -1 as well, with suitable choices of the parameters a, b, c; see for instance [Rainville 1960, Theorem 26]. More precisely, when we compute an interpolating polynomial $p_n(x)$ of a hypergeometric function ${}_2F_1[a, b; c; x]$ on $[0, 1]$ we take the value ${}_2F_1[a, b; c; 1]$ in the sense that the hypergeometric function defined at $x = 1$ by means of its asymptotic behavior at $x = 1$; see (1-1). Several hypergeometric functional identities also play a crucial role in determining functional values at the interpolating points.

The following lemmas are useful in describing the error analysis for the interpolating polynomials that we obtained in this paper. Our subsequent paper(s) in this series will cover the study of interpolating polynomials using other techniques.

Lemma 1.1 [Anderson et al. 1997, Lemma 1.33(1), p. 13; see also Lemma 1.35(2)]. *If* $a, b, c \in (0, \infty)$, *then* ${}_2F_1[a, b; c; x]$ *is strictly increasing on* $[0, 1)$. *In particular, if* $c > a + b$ *then for* $x \in [0, 1]$ *we have*

$$ {}_2F_1[a, b; c; x] \le \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

Lemma 1.2 [Anderson et al. 1997, Lemma 2.16(2), p. 36]. *The gamma function* $\Gamma(x)$ *is a log-convex function on* $(0, \infty)$. *In other words, the logarithmic derivative,* $\Gamma'(x)/\Gamma(x)$, *of the gamma function is increasing on* $(0, \infty)$.

Note that in all the plots in this paper, graphs drawn in blue represent the original functions and graphs drawn in red represent interpolating polynomials.

2. Linear interpolation on ${}_2F_1[a, b; c; x]$

For performing linear interpolation of the function ${}_2F_1[a, b; c; x] = f(x)$, we consider the end points $x_0 = 0$ and $x_1 = 1$ of the interval [0, 1]. The functional values at these points are respectively $f(0) = 1$ and $f(1)$, described in (1-1). Hence, the equation of the segment of the straight line joining 0 and 1 is

$$ P_l(x) = f(x_0) + \frac{x - x_0}{x_1 - x_0}(f(x_1) - f(x_0)) = \frac{\Gamma(c)\Gamma(c-a-b) - \Gamma(c-a)\Gamma(c-b)}{\Gamma(c-a)\Gamma(c-b)} x + 1, $$

when $c - a - b > 0$ and $c \neq 0, -1, -2, -3, \ldots$. The polynomial $P_l(x)$ represents the linear interpolation of ${}_2F_1[a, b; c; x]$ interpolating at 0 and 1.

Using Lemma 1.1, we obtain the following error estimate:

Lemma 2.1. *Let* $a, b, c \in (-2, \infty)$ *with* $c - a - b > 2$. *The deviation of the given function* $f(x) = {}_2F_1[a, b; c; x]$ *from the approximating function* $P_l(x)$ *for all values of* $x \in [0, 1]$ *is estimated by*

$$ |E_l(f, x)| = |f(x) - P_l(x)| \le \frac{|a(a+1)b(b+1)|}{8} \frac{\Gamma(c)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}. $$

Proof. Maximizing

$$ |E_l(f, x)| = \tfrac{1}{2} x(1-x)|f''(x)| $$

in [0, 1] yields

$$ \tfrac{1}{8}(1-0)^2 \max_{0 \le x \le 1} |f''(x)|, $$

where $f(x) = {}_2F_1[a, b; c; x]$. The following well-known derivative formula is useful:

$$ \frac{d}{dx}\, {}_2F_1[a, b; c; x] = \frac{ab}{c}\, {}_2F_1[a+1, b+1; c+1; x]. \tag{2-1} $$

The proof follows from (1-1), Lemma 1.1, (2-1), and the fact that

$$ \Gamma(x+1) = x\Gamma(x). $$

□

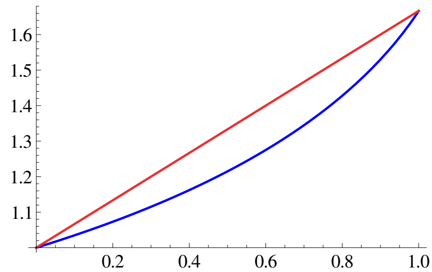

Figure 1. Linear interpolation of ${}_2F_1[1, 2; 6; x]$ at 0 and 1.

nodes x_i	0	0.25	0.5	0.75	1
actual values ${}_2F_1[1,2;6;x_i]$	1	1.0936	1.2149	1.3843	1.6667
polynomial approx. by $P_l(x_i)$	1	1.1667	1.3333	1.5000	1.6667
validity of error bounds by $E_l(f, x_i)$	0	$0.0731 < 1.25$	$0.1184 < 1.25$	$0.1157 < 1.25$	0

Table 1. Comparison of the functional and linear polynomial values.

Remark 2.2. It follows from Lemma 2.1 that there is no error for the choices $a = 0$, $a = -1$, $b = 0$, $b = -1$. In other words, for these choices $E_l(f, x)$ vanishes.

