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Pythagorean orthogonality of compact sets

Pallavi Aggarwal, Steven Schlicker and Ryan Swartzentruber

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The Hausdorff metric h is used to define the distance between two elements of $\mathcal{H}(\mathbb{R}^n)$, the hyperspace of all nonempty compact subsets of $\mathbb{R}^n$. The geometry this metric imposes on $\mathcal{H}(\mathbb{R}^n)$ is an interesting one — it is filled with unexpected results and fascinating connections to number theory and graph theory. Circles and lines are defined in this geometry to make it an extension of the standard Euclidean geometry. However, the behavior of lines and segments in this extended geometry is much different from that of lines and segments in Euclidean geometry. This paper presents surprising results about rays in the geometry of $\mathcal{H}(\mathbb{R}^n)$, with a focus on attempting to find well-defined notions of angle and angle measure in $\mathcal{H}(\mathbb{R}^n)$.

1. Background

In this section we provide the definition of the Hausdorff metric and some known results about lines and segments of compact sets. The Hausdorff metric h was introduced by Felix Hausdorff in the early twentieth century as a way to measure the distance between compact sets. The space $\mathbb{R}^n$ will be our underlying space, and we will denote by $\mathcal{H}(\mathbb{R}^n)$ the hyperspace of all nonempty compact subsets of $\mathbb{R}^n$. The standard Euclidean metric on $\mathbb{R}^n$ will be denoted by d_E. The Hausdorff metric on $\mathcal{H}(\mathbb{R}^n)$ is defined as follows.

Definition 1.1. The *Hausdorff distance* $h(A, B)$ between sets A and B in $\mathcal{H}(\mathbb{R}^n)$ is

$$h(A, B) = \max\{d(A, B), d(B, A)\},$$

where

$$d(a, B) = \min_{b\in B}\{d_E(a, b)\}$$

for $a \in A$ and

$$d(A, B) = \max_{a\in A}\{d_E(a, B)\}.$$

Example 1.2. Let A be the closed interval $[0, 2]$ and B be the closed interval $[3, 4]$ in $\mathcal{H}(\mathbb{R})$. Then $d(A, B) = |0 - 3| = 3$ and $d(B, A) = |4 - 2| = 2$. This example

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illustrates that it is possible to have $d(A, B) \neq d(B, A)$, which necessitates using the maximum of $d(A, B)$ and $d(B, A)$ as the Hausdorff distance between A and B. Thus we have that $h(A, B) = d(A, B) = 3$.

The proof that h is a metric can be found in many topology texts; see [Barnsley 1993; Edgar 1990] for example.

Line segments, lines, and rays in $\mathcal{H}(\mathbb{R}^n)$ can be defined in a way that makes them analogous to segments, lines, and rays in $\mathbb{R}^n$. In $\mathbb{R}^n$ we can think of the line segment $\overline{ab}$ as the set of all points c that lie between a and b, that is, the points c that satisfy $d_E(a, b) = d_E(a, c) + d_E(c, b)$. We follow the convention in [Blumenthal 1953] and write acb to indicate that c lies between a and b. Similarly, the ray $\overrightarrow{ab}$ can be thought of as all points c such that acb or abc, and the line $\overleftrightarrow{ab}$ is the set of all points c such that cab, acb, or abc. We can naturally extend these notions to define segments, lines, and rays in $\mathcal{H}(\mathbb{R}^n)$.

Definition 1.3. The set $C \in \mathcal{H}(\mathbb{R}^n)$ lies *between* the sets A and B in $\mathcal{H}(\mathbb{R}^n)$ if

$$h(A, C) + h(C, B) = h(A, B).$$

We write ACB to signify that C lies between A and B.

The definition of betweenness in $\mathcal{H}(\mathbb{R}^n)$ then allows us to define segments, lines, and rays in $\mathcal{H}(\mathbb{R}^n)$.

Definition 1.4. Let A and B be distinct sets in $\mathcal{H}(\mathbb{R}^n)$:

(1) The *segment* $\overline{AB}$ is the collection of all sets $C \in \mathcal{H}(\mathbb{R}^n)$ that satisfy ACB.

(2) The *ray* $\overrightarrow{AB}$ is the collection of all sets $C \in \mathcal{H}(\mathbb{R}^n)$ that satisfy ACB or ABC.

(3) The *line* $\overleftrightarrow{AB}$ is the collection of all sets $C \in \mathcal{H}(\mathbb{R}^n)$ that satisfy CAB, ACB, or ABC.

Example 1.5. Let A be the circle of radius 1 and B be the circle of radius 2 in $\mathcal{H}(\mathbb{R}^2)$, both centered at the origin. Then $h(A, B) = d(A, B) = d(B, A) = 1$. Figure 1 (left image) illustrates a set C satisfying CAB (the shaded set), a set C

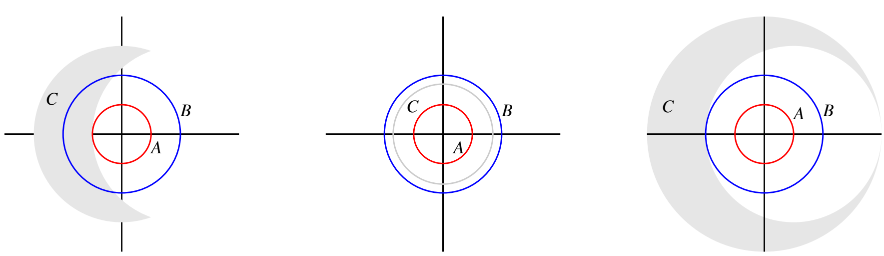

Figure 1. Sets C satisfying CAB, ACB, and ABC.