Figure 1 shows linear interpolation of the hypergeometric function at 0 and 1, whereas Table 1 compares the values of the hypergeometric function up to four decimal places with its interpolating polynomial values in the interval [0, 1] for the choice of parameters $a = 1$, $b = 2$ and $c = 6$. Figure 1 and Table 1 also indicate errors at various points within the unit interval except at the end points.

3. Quadratic interpolation on ${}_2F_1[a, b; c; x]$

Let the three points in consideration for quadratic interpolation be $x_0 = 0$, $x_1 = 0.5$ and $x_2 = 1$. The functional values at $x_0 = 0$ and $x_2 = 1$ can be found easily in terms of the parameters but the functional value at $x_1 = 0.5$ can be obtained through different identities involving hypergeometric functions ${}_2F_1[a, b; c; x]$ dealing with various constraints on the parameters a, b, c. This section consists of two subsections and in each subsection the method to obtain the functional value of ${}_2F_1[a, b; c; x]$ at $x = 0.5$ uses three different identities. Finally, we compare the resultant interpolations. In fact we observe that the interpolating polynomial remains unchanged in two cases, although the approaches are different (see the subsection on page 630 for more details).

Quadratic interpolation on ${}_2F_1[a, 1-a; c; x]$**.** This section deals with the value ${}_2F_1\left[a, b; c; \frac{1}{2}\right]$, where $a + b = 1$ due to the following identity of Bailey [1935,

p. 11] (see also [Rainville 1960, p. 69]):

$$
\begin{aligned}
&{}_2F_1\big[a,\,1-a;\,c;\,\tfrac{1}{2}\big] \\
&\qquad = \frac{2^{1-c}\,\Gamma(c)\,\Gamma\big(\frac{1}{2}\big)}{\Gamma\big(\frac{1}{2}(c+a)\big)\,\Gamma\big(\frac{1}{2}(1+c-a)\big)} = \frac{\Gamma\big(\frac{1}{2}c\big),\,\Gamma\big(\frac{1}{2}(1+c)\big)}{\Gamma\big(\frac{1}{2}(c+a)\big)\,\Gamma\big(\frac{1}{2}(1+c-a)\big)}, \qquad (3\text{-}1)
\end{aligned}
$$

where c is a positive integer. It follows from (3-1) that

$$\Gamma\big(\tfrac{1}{2}c\big)\Gamma\big(\tfrac{1}{2}(1+c)\big) = 2^{1-c}\sqrt{\pi}\,\Gamma(c), \qquad (3\text{-}2)$$

since $\Gamma\big(\frac{1}{2}\big)=\sqrt{\pi}$. In this case, we obtain

$$
\begin{aligned}
f(x_0) &= f(0) = {}_2F_1[a,\,1-a;\,c;\,0] = 1,\\
f(x_1) &= f(0.5) = {}_2F_1\big[a,\,1-a;\,c;\,\tfrac{1}{2}\big] = \frac{\Gamma\big(\frac{1}{2}c\big)\,\Gamma\big(\frac{1}{2}(1+c)\big)}{\Gamma\big(\frac{1}{2}(c+a)\big)\,\Gamma\big(\frac{1}{2}(1+c-a)\big)},\\
f(x_2) &= f(1) = {}_2F_1[a,\,1-a;\,c;\,1] = \frac{\Gamma(c)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c+a-1)} \quad (c>1).
\end{aligned}
$$

Consider the well-known Lagrange fundamental polynomials

$$
\begin{aligned}
L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)},\\
L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)},\\
L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}.
\end{aligned}
$$

Then the quadratic interpolation of $f(x) = {}_2F_1[a,\,1-a;\,c;\,x]$ becomes

$$
\begin{aligned}
P_{q_3}(x) &= f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x)\\
&= (2x^2-3x+1) + (-4x^2+4x)\frac{\Gamma\big(\frac{1}{2}c\big)\,\Gamma\big(\frac{1}{2}(1+c)\big)}{\Gamma\big(\frac{1}{2}(c+a)\big)\,\Gamma\big(\frac{1}{2}(1+c-a)\big)}\\
&\qquad\qquad + (2x^2-x)\frac{\Gamma(c)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c+a-1)}.
\end{aligned}
$$

This leads to the following result.

Theorem 3.1. *Let $a, b, c \in \mathbb{R}$ be such that $c > 1$. Then*

$$
\begin{aligned}
P_{q_1}(x) &= \left(2 - \frac{4\Gamma\big(\frac{1}{2}c\big)\Gamma\big(\frac{1}{2}(1+c)\big)}{\Gamma\big(\frac{1}{2}(c+a)\big)\Gamma\big(\frac{1}{2}(1+c-a)\big)} + \frac{2\Gamma(c)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c+a-1)}\right)x^2\\
&\quad + \left(\frac{4\Gamma\big(\frac{1}{2}c\big)\Gamma\big(\frac{1}{2}(1+c)\big)}{\Gamma\big(\frac{1}{2}(c+a)\big)\Gamma\big(\frac{1}{2}(1+c-a)\big)} - \frac{\Gamma(c)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c+a-1)} - 3\right)x + 1.
\end{aligned}
$$

is a quadratic interpolation of ${}_2F_1[a,\,1-a;\,c;\,x]$ in $[0,\,1]$.

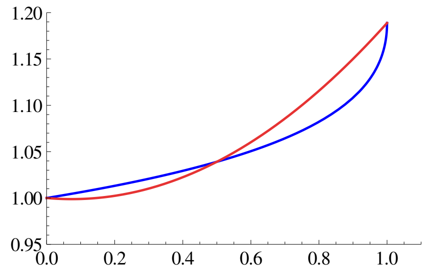

Figure 2. The quadratic interpolation of ${}_2F_1[0.9, 0.1; 1.5; x]$ at $0, 0.5$, and 1.

Remark 3.2. It is evident that when $a = 0, 1$, then $P_{q_1}(x) = {}_2F_1[a, 1-a; c; x] = 1$ for all $x \in [0, 1]$ and for all $c > 1$. Moreover, for all $c > 1$, we have the following three natural observations:

(i) If $-1 < a < 0$, then $P_{q_1}(x)$ and ${}_2F_1[a, 1-a; c; x]$ decrease together in $[0, 1]$.

(ii) If $0 < a < 1$, then $P_{q_1}(x)$ and ${}_2F_1[a, 1-a; c; x]$ increase together in $[0, 1]$.

(iii) If $1 < a < 2$, then $P_{q_1}(x)$ and ${}_2F_1[a, 1-a; c; x]$ decrease together in $[0, 1]$.

Indeed, these follow from derivative test. More observations are stated later while estimating the error (see Remark 3.10).

An interpolating polynomial $P_{q_1}(x)$ of ${}_2F_1[a, 1-a; c; x]$ for certain choices of parameters a and c is as shown in Figure 2.

Remark 3.3. Note that in Theorem 3.1, the parameter c cannot be chosen such that $c \le \frac{1}{2}(a+b+1)$ since the choice $b = 1-a$ results in $c \le 1$, which is a contradiction to the assumption that $c > 1$. In particular, $c \neq \frac{1}{2}(a+b+1)$ in Theorem 3.1, which is the negation of a constraint that will be considered in the next subsection.

Quadratic interpolation on $\boldsymbol{{}_2F_1\big[a, b; \frac{1}{2}(a+b+1); x\big]}$**.** In this section, $f(x) = {}_2F_1[a, b; c; x]$, $c = \frac{1}{2}(a+b+1)$, is first interpolated using the following quadratic transformation obtained from [Andrews et al. 1999, (3.1.3)]; see also [Rainville 1960, Theorem 2.5].

Lemma 3.4. *If* $\frac{1}{2}(a+b+1)$ *is a positive integer, and if* $|x| < 1$ *and* $|4x(1-x)| < 1$, *then*

$$ {}_2F_1\big[a, b; \tfrac{1}{2}(a+b+1); x\big] = {}_2F_1\big[\tfrac{1}{2}a, \tfrac{1}{2}b; \tfrac{1}{2}(a+b+1); 4x(1-x)\big]. \tag{3-3}$$

If we choose $x = 0.5$ then the right-hand side of (3-3) computes the asymptotic behavior of the hypergeometric function at 1. Hence the functional value at $x = 0.5$ of the function $f(x) = {}_2F_1\big[a, b; \frac{1}{2}(a+b+1); x\big]$ can be obtained with the help of (1-1). Due to Lemma 3.4 and (1-1), in this case, the constraints on the parameters are computed as

- $a+b<1$;
- $a+b \neq -(2n+1)$ for $n \in \mathbb{N} \cup \{0\}$.

One can easily obtain that

$$f(x_0) = {}_2F_1\big[a, b; \tfrac{1}{2}(a+b+1); 0\big] = 1;$$

$$f(x_1) = {}_2F_1\big[a, b; \tfrac{1}{2}(a+b+1); \tfrac{1}{2}\big] = \frac{\sqrt{\pi}\,\Gamma\big(\frac{1}{2}(a+b+1)\big)}{\Gamma\big(\frac{1}{2}(a+1)\big)\Gamma\big(\frac{1}{2}(b+1)\big)},$$

$$f(x_2) = {}_2F_1\big[a, b; \tfrac{1}{2}(a+b+1); 1\big] = \frac{\Gamma\big(\frac{1}{2}(1-a-b)\big)\Gamma\big(\frac{1}{2}(a+b+1)\big)}{\Gamma\big(\frac{1}{2}(a+1-b)\big)\Gamma\big(\frac{1}{2}(b+1-a)\big)} = \frac{\cos\big(\frac{\pi}{2}(b-a)\big)}{\cos\big(\frac{\pi}{2}(b+a)\big)},$$

where $f(x_2)$ is obtained by the well-known Euler's reflection formula (in nonintegral variable x)

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

This leads to the additional constraints on the parameters

$$\begin{aligned} &a+b \neq 1 \pm 2n \quad \text{and} \quad a-b \neq -1 \pm 2n, \quad n \in \mathbb{Z} \quad \text{or} \\ &a+b \neq -1 \pm 2n \quad \text{and} \quad a-b \neq 1 \pm 2n, \quad n \in \mathbb{Z}. \end{aligned} \tag{3-4}$$

(These constraints may be relaxed when one does not use Euler's reflection formula!)