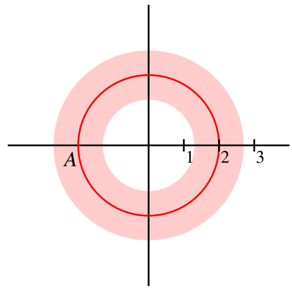

Figure 2. A dilation of a circle.

(middle image) satisfying ACB (the circle C centered at the origin of radius $1+s$ for any $0 < s < 1$), and a set C (right image) satisfying ABC (the shaded set).

It is reasonable to ask how one goes about finding a set C that satisfies CAB, ACB, or ABC. The key lies in the dilation of a set.

Definition 1.6. Given $A \in \mathcal{H}(\mathbb{R}^n)$ and $s \in \mathbb{R}$ with $s \geq 0$, the *s-dilation* of A is the set

$$(A)_s = \{x \in \mathbb{R}^n \mid d(x, A) \leq s\}.$$

As an example, the 0.7-dilation of the circle of radius 2 in $\mathcal{H}(\mathbb{R}^2)$ is the shaded region shown in Figure 2.

Dilations are useful mainly because $h(A, (A)_s) = s$ and any set C that satisfies $h(A, C) = s$ is a subset of $(A)_s$ [Braun et al. 2005, Theorem 4]. Thus when we want to find a set C that satisfies ACB with $h(A, C) = s$, for example, we can restrict our search to subsets of $(A)_s \cap (B)_{h(A,B)-s}$. In fact, the set $X = (A)_s \cap (B)_{h(A,B)-s}$ itself satisfies AXB with $h(A, X) = s$ for any $0 < s < h(A, B)$, as the following lemma attests.

Lemma 1.7 [Bogdewicz 2000, Lemma 3.6]. *Let $A, C \in \mathcal{H}(\mathbb{R}^n)$, $h(A, C) = q$ and let*

$$W = (A)_s \cap (C)_{q-s}$$

for each $s \in [0, q]$. Then $h(A, W) = s$ and $h(W, C) = q - s$.

If we restrict ourselves to the subspace of single point sets, then the Hausdorff metric is just the Euclidean metric. In this way, the standard Euclidean geometry can be embedded in the geometry of $\mathcal{H}(\mathbb{R}^n)$. In general, though, lines and segments behave quite differently in $\mathcal{H}(\mathbb{R}^n)$ than they do in Euclidean geometry. For example, in Euclidean geometry, given two points a and b, for any $s \geq 0$ there is exactly one point c on $\overrightarrow{ab}$ (and $\overrightarrow{ba}$) with $d_E(a, c) = s$. It is demonstrated in [Bay et al. 2005] that, under certain circumstances, there are no sets C that satisfy BAC with $h(A, C) = s$ for all s larger than some real number s_0. Hence some lines in $\mathcal{H}(\mathbb{R}^n)$ are actually just halflines. On the other hand, if there exists $a \in A$ such that $d(a, B) \neq h(A, B)$ or $b \in B$ such that $d(b, A) \neq h(A, B)$, then there are infinitely many sets C that satisfy ACB and $h(A, C) = s$ for any $0 < s < h(A, B)$ [Blackburn et al. 2009, Lemma 2.3]. This prompts the following definition:

Definition 1.8. Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$. The elements $C, C' \in \mathcal{H}(\mathbb{R}^n)$ are *at the same location* on $\overleftrightarrow{AB}$ if C and C' satisfy

- ACB and $AC'B$,
- CAB and $C'AB$, or
- ABC and ABC',

with $h(A, C) = h(A, C') = s$ for some s.

2. Some results about rays in $\mathcal{H}(\mathbb{R}^n)$

The discussion in the previous section indicates that the geometry of the Hausdorff metric is often surprising and counterintuitive. To develop the geometry more fully, we are interested in defining and measuring angles in $\mathcal{H}(\mathbb{R}^n)$. As in Euclidean geometry, we can consider an angle as being formed by two rays with a common endpoint. In Euclidean geometry, however, if the point c is on the ray $\overrightarrow{ab}$, then the rays $\overrightarrow{ac}$ and $\overrightarrow{ab}$ are the same. There is no guarantee that the same result is true in $\mathcal{H}(\mathbb{R}^n)$. In this section we examine some behavior of rays in $\mathcal{H}(\mathbb{R}^n)$ that will be pertinent if we want to define angle measure.

One result that we will use comes from [Montague 2008].

Lemma 2.1. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$. Fix $s, t \in \mathbb{R}$ with $s, t > 0$, and $s + t = h(A, B)$. Then, for $C \in \mathcal{H}(\mathbb{R}^n)$, the following statements are equivalent*:

(1) *The set C is a subset of $(A)_s \cap (B)_t$, $A \subseteq (C)_s$, and $B \subseteq (C)_t$.*

(2) *The set C is between A and B, and $h(A, C) = s$.*

Proof. (1) $\Rightarrow$ (2). Suppose $C \subseteq (A)_s \cap (B)_t$, $A \subseteq (C)_s$, and $B \subseteq (C)_t$. Since $C \subseteq (A)_s$, we know $d(C, A) \leq s$, and since $A \subseteq (C)_s$, we know $d(A, C) \leq s$. Similarly, $d(B, C) \leq t$ and $d(C, B) \leq t$. Thus $h(A, C) \leq s$ and $h(C, B) \leq t$. Since $s + t = h(A, B) \leq h(A, C) + h(C, B)$, and $h(A, C) \leq s$ and $h(C, B) \leq t$, we conclude that $h(A, C) = s$ and $h(C, B) = t$. Therefore, C is between A and B and $h(A, C) = s$.