Thus, the first quadratic interpolation of $f(x) = {}_2F_1\big[a, b; \frac{1}{2}(a+b+1); x\big]$ becomes

$$\begin{aligned} P_{q_2}(x) &= f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x) \\ &= (2x^2-3x+1) + (-4x^2+4x)\frac{\sqrt{\pi}\,\Gamma\big(\frac{1}{2}(a+b+1)\big)}{\Gamma\big(\frac{1}{2}(a+1)\big)\Gamma\big(\frac{1}{2}(b+1)\big)} \\ &\qquad + (2x^2-x)\frac{\cos\big(\frac{\pi}{2}(b-a)\big)}{\cos\big(\frac{\pi}{2}(b+a)\big)}. \end{aligned}$$

This leads to the following result.

Theorem 3.5. *Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{0\}$ be such that $a+b \neq -(2n+1)$ and $a+b<1$. If either $a+b \neq 1 \pm 2n$ and $a-b \neq -1 \pm 2n$, or $a+b \neq -1 \pm 2n$ and $a-b \neq 1 \pm 2n$ hold, then*

$$\begin{aligned} P_{q_2}(x) = &\left(2 - \frac{4\sqrt{\pi}\,\Gamma\big(\frac{1}{2}(a+b+1)\big)}{\Gamma\big(\frac{1}{2}(a+1)\big)\Gamma\big(\frac{1}{2}(b+1)\big)} + \frac{2\cos\big(\frac{\pi}{2}(b-a)\big)}{\cos\big(\frac{\pi}{2}(b+a)\big)}\right)x^2 \\ &+ \left(\frac{4\sqrt{\pi}\,\Gamma\big(\frac{1}{2}(a+b+1)\big)}{\Gamma\big(\frac{1}{2}(a+1)\big)\Gamma\big(\frac{1}{2}(b+1)\big)} - \frac{\cos\big(\frac{\pi}{2}(b-a)\big)}{\cos\big(\frac{\pi}{2}(b+a)\big)} - 3\right)x + 1 \end{aligned}$$

is a quadratic interpolation of ${}_2F_1\big[a, b; \frac{1}{2}(a+b+1); x\big]$ in [0, 1].

Secondly, we discuss quadratic interpolation of the same function ${}_2F_1[a, b; c; x]$, $c = \frac{1}{2}(a+b+1)$, in [0, 1], but using a different hypergeometric identity. Finally, we observe that both the interpolations are same except at a minor difference in one of the constraints.

Recall the transformation formula [Rainville 1960, Theorem 20, p. 60]:

Lemma 3.6. *If* $|x| < 1$ *and* $|x/(1-x)| < 1$, *then we have*

$$ {}_2F_1[a, b; c; x] = (1-x)^{-a}{}_2F_1\left[a, c-b; c; \frac{-x}{1-x}\right]. $$

Note that $-x/(1-x) = -1$ for $x = 0.5$. This suggests that to find the value $f(0.5) = 2^a{}_2F_1[a, c-b; c; -1]$ we can use the following identity [Rainville 1960, Theorem 26, p. 68]; see also [Beals and Wong 2010].

Lemma 3.7. *Let* $a', b' \in \mathbb{R}$. *If* $1 + a' - b' \neq \{0, -1, -2, -3, \dots\}$ *and* $b' < 1$, *then we have*

$$ {}_2F_1[a', b'; a'-b'+1; -1] = \frac{\Gamma(a'-b'+1)\Gamma\left(\frac{1}{2}a'+1\right)}{\Gamma(a'+1)\Gamma\left(\frac{1}{2}a'-b'+1\right)}. $$

Comparison of the parameters $a' = a,\ b' = c-b$ and $a'-b'+1 = c$ leads to

$$ {}_2F_1[a, c-b; c; -1] = \frac{\Gamma(a-c+b+1)\Gamma\left(\frac{1}{2}a+1\right)}{\Gamma(a+1)\Gamma\left(\frac{1}{2}a-c+b+1\right)} \tag{3-5} $$

with the constraints

- $2c = a+b+1$;
- $c \neq \{0, -1, -2, -3, \dots\} \iff a+b \neq -(2n+1),\ n \in \mathbb{N} \cup \{0\}$;
- $c - b < 1 \iff a - b < 1$.

Under these conditions, (3-5) leads to

$$ \begin{aligned} f(x_1) = f(0.5) = {}_2F_1\left[a, b; \tfrac{1}{2}(a+b+1); \tfrac{1}{2}\right] &= 2^a \frac{\Gamma\left(\frac{1}{2}(a+b+1)\right)\Gamma\left(\frac{1}{2}a+1\right)}{\Gamma(a+1)\Gamma\left(\frac{1}{2}(b+1)\right)} \\ &= \frac{2^{a-1}\,\Gamma\left(\frac{1}{2}(a+b+1)\right)\,\Gamma\left(\frac{1}{2}a\right)}{\Gamma(a)\,\Gamma\left(\frac{1}{2}(b+1)\right)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(a+b+1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(b+1)\right)}, \end{aligned} $$

where the last equality holds by (3-2). Also as discussed at the beginning of this subsection, we have

$$ \begin{aligned} f(x_0) &= f(0) = {}_2F_1\left[a, b; \tfrac{1}{2}(a+b+1); 0\right] = 1, \\ f(x_2) &= f(1) = {}_2F_1\left[a, b; \tfrac{1}{2}(a+b+1); 1\right] = \frac{\cos\left(\frac{\pi}{2}(b-a)\right)}{\cos\left(\frac{\pi}{2}(b+a)\right)}, \quad a+b < 1, \end{aligned} $$

with additional constraints obtained in (3-4) (here also (3-4) may be relaxed!).