(2) $\Rightarrow$ (1). Suppose C is between A and B, and $h(A, C) = s$. Since C is a distance s from A, by [Braun et al. 2005, Theorem 2], we know that $C \subseteq (A)_s$. Given ACB and $h(A, C) = s$, we also have that $h(C, B) = t = h(A, B) - s$. Then, we know that $C \subseteq (B)_t$, by [Braun et al. 2005, Theorem 2]. Thus, $C \subseteq (A)_s \cap (B)_t$. Also, we know $d(a, C) \leq d(A, C) \leq h(A, C) = s$ for all $a \in A$, so $A \subseteq (C)_s$. Likewise, $d(b, C) \leq h(B, C) = t$ for all $b \in B$, so $B \subseteq (C)_t$. □

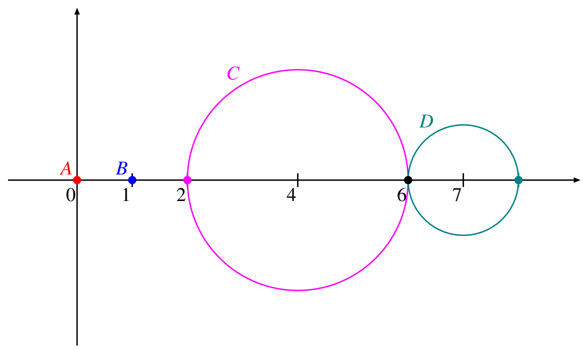

Figure 3. D in $\overrightarrow{AC}$.

Our first result addresses the question of whether rays $\overrightarrow{AC}$ and $\overrightarrow{AD}$ must be the same if C and D both lie on ray $\overrightarrow{AB}$.

Proposition 2.2. *For $A, B, C, D \in \mathcal{H}(\mathbb{R}^n)$, if $C \in \overrightarrow{AB}$ with ABC, and $D \in \overrightarrow{AC}$ with ACD, then $D \in \overrightarrow{AB}$ with ABD and BCD.*

Proof. It is not difficult to show that ABD is equivalent to BCD, so we focus on the latter. Let $s = h(A, B)$ and $t = h(B, C)$. Then ABC implies $h(A, C) = s + t$. Lemma 2.1 tells us that

$$B \subseteq (A)_s \cap (C)_t, \quad A \subseteq (B)_s, \quad \text{and} \quad C \subseteq (B)_t.$$

Similarly, since we have ACD, for some fixed $x, y \in \mathbb{R}$ with $x, y > 0$, $x + y = h(A, D)$, and $h(A, C) = x$, it follows that

$$C \subseteq (A)_x \cap (D)_y, \quad A \subseteq (C)_x, \quad \text{and} \quad D \subseteq (C)_y.$$

Thus C must be a subset of both $(B)_t$ and $(A)_x \cap (D)_y$, from which it follows that

$$C \subseteq (B)_t \cap (A)_x \cap (D)_y.$$

Hence we have $C \subseteq (B)_t \cap (D)_y$, $B \subseteq (C)_t$, and $D \subseteq (C)_y$. We conclude that C is between B and D by Lemma 2.1. □

One consequence of Proposition 2.2 is the following corollary, whose proof is left to the reader.

Corollary 2.3. *For $A, B, C, D \in \mathcal{H}(\mathbb{R}^n)$, if $C \in \overrightarrow{AB}$ with ACB, and $D \in \overrightarrow{AC}$ with ADC, then $D \in \overrightarrow{AB}$.*

It is not always true that $\overrightarrow{AB}$ contains the same sets as $\overrightarrow{AC}$. The following example demonstrates that $C \in \overrightarrow{AB}$ with ABC and $D \in \overrightarrow{AB}$ does not necessarily imply that $D \in \overrightarrow{AC}$.

Example 2.4. Let $A = \{(0, 0)\}$, $B = \{(1, 0)\}$, C be the circle of radius 2 centered at $(4, 0)$, and D be the unit circle centered at $(7, 0)$ in $\mathcal{H}(\mathbb{R}^2)$ as illustrated in Figure 3.

Note that $h(A, B) = 1$, $h(B, C) = 5$, and $h(A, C) = 6$, so $C \in \overrightarrow{AB}$. Similarly, $h(B, D) = 7$, and $h(A, D) = 8$, so $D \in \overrightarrow{AB}$. However, $h(C, D) = 4$, so $D \notin \overrightarrow{AC}$. Therefore, $D \in \overrightarrow{AB}$ does not necessarily imply that $D \in \overrightarrow{AC}$.

3. Pythagorean triples in $\mathcal{H}(\mathbb{R}^n)$

The results in the previous section about rays provide some caution about the idea of defining angle measure. We can define an angle to be a union of two rays that emanate from a given point, but we might expect the measure of an angle, if possible, to be more complicated than in Euclidean geometry.

To consider angle measure in $\mathcal{H}(\mathbb{R}^n)$, we are motivated by an approach used by Wildberger [2005], who presents an alternative to classical trigonometry that does not rely on the general notion of angle. The concept of *spread*, or proportion of distance, utilizes the idea of orthogonality. In order to use this approach in $\mathcal{H}(\mathbb{R}^n)$, we will need to define and understand orthogonality. We define orthogonality in $\mathcal{H}(\mathbb{R}^n)$ as Pythagorean orthogonality.

Definition 3.1. The sets A, B, and Q in $\mathcal{H}(\mathbb{R}^n)$ form a *Pythagorean triple* (or right triangle with $\overline{AQ}$ as hypotenuse) if

$$h(A, B)^2 + h(B, Q)^2 = h(A, Q)^2.$$

Example 3.2. Let $a, b, q > 0$ with $q^2 = a^2 + b^2$, and let $A = \{(a, 0)\}$, $B = \{(0, 0)\}$, and $Q = \{(0, b)\}$. Note that $h(A, B) = a$, $h(B, Q) = b$, and $h(A, Q) = q$. In this case the sets A, B, and Q form a Pythagorean triple, and we can see that the idea of Pythagorean triples in $\mathcal{H}(\mathbb{R}^n)$ is really a generalization of the concept of Pythagorean orthogonality in $\mathbb{R}^n$. An example of infinite sets that form a Pythagorean triple is the collection A, B, and Q, where A is the circle centered at the origin of radius 1, B is the circle centered at the origin of radius 4, and Q is the disk centered at the origin of radius 6. In this case we have $h(A, B) = 3$, $h(B, Q) = d(Q, B) = 4$, and $h(A, Q) = d(Q, A) = 5$.

The question we want to address now is, given sets $A, B \in \mathcal{H}(\mathbb{R}^n)$, must there exist a set (or sets) Q that forms a Pythagorean triple with $\overline{AB}$ as hypotenuse such that $h(A, Q) = s$? In fact, we will prove that there are infinitely many such Q at a fixed location s from A. It is important to note that any such set Q must lie within the intersection $(A)_s \cap (B)_{\sqrt{r^2-s^2}}$. For the remainder of this section we set the conditions that

- A and B are in $\mathcal{H}(\mathbb{R}^n)$ with $r = h(A, B) = d(A, B) > 0$;
- s and t are positive numbers with $r^2 = s^2 + t^2$ (note that this implies $r > s$, $r > t$, and $s + t > r$); and
- $Q_s = (A)_s \cap (B)_t$.

We will refer to these conditions as our *Pythagorean conditions*. We present several lemmas that we will use to establish our first result about Pythagorean triples in $\mathcal{H}(\mathbb{R}^n)$. By $N_\epsilon(q)$ we mean the open ϵ-neighborhood $\{x \in \mathbb{R}^n : d_E(x, q) < \epsilon\}$ centered at point q. We denote the closure of a set S as $\bar{S}$ and the boundary of S as ∂S.

We state the first lemma, which is [Blackburn et al. 2009, Lemma 2.3].

Lemma 3.3. *Let A and B be elements of $\mathcal{H}(\mathbb{R}^n)$. If $d(B, A) > 0$, then there exist $b \in B$ and $a \in \partial A$ such that $d_E(b, a) = d(b, A) = d(B, A)$.*

Lemma 3.4. *Let $B \in \mathcal{H}(\mathbb{R}^n)$ and let $t > 0$. If $y \in \partial(B)_t$, then $d(y, B) = t$.*

The proof of Lemma 3.4 is straightforward and is left to the reader.

We know that if $h(A, B) = r$ and $u + v = r$, then $h(A, (A)_u \cap (B)_v) = u$ and $h(B, (A)_u \cap (B)_v) = v$. However, in the case where $r^2 = s^2 + t^2$ (so that $s + t \neq r$), we cannot conclude that $h(A, Q_s) = s$ and $h(B, Q_s) = t$, but we do have the inequalities.

Lemma 3.5. *Given the Pythagorean conditions,*

$$h(A, Q_s) \leq s \quad \text{and} \quad h(B, Q_s) \leq t.$$

Proof. We will demonstrate that $h(A, Q_s) \leq s$. The argument that $h(B, Q_s) \leq t$ is similar and is left to the reader. We first show that $d(A, Q_s) \leq s$. Let $a \in A$, and let $b \in B$ such that $d_E(a, b) = d(a, B) \leq d(A, B) = r$. Let $x \in \overrightarrow{ab}$ with $d_E(a, x) = s$. If $d_E(a, x) \geq d_E(a, b)$, then $b \in (A)_s$ and so $b \in Q_s$. If $d_E(a, x) < d_E(a, b)$, then $x \in \overline{ab}$. So $d_E(b, x) = d_E(a, b) - s \leq r - s < t$ and $x \in (A)_s \cap (B)_t$. In either case we have $d(a, Q_s) \leq s$, which demonstrates that $d(A, Q_s) \leq s$.

The fact that $Q_s \subseteq (A)_s$ implies that $d(Q_s, A) \leq s$. Therefore, we have $h(A, Q_s) \leq s$. □

To demonstrate that there are infinitely many sets Q such that A, B, and Q form a Pythagorean triple with $\overline{AB}$ as hypotenuse, we will next show that we can remove a small neighborhood from Q_s without affecting the inequalities in Lemma 3.5.

Lemma 3.6. *Given the Pythagorean conditions, let $a \in A$ and $b \in B$ such that $d_E(a, b) = r$. Then $N_\epsilon(q)$ is in the interior of Q_s, where $\epsilon = \frac{1}{2}(s + t - r)$ and $q \in \overline{ab}$ with $d_E(q, b) = t - \epsilon$.*

Proof. It is not difficult to show that $r > t - \epsilon > 0$ and so q exists. Let $x \in N_\epsilon(q)$. Then

$$d_E(x, b) \leq d_E(x, q) + d_E(q, b) < \epsilon + (t - \epsilon) = t.$$

Now

$$d_E(a, q) = d_E(a, b) - d_E(b, q) = s - \epsilon,$$

so it follows that $d_E(x, a) < s$. □

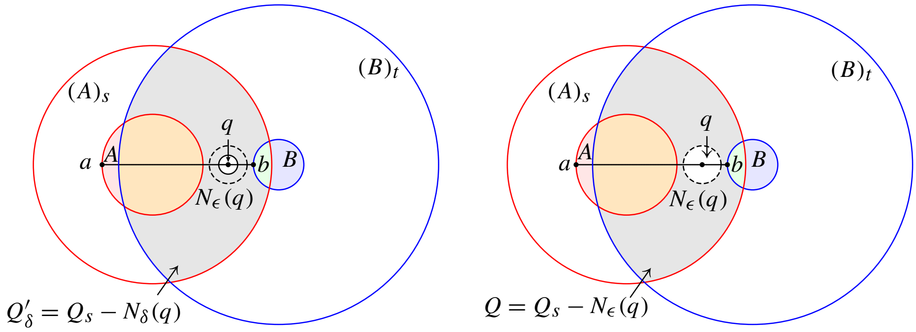

Figure 4. Left: $Q'_\delta = Q_s \setminus N_\delta(q)$. Right: $Q = Q_s \setminus N_\epsilon(q)$.