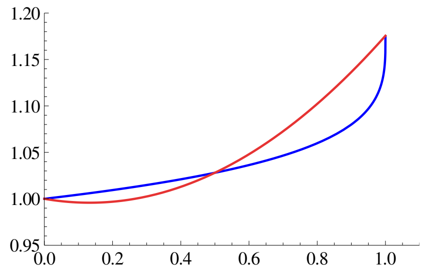

Figure 3. The quadratic interpolation of ${}_2F_1[0.1, 0.3; 0.7; x]$ at 0, 0.5, and 1.

Thus, the second quadratic interpolation of $f(x) = {}_2F_1\big[a, b; \frac{1}{2}(a+b+1); x\big]$ remains same as the first quadratic interpolation obtained in Theorem 3.5 but with an additional constraint $a-b<1$. This shows that the quadratic interpolation obtained by Theorem 3.5 is stronger than what was discussed so far using Lemmas 3.6 and 3.7. A quadratic interpolation of ${}_2F_1\big[a, b; \frac{1}{2}(a+b+1); x\big]$ is shown in Figure 3.

Error estimates. The error estimate in quadratic interpolation of ${}_2F_1[a, b; c; x]$ interpolating at 0, 0.5, 1 in [0, 1] is formulated as below:

Lemma 3.8. *Let $P_q(x)$ be a quadratic interpolation of $f(x) = {}_2F_1[a, b; c; x]$ interpolating at* 0, 0.5, 1 *in* [0, 1]. *If $a, b, c \in (-3, \infty)$ with $c-a-b>3$, then the deviation of $f(x)$ from $P_q(x)$ is estimated by*

$$\begin{aligned}|E_q(f, x)| &= |f(x) - P_q(x)|\\ &\le \tfrac{1}{6}M\big|a(a+1)(a+2)b(b+1)(b+2)\big|\,\frac{\Gamma(c)\Gamma(c-a-b-3)}{\Gamma(c-a)\Gamma(c-b)}\end{aligned}$$

for all values of $x \in [0, 1]$, where M is defined by

$$M := \begin{cases} \frac{1}{12}(3-\sqrt{3})\big(-1+\frac{1}{6}(3-\sqrt{3})\big)\big(-1+\frac{1}{3}(3-\sqrt{3})\big), & x<\frac{1}{2},\\ -\frac{1}{12}(3+\sqrt{3})\big(-1+\frac{1}{6}(3+\sqrt{3})\big)\big(-1+\frac{1}{3}(3+\sqrt{3})\big), & x>\frac{1}{2}.\end{cases} \tag{3-6}$$

Proof. We need to estimate

$$\max_{0\le x\le 1} \tfrac{1}{6}\big|x(x-0.5)(x-1)\big| \max_{0\le x\le 1} |f'''(x)|,$$

where $f(x) = {}_2F_1[a, b; c; x]$. Note that

$$\max_{0\le x\le 1} |x(x-0.5)(x-1)| = M \ (\approx 0.0481125\ldots)$$

by (3-6). We apply the well known derivative formula (2-1) to maximize $|f'''(x)|$, $0 \le x \le 1$. The proof follows from (1-1), Lemma 1.1, (2-1), and the fact that

$$\Gamma(x+1) = x\Gamma(x). \qquad \square$$

The following result is an immediate consequence of Lemma 3.8 which estimates the difference $E_{q_1}(f, x) = {}_2F_1[a, 1-a; c; x] - P_{q_1}(x)$ in [0, 1].

Corollary 3.9. *Let $a, c \in \mathbb{R}$ be such that $-3 < a < 4$ and $c > 4$. Then the deviation of ${}_2F_1[a, 1-a; c; x]$ from $P_{q_1}(x)$ is estimated by*

$$\begin{aligned}|E_{q_1}(f, x)| &= |f(x) - P_{q_1}(x)| \\ &\le \tfrac{1}{6}M\big|a(a+1)(a+2)(1-a)(2-a)(3-a)\big| \frac{\Gamma(c)\Gamma(c-4)}{\Gamma(c-a)\Gamma(c+a-1)}\end{aligned}$$

for all values of $x \in [0, 1]$, where M is obtained by (3-6).

Remark 3.10. It follows from Corollary 3.9 that there is no error for any of the choices $a = -2, -1, 0, 1, 2, 3$. In other words, for any of these choices, $E_{q_1}(f, x)$ vanishes.

Similarly, as a consequence of Lemma 3.8, we obtain:

Corollary 3.11. *Let $a, b \in \mathbb{R}$ be such that $-7 < a + b < -5$. Then the deviation of ${}_2F_1\big[a, b; \frac{1}{2}(a+b+1); x\big]$ from $P_{q_2}(x)$ is estimated by*

$$\begin{aligned}|E_{q_2}(f, x)| &= |f(x) - P_{q_2}(x)| \\ &\le \tfrac{1}{6}M\big|a(a+1)(a+2)b(b+1)(b+2)\big| \frac{\Gamma\big(\frac{1}{2}(a+b+1)\big)\Gamma\big(\frac{1}{2}(-a-b-5)\big)}{\Gamma\big(\frac{1}{2}(b-a+1)\big)\Gamma\big(\frac{1}{2}(a-b+1)\big)}\end{aligned}$$

for all values of $x \in [0, 1]$, where M is obtained by (3-6).