Lemma 3.7. *Given the Pythagorean conditions, let $a \in A$ and $b \in B$ such that $d_E(a, b) = r$, let $\epsilon = \frac{1}{2}(s+t-r)$, and let $q \in \overline{ab}$ with $d_E(q, b) = t$. Let $0 \leq \delta < \epsilon$ and let $Q'_\delta = Q_s \setminus N_\delta(q)$. Then $h(A, Q'_\delta) \leq s$ and $h(B, Q'_\delta) \leq t$.*

Proof. A picture illustrating the theorem is shown in Figure 4, left (where A is the disk centered at $(-3, 0)$ of radius 2 and B is the disk centered at $(2, 0)$ with radius 1 in $\mathbb{R}^2$). First note that $\epsilon > 0$. Also note that Q'_δ is closed and a subset of Q_s (that is, Q'_δ is just Q_s with a neighborhood around a point in its interior removed), so it is an element in $\mathcal{H}(\mathbb{R}^n)$. We know that $h(A, Q_s) \leq s$ by Lemma 3.5. Since $Q'_\delta \subseteq Q_s$ and $d(x, A) \leq s$ for every $x \in Q_s$, we have $d(x, A) \leq s$ for every $x \in Q'_\delta$. So $d(Q'_\delta, A) \leq d(Q_s, A) \leq s$. Now we show that $d(A, Q'_\delta) \leq s$.

Let $x \in A$. We consider two cases. First, suppose that $x \in N_\delta(q)$. Let q' be the point on the boundary of $\overline{N_\delta(q)}$ closest to x. Since Q'_δ is closed, it follows that $\partial \overline{N_\delta(q)} \subseteq Q'_\delta$. So $q' \in Q'_\delta$ and $d_E(x, q') \leq \delta \leq s$ (note that $t < r$ and so $s > \frac{1}{2}(s+t-r) = \epsilon$). Thus, $d(x, Q'_\delta) \leq s$. For the second case, assume that $x \notin N_\delta(q)$. Let $q_x \in Q_s$ such that $d_E(x, q_x) = d(x, Q_s) \leq h(A, Q_s) \leq s$. If $q_x \notin N_\delta(q)$, then $q_x \in Q'_\delta$ and $d(x, Q'_\delta) \leq s$. If $q_x \in N_\delta(q)$, let q' be the point on $\partial N_\delta(q) \cap \overline{xq_x}$. Then $d_E(x, q') < d_E(x, q_x) \leq s$ and $d(x, Q'_\delta) \leq s$. Therefore, $d(x, Q'_\delta) \leq s$ for every $x \in A$ and $d(A, Q'_\delta) \leq s$. A similar argument shows $h(B, Q'_\delta) \leq t$. □

Theorem 3.8 will demonstrate that, under certain conditions, we have infinitely many Pythagorean triples with $\overline{AB}$ as hypotenuse.

Theorem 3.8. *Given the Pythagorean conditions, let $a \in A$ and $b \in B$ such that $d_E(a, b) = r$, let $\epsilon = \frac{1}{2}(s+t-r)$, and let $q \in \overline{ab}$ with $d_E(q, b) = t$. Let $0 \leq \delta < \epsilon$ and let $Q'_\delta = Q_s \setminus N_\delta(q)$. If $Q_s \cap \partial(A)_s \neq \varnothing$ and $Q_s \cap \partial(B)_t \neq \varnothing$, then A, B, and Q'_δ form a Pythagorean triple with $h(A, Q'_\delta) = s$.*

Proof. An illustration of Theorem 3.8 is shown in Figure 4. Lemma 3.7 shows that $h(A, Q'_\delta) \leq s$. To verify that $h(A, Q'_\delta) = s$ we use the hypothesis that Q_s

contains a point $z \in \partial(A)_s$. Lemma 3.4 shows that $d(z, A) = s$. The proof of Lemma 3.7 demonstrated that if $x \in N_\delta(q)$, then $d(x, A) < s$. Therefore, $z \in Q'_\delta$ and $d(Q'_\delta, A) = s$. A similar argument shows that $h(Q'_\delta, B) = t$. □

Theorem 3.8 shows that under certain conditions, there are infinitely many sets Q'_δ such that A, B, and Q'_δ form a Pythagorean triple with $h(A, Q'_\delta)$ the same for every δ. In other words, there can be infinitely many different Pythagorean triples with a fixed $\overline{AB}$ as hypotenuse. The next question we address is if this is always the case. In other words, can we find Pythagorean triples with $\overline{AB}$ as hypotenuse if Q_s does not contain boundary points of $(A)_s$ or $(B)_t$?

Lemma 3.9 shows that we cannot have $Q_s \cap \partial(B)_t = \varnothing$, and helps us understand when $Q_s \cap \partial(A)_s \neq \varnothing$.

Lemma 3.9. *Assume the Pythagorean conditions*:

(1) *If* $0 < s < d(B, A)$, *then* $Q_s \cap \partial(A)_s \neq \varnothing$.

(2) $Q_S \cap \partial(B)_t \neq \varnothing$.

Proof. Assume $0 < s < d(B, A)$. Let $b \in B$ and $a \in A$ such that $d_E(b, a) = d(b, A) = d(B, A) > s$. Let $x \in \overline{ba}$ such that $d_E(a, x) = s$. Thus, $x \in (A)_s$ and $x \in (B)_t$, and $x \in Q_s$. Now let $z \in \overline{bx}$ with $d_E(z, x) > 0$. We will show that $z \notin (A)_s$. Suppose to the contrary that $z \in (A)_s$. Then there is an $a_z \in A$ with $d_E(z, a_z) \leq s$. Applying the triangle inequality along with the fact that $d_E(b, x) = d_E(b, a) - s = d(b, A) - s$ shows that

$$d_E(b, a_z) < d(b, A) - s + s = d(b, A),$$

which is impossible. We conclude that $z \notin (A)_s$ and so every neighborhood around x contains a point in $(A)_s$ and a point not in $(A)_s$. Therefore, $x \in \partial(A)_s$ and $Q_s \cap \partial(A)_s \neq \varnothing$.

The proof of the second assertion follows the same argument as part (1), noting that t is always between 0 and $r = d(A, B)$. □

Note that we cannot draw any conclusions about $Q_s \cap \partial(A)_s$ if $s \geq d(B, A)$, as the next examples illustrate.

Example 3.10. Let A be an annulus centered at the origin with inner radius r_A and outer radius R_A and let B be the single point set consisting of the point at the origin, as shown in Figure 5, left. Then $0 < d(B, A) = r_A < d(A, B) = R_A$. In this case, if $s \geq d(B, A)$, then $Q_s = (B)_t$ and $Q_s \cap \partial(A)_s = \varnothing$.

Example 3.11. Let $A = \{(0, 0), (1, 0)\}$ and $B = \{(-1, 0)\}$ in $\mathbb{R}^2$, as shown in Figure 5, right. Then $0 < d(B, A) = 1 < d(A, B) = 2$. In this case, $s \geq d(B, A)$ does not imply $Q_s \cap \partial(A)_s = \varnothing$. In fact, here we will have $Q_s \cap \partial(A)_s = \varnothing$ only when $1 + s > t$.

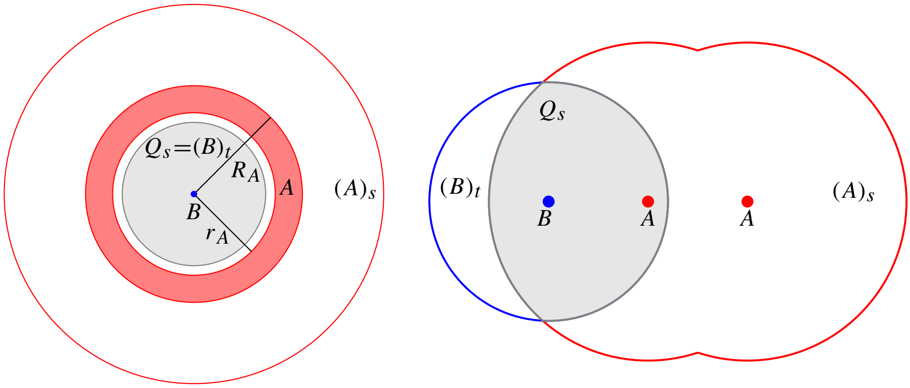

Figure 5. Left: $Q_s \cap \partial(A)_s = \varnothing$. Right: $Q_s \cap \partial(A)_s \neq \varnothing$.

As a consequence of Theorem 3.8 and Lemma 3.9, to determine if we can always find a Pythagorean triple with $\overline{AB}$ as hypotenuse we only have to consider the remaining case when $s \geq d(B, A)$ and $Q_s \cap \partial(A)_s = \varnothing$. The next theorem shows what happens when Q_s does not contain a boundary point of $(A)_s$.

Theorem 3.12. *Assume the Pythagorean conditions. If $s \geq d(B, A)$ and Q_s does not contain a boundary point of $(A)_s$, then $(B)_t \subseteq (A)_s$.*

Proof. Assume that $s \geq d(A, B)$ and that Q_s does not contain a boundary point of $(A)_s$. Now suppose to the contrary that $(B)_t \not\subseteq (A)_s$. Then there is a point $y \in (B)_t - (A)_s$. Since $y \in (B)_t$, there exists $b \in B$ such that $d_E(y, b) = d(y, B) \leq t$. Also, $b \in B$ implies there is a point $a \in A$ such that $d_E(b, a) = d(b, A) \leq d(B, A)$. Since $y \notin (A)_s$, we know that $d_E(y, a) > s$. Let $x \in \overline{ay}$ with $d_E(a, x) = s$.

Claim. $\overline{xy} \subseteq (\{b\})_{d_E(b,y)}$.

Proof of the claim. Let $z \in \overline{xy}$ and construct the triangles $\triangle ayb$ and $\triangle azb$ as shown in Figure 6. Let θ be the angle formed at point a. The law of cosines shows that

$$d_E(b, y)^2 = d_E(b, a)^2 + d_E(a, y)^2 - 2d_E(b, a)d_E(a, y)\cos\theta.$$

Substituting $d_E(a, z) + d_E(z, y)$ for $d_E(a, y)$ and using the law of cosines again yields

$$d_E(b, y)^2 = d_E(b, z)^2 + d_E(z, y)\big(2d_E(a, z) + d_E(z, y) - 2d_E(b, a)\cos\theta\big).$$

Now $d_E(a, z) \geq s \geq d(B, A) \geq d_E(b, a)$, so

$$\begin{aligned} 2d_E(a, z) + d_E(z, y) - 2d_E(b, a)\cos\theta &\geq 2s + d_E(z, y) - 2d_E(b, a)\cos\theta \\ &\geq 2d_E(b, a) + d_E(z, y) - 2d_E(b, a)\cos\theta \\ &= d_E(z, y) + 2d_E(b, a)(1 - \cos\theta) > 0. \end{aligned}$$

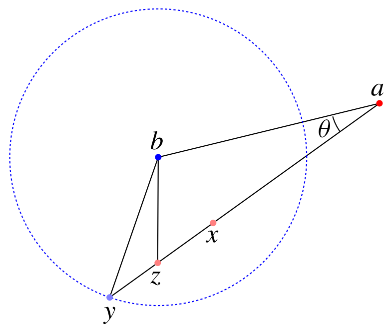

Figure 6. Showing $d_E(b, x) \le t$.