Remark 3.12. It follows from Corollary 3.11 that since $E_{q_2}(f, x)$ vanishes for the choices $a = -2, -1, 0$ and $b = -2, -1, 0$, there is no error for these choices of the parameters a and b.

Now we describe a slightly deeper analysis on the error obtained in Corollary 3.9 through the following lemma, which is a consequence of Lemma 1.2. A similar analysis can be described for Corollary 3.11.

Lemma 3.13. *Let $a, c \in \mathbb{R}$ be such that $c > 4$. If either $1 < a < 4$ or $-3 < a < 0$ holds, then the quotient*

$$\frac{\Gamma(c)\Gamma(c-4)}{\Gamma(c-a)\Gamma(c+a-1)}$$

decreases when c increases.

Proof. We use Lemma 1.2. Since $c - a > c - 4 > 0$, on one hand we have

$$\frac{\Gamma'(c-4)}{\Gamma(c-4)} - \frac{\Gamma'(c-a)}{\Gamma(c-a)} < 0.$$

On the other hand, since $c < c + a - 1$, we have

$$\frac{\Gamma'(c)}{\Gamma(c)} - \frac{\Gamma'(c+a-1)}{\Gamma(c+a-1)} < 0.$$

Thus, if

$$g(c)=\frac{\Gamma(c)\Gamma(c-4)}{\Gamma(c-a)\Gamma(c+a-1)},$$

it follows that

$$\begin{aligned}\frac{g'(c)}{g(c)}&=\frac{\Gamma'(c)}{\Gamma(c)}+\frac{\Gamma'(c-4)}{\Gamma(c-4)}-\frac{\Gamma'(c-a)}{\Gamma(c-a)}-\frac{\Gamma'(c+a-1)}{\Gamma(c+a-1)}\\&=\left(\frac{\Gamma'(c-4)}{\Gamma(c-4)}-\frac{\Gamma'(c-a)}{\Gamma(c-a)}\right)+\left(\frac{\Gamma'(c)}{\Gamma(c)}-\frac{\Gamma'(c+a-1)}{\Gamma(c+a-1)}\right)<0.\end{aligned}$$

By the definition of the gamma function, obviously, one can see that $\Gamma(x)>0$ for $x>0$. This shows that $g(c)>0$ and hence $g'(c)<0$. Thus, $g(c)$ decreases for $1<a<4<c$.

For $c>4$, if $-3<a<0$ holds then we consider the rearrangement

$$\frac{g'(c)}{g(c)}=\left(\frac{\Gamma'(c)}{\Gamma(c)}-\frac{\Gamma'(c-a)}{\Gamma(c-a)}\right)+\left(\frac{\Gamma'(c-4)}{\Gamma(c-4)}-\frac{\Gamma'(c+a-1)}{\Gamma(c+a-1)}\right)$$

and show that $g'(c)/g(c)<0$. □

Using Mathematica or other similar tools, one can see that Lemma 3.13 even holds true for the remaining range $0\le a\le 1$. This suggests the following conjecture.

Conjecture 3.14. *Let $a, c\in\mathbb{R}$ be such that $0\le a\le 1$ and $c>4$. Then the quotient*

$$\frac{\Gamma(c)\Gamma(c-4)}{\Gamma(c-a)\Gamma(c+a-1)}$$

decreases when c increases.

Thus, we observe that when $c>4$ increases then the error $E_{q_1}(f,x)$ estimated in Corollary 3.9 decreases (see also Figures 4 and 5).

Figures 4 and 5 describe the quadratic interpolation of the hypergeometric functions ${}_2F_1[a, 1-a, c, x]$ at 0, 0.5 and 1, whereas Tables 2 and 3 compare the values of the hypergeometric function up to four decimal places with its interpolating

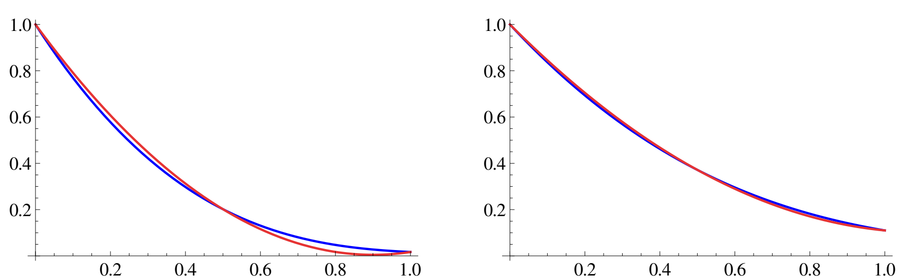

Figure 4. The error estimate $E_{q_1}(f,x)$ when $a=3.9$ and c increases from 4.5 to 6.5.