Therefore,

$$d_E(b, y)^2 > d_E(b, z)^2$$

and $d_E(b, z) < d_E(b, y)$ as desired, completing the proof of the claim. □

We proceed with the proof of Theorem 3.12. The set $G = (A)_s \cap \overline{xy}$ is the intersection of two compact sets and so is compact. Moreover, since $x \in (A)_s$, the set G is nonempty. Now let $w \in G$ such that $d_E(a, w) = \max_{g \in G}\{d_E(a, g)\}$. In other words, w is the point on $\overline{xy}$ in $(A)_s$ closest to y. Since $y \notin (A)_s$ we know that $w \neq y$. Therefore, no point in $\overline{yw}$ other than w can be in $(A)_s$. Thus, we have that $w \in \partial(A)_s$. Since $w \in (\{b\})_{d_E(b,y)}$ as well, we have found a point in $Q_s \cap \partial(A)_s$, a contradiction. We conclude that $(B)_t \subseteq (A)_s$. □

Theorem 3.8, Lemma 3.9, and Theorem 3.12 combine to leave one remaining case to consider to determine if we can always find a Pythagorean triple with $\overline{AB}$ as hypotenuse: when Q_s does not contain a boundary point of $(A)_s$, or $(B)_t \subseteq (A)_s$.

Theorem 3.13. *Assume the Pythagorean conditions. Let $a \in A$ and $b \in B$ such that $d_E(a, b) = d(a, B) = d(A, B) = r$. Let $Q = (B)_t \setminus N_s(a)$. If $(B)_t \subseteq (A)_s$, then $h(A, Q) = s$ and $h(B, Q) = t$ and the sets A, B, and Q form a Pythagorean triple with $\overline{AB}$ as hypotenuse.*

Proof. To show that A, B, and Q form a Pythagorean triple we need to know that Q is not empty, $h(A, Q) = s$, and $h(B, Q) = t$. First we show that Q is not empty.

Since $r > s$ there exists a $c \in \overline{ab}$ such that $d_E(a, c) = s$. Then $c \notin N_s(a)$. We also have

$$d_E(b, c) = r - s < t,$$

and $c \in (B)_t$. Therefore, $Q \neq \varnothing$.

Next we prove that $h(A, Q) = s$. The fact that $Q \subseteq (B)_t \subseteq (A)_s$ implies that $d(Q, A) \le s$.

Now we demonstrate that $d(A, Q) \le s$. Let $a' \in A$. There exists $b' \in B$ such that $d_E(a', b') = d(a', B) \le \max_{x \in A}\{d(x, B)\} = d(a, B) = r$. Note that $d_E(a', b') \le r = d(a, B) \le d_E(a, b')$. Since $d(a, B) = r$ and $N_s(a) \subseteq N_r(a)$, we know that $b' \notin N_s(a)$ and $b' \in Q$. If $d_E(a', b') \le s$, then $d(a', Q) \le s$. Now assume that $d_E(a', b') > s$. Let $c' \in \overline{a'b'}$ such that $d_E(a', c') = s$. Then

$$d_E(a, b') \le d_E(a, c') + d_E(c', b')$$

so

$$d_E(a, c') \ge d_E(a, b') - d_E(c', b') \ge d_E(a', b') - d_E(c', b') = d_E(a', c') = s.$$

Thus, $c' \notin N_s(a)$. Also,

$$d_E(b', c') = d_E(a', b') - d_E(a', c') \le r - s < t$$

and so $c' \in Q$. Therefore, $d(a', Q) \le s$. We conclude that $d(A, Q) \le s$.

Now, we show that $d(a, Q) = s$. Since $Q = (B)_t \setminus N_s(a)$, we see that $d(a, Q) \ge s$. Recall $c \in Q$ with $d_E(a, c) = s$. So $d(a, Q) = d_E(a, c) = s$. Therefore $d(A, Q) = s$ and $h(A, Q) = s$.

Next we prove that $h(B, Q) = t$. Recall that $Q \subseteq (B)_t \subseteq (A)_s$, so it follows that $d(Q, B) \le t$.

Now we demonstrate that $d(B, Q) \le t$. Let $b^* \in B$. There exists $a^* \in A$ such that $d_E(b^*, a^*) = d(b^*, A) \le r$. Note that $d_E(b^*, a^*) \le r = d_E(a, b) \le d_E(a, b^*)$. Since $N_s(a) \subseteq N_r(a)$, we know that $b^* \notin N_s(a)$ and $b^* \in Q$. So $d(b^*, Q) = 0$ and $d(B, Q) = 0 < t$.