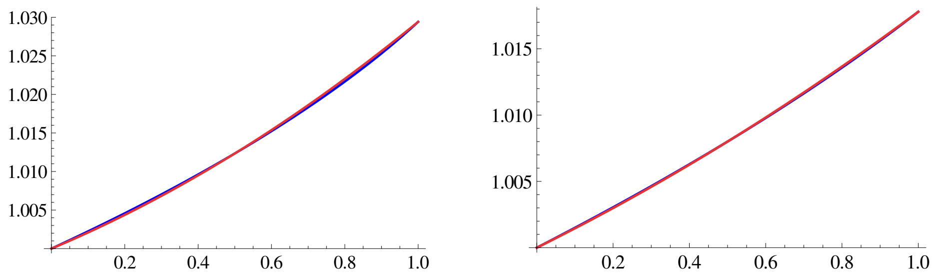

Figure 5. The error estimate $E_{q_1}(f, x)$ when $a = 0.9$ and c increases from 4.1 to 6.1.

nodes x_i	0	0.25	0.5	0.75	1
actual values ${}_2F_1[3.9, -2.9; 5; x_i]$	1	0.5372	0.2516	0.0998	0.0367
polynomial approx. by $P_{q_1}(x_i)$	1	0.5591	0.2516	0.0775	0.0367
validity of error bounds by $E_{q_1}(f, x_i)$	0	$0.0219 < 0.0274$	0	$0.0223 < 0.0274$	0
actual values ${}_2F_1[3.9, -2.9; 6; x_i]$	1	0.6027	0.3358	0.1724	0.0845
polynomial approx. by $P_{q_1}(x_i)$	1	0.6163	0.3358	0.1585	0.0845
validity of error bounds by $E_{q_1}(f, x_i)$	0	$0.0136 < 0.0158$	0	$0.0139 < 0.0158$	0

Table 2. Comparison of the functional and quadratic polynomial values.

nodes x_i	0	0.25	0.5	0.75	1
actual values ${}_2F_1[0.9, 0.1; 5; x_i]$	1	1.0047	1.0099	1.0158	1.0227
polynomial approx. by $P_{q_1}(x_i)$	1	1.0046	1.0099	1.0160	1.0227
validity of error bounds by $E_{q_1}(f, x_i)$	0	$0.0001 < 0.0016$	0	$0.0002 < 0.0016$	0
actual values ${}_2F_1[0.9, 0.1; 6; x_i]$	1	1.0039	1.0082	1.0128	1.0182
polynomial approx. by $P_{q_1}(x_i)$	1	1.0038	1.0082	1.0129	1.0182
validity of error bounds by $E_{q_1}(f, x_i)$	0	$0.0001 < 0.0004$	0	$0.0001 < 0.0004$	0

Table 3. Comparison of the functional and quadratic polynomial values.

polynomial values in the interval [0, 1] for the choice of parameters $a = 3.9$, $c = 5$ and $a = 0.9$, $c = 6$ respectively. Figures 4 and 5 and Tables 2 and 3 also indicate errors at various points within the unit interval except at the interpolating points at $x = 0, 0.5, 1$.

The error estimate $|E_{q_2}(f, x)|$ for the function ${}_2F_1\bigl[a, b; \frac{1}{2}(a+b+1); x\bigr]$ can be analyzed in a similar way, and hence we omit the proof.

4. An application

In this section, we briefly consider interpolation of a continued fraction that converges to a quotient of two hypergeometric functions. Gauss used the contiguous relations to give several ways to write a quotient of two hypergeometric functions as a continued fraction. For instance, it is well known that

$$\frac{{}_2F_1[a+1, b; c+1; x]}{{}_2F_1[a, b; c; x]} = \cfrac{1}{1 + \cfrac{\frac{(a-c)b}{c(c+1)}x}{1 + \cfrac{\frac{(b-c-1)(a+1)}{(c+1)(c+2)}x}{1 + \cfrac{\frac{(a-c-1)(b+1)}{(c+2)(c+3)}x}{1 + \cfrac{\frac{(b-c-2)(a+2)}{(c+3)(c+4)}x}{1 + \ddots}}}}}, \quad |x| < 1. \tag{4-1}$$

On one hand, if we adopt the basic linear interpolation method that we discussed in Section 2 (that is, linear interpolation directly) to the function

$$g(x) = \frac{{}_2F_1[a+1, b; c+1; x]}{{}_2F_1[a, b; c; x]}$$

at $x_0 = 0$ and $x_1 = 1$, we obtain the linear interpolation of the above continued fraction in the form

$$R_l(x) = g(x_0) + \frac{x - x_0}{x_1 - x_0}(g(x) - g(x_0)) = 1 + \left(\frac{b}{c-b}\right)x, \quad c - b > a,$$

since $g(x_0) = 1$ and $g(x_1) = c/(c-b)$. For the choice $a = 1$, $b = 2$, $c = 6$, this approximation is also shown in Figure 6.

On the other hand, an application of linear interpolation of ${}_2F_1[a, b; c; x]$ obtained in Section 2 leads to the following approximation of the above continued

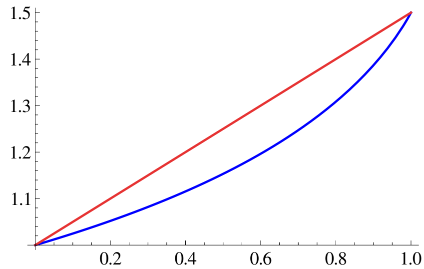

Figure 6. Approximation of ${}_2F_1[a+1,b;c+1;x]/{}_2F_1[a,b;c;x]$ through $R_l(x)$.

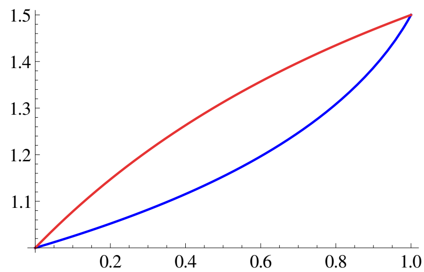

Figure 7. Approximation of ${}_2F_1[a+1,b;c+1;x]/{}_2F_1[a,b;c;x]$ through $R_r(x)$.

fraction in terms of ratio of polynomial approximation (we call this *rational interpolation*):

$$\begin{aligned} R_r(x) &= \frac{1}{P_l(x)}\left(\frac{\Gamma(c+1)\Gamma(c-a-b)-\Gamma(c-a)\Gamma(c-b+1)}{\Gamma(c-a)\Gamma(c-b+1)}x+1\right) \\ &= \frac{\big[c\Gamma(c)\Gamma(c-a-b)/(c-b)-\Gamma(c-a)\Gamma(c-b)\big]x+\Gamma(c-a)\Gamma(c-b)}{\big[\Gamma(c)\Gamma(c-a-b)-\Gamma(c-a)\Gamma(c-b)\big]x+\Gamma(c-a)\Gamma(c-b)} \\ &= 1+\frac{b}{c-b}\left[\frac{\Gamma(c-a-b)\Gamma(c)\,x}{\big[\Gamma(c)\Gamma(c-a-b)-\Gamma(c-a)\Gamma(c-b)\big]x+\Gamma(c-a)\Gamma(c-b)}\right], \end{aligned}$$

where $c-a-b>0$. For the choice $a=1,\ b=2,\ c=6$, this approximation is also shown in Figure 7.

Observe that

$$R_r(x_0)=1=R_l(x_0) \quad \text{and} \quad R_r(x_1)=\frac{c}{c-b}=R_l(x_1)$$

and hence R_r also interpolates the continued fraction under consideration at 0 and 1. Further we observe that both the approximations $R_l(x)$ and $R_r(x)$ of the continued fraction are easy to obtain and the first approximation, i.e., $R_l(x)$, is in a simpler form than $R_r(x)$, as expected. Now, it would be interesting to know which one would give the best approximation to the continued fraction under consideration. With the special choice $a=1,\ b=2,\ c=6$, we see from Figures 6 and 7 that $R_l(x)$ is a better approximation than $R_r(x)$. One may ask: does it happen for arbitrary parameters a, b, c? Since $R_l(x) = R_r(x)$ if and only if $\Gamma(c)\Gamma(c-a-b) = \Gamma(c-a)\Gamma(c-b)$, the answer to this question is yes except when $\Gamma(c)\Gamma(c-a-b) = \Gamma(c-a)\Gamma(c-b)$.

This leads to the following result:

Theorem 4.1. *Let $R_l(x)$ and $R_r(x)$ be respectively the linear interpolation and the rational interpolation of the quotient ${}_2F_1[a+1, b; c+1; x]/{}_2F_1[a, b; c; x]$ (equivalently, of the continued fraction* (4-1)*). Then $R_l(x)$ and $R_r(x)$ coincide with each other if and only if $\Gamma(c)\Gamma(c-a-b) = \Gamma(c-a)\Gamma(c-b)$ holds for $c-a-b > 0$.*

5. Concluding remarks and future scope

Recall that, in this paper, we use some standard interpolation techniques to approximate the hypergeometric function

$$ {}_2F_1[a, b; c; x] = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots $$

for a range of parameter triples (a, b, c) on the interval $0 < x < 1$. Some of the familiar hypergeometric functional identities and asymptotic behavior of the hypergeometric function at $x = 1$ played crucial roles in deriving the formula for such approximations. One can expect similar formulae using other well-known interpolations and obtain better approximations for the hypergeometric function; however, we discuss such results in an upcoming manuscript(s). Different numerical methods for the computation of the confluent and Gauss hypergeometric functions were studied recently in [Pearson et al. 2017]. Such investigation may be extended to the q-analog of the hypergeometric functions, namely, Heine's basic hypergeometric functions; for instance refer to [Chen and Fu 2011] for similar discussions.

We also focus on error analysis of the numerical approximations leading to monotone properties of quotients of gamma functions in parameter triples (a, b, c). Monotone properties of the gamma function and its quotients in different forms are of recent interest to many researchers; see for instance [Alzer 1993; Anderson and Qiu 1997; Bustoz and Ismail 1986; Chen and Zhou 2014; Giordano and Laforgia 2001; Gautschi 1959; Luo et al. 2017; Mortici and Dumitrescu 2017]. In this paper, we also studied and stated a conjecture (see Conjecture 3.14) related to monotone properties of quotients of gamma functions to analyze the error estimate of the numerical approximations under consideration.

Finally, an application to continued fractions of Gauss is also discussed. Approximations of continued fractions in different forms are also attractive to many researchers; see [Lu et al. 2017; 2016].

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