Finally, we show that $d(Q, B) = t$. Let $W = \overrightarrow{ab} \cap (B)_t$ and let $w \in W$ such that $d_E(a, w)$ is a maximum. By definition, $w \in (B)_t$. Let $x \in \overrightarrow{bw}$ such that $d_E(b, x) = t$. (Note that $d_E(b, x) \le d_E(b, w)$.) Then $x \in W$ and so $d_E(a, w) \ge d_E(a, b) + d_E(b, x) = r + t > s$. Thus, $w \notin N_s(a)$ and $w \in Q$. If $w' \in \overrightarrow{ab}$ with $d_E(a, w') > d_E(a, w)$, then $w' \notin W$ and so $w' \notin (B)_t$. Thus, $w \in \partial(B)_t$. Lemma 3.4 shows that $d(w, B) = t$ and so $d(Q, B) = t$. □

Corollary 3.14. *Assume the Pythagorean conditions. If $(B)_t \subseteq (A)_s$, then there are infinitely many sets $Q \in \mathcal{H}(\mathbb{R}^n)$ that form a Pythagorean triple with A and B with $\overline{AB}$ as hypotenuse and $h(A, Q) = s$.*

Proof. Figure 7 will be a useful reference for this proof. Let $a \in A$ and $b \in B$ such that $d_E(a, b) = d(A, B) = h(A, B) = r$. Let $C = (B)_t \setminus N_s(a)$. Let $\mu = \min\left\{\frac{1}{2}(s + t - r), r - s\right\}$ and $q \in \overline{ab}$ such that $d_E(a, q) = s + \mu$. Thus, $q \notin N_s(a)$. Since $d_E(a, q) + d_E(q, b) = d_E(a, b) = r$ and $d_E(a, q) = s + \mu$, we have that

$$d_E(b, q) = d_E(a, b) - d_E(a, q) = r - (s + \mu) < t - \mu < t.$$

We conclude that $q \in N_t(b) \subset (B)_t$ and $q \notin N_s(a)$. It follows $q \in (B)_t \setminus N_s(a) = C$.

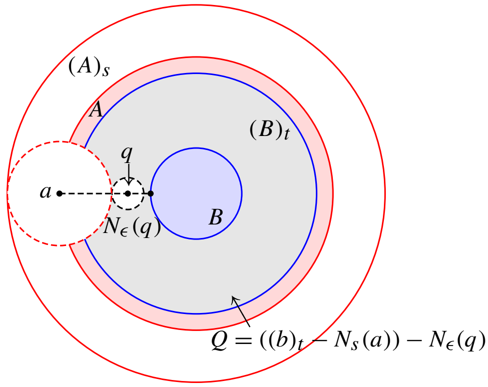

Figure 7. The shaded set is $Q = C - N_\epsilon(q)$.

Now we demonstrate that $N_\mu(q) \subseteq N_t(b) \subseteq (B)_t$. Let $z \in N_\mu(q)$. Two applications of the triangle inequality show that $d_E(a, z) > s$ and $d_E(b, z) < t$. It follows that $z \notin N_s(a)$ and $N_\mu(q) \cap N_s(a) = \varnothing$, and that $z \in N_t(b)$ and $N_\mu(q) \subseteq N_t(b) \subset (B)_t$. Now let $0 < \epsilon < \mu$ and let $Q = C - N_\epsilon(q)$. We will demonstrate that A, B, and Q form a Pythagorean triple.

Because Q is a closed subset of C, it is an element of $\mathcal{H}(\mathbb{R}^n)$. Theorem 3.13 shows that $h(A, C) = s$ and $h(B, C) = t$. Let $x \in C$. Since $Q \subseteq C$ and $d(x, A) \leq s$, we have that $d(Q, A) \leq d(C, A) \leq s$. Likewise, $d(x, B) \leq t$, so $d(Q, B) \leq d(C, B) \leq t$.

Let $x \in A$. As in the proof of Lemma 3.7, we can show that $d(x, Q) \leq s$ and $d(A, Q) \leq s$. We proceed to prove that $h(A, Q) = s$. Theorem 3.13 shows that for $c \in \overline{ab}$ with $d_E(a, c) = s$, we have $d_E(a, c) = d(a, C) = s$. Now

$$d_E(c, q) = d_E(a, q) - d_E(a, c) = (s + \mu) - s = \mu > \epsilon,$$

so $c \in Q$ and $d(A, Q) = s$.

The proof that $h(B, Q) = t$ is similar and is left to the reader. □

In summary, in this section we have demonstrated that for any distinct sets A and B in $\mathcal{H}(\mathbb{R}^n)$ and any $0 < s < h(A, B) = d(A, B)$, there are infinitely many sets Q such that A, B, and Q form a Pythagorean triple with $\overline{AB}$ as hypotenuse and $h(A, Q) = s$.

4. Projections

To continue our attempt to develop angle measure in $\mathcal{H}(\mathbb{R}^n)$, we return to Wildberger's approach. To measure *spread*, we will need to determine if, given two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in $\mathcal{H}(\mathbb{R}^n)$, we can find a set P on the ray $\overrightarrow{AC}$ that creates a Pythagorean triple with $\overline{AB}$ as hypotenuse. We will call such a set P a *projection* of B onto $\overleftrightarrow{AC}$.

The previous section shows that there are infinitely many Pythagorean triples to consider, but we need to know if there is a specific one that can be found on a given ray.

Unfortunately, Example 4.1 will demonstrate that our quest for projections will not always be successful.

Example 4.1. Let $A = \{a\}$, $B = \{b\}$ with $h(a, b) = d_E(a, b) = r > 0$, and let c be a point such that $\overline{ab} \perp \overline{ac}$. Let $q = d_E(a, c)$. Choose c_1 so that bac_1 and $\sqrt{r^2+q^2} - r < d_E(a, c_1) < q$ (since $q > \sqrt{r^2+q^2} - r$ we can find many such c_1), and let $C = \{c, c_1\}$. We will show that there is no set P that lies on $\overleftrightarrow{AC}$ and forms a Pythagorean triple with A and B.

Observe that $h(A, B) = d_E(a, b) = r$ and that $h(A, C) = d(C, A) = d_E(c, a) = q$. Note that

$$h(B, C) = d_E(c_1, b) = r + d_E(a, c_1) > \sqrt{r^2+q^2}$$

so A, B, and C themselves do not form a Pythagorean triple. Suppose to the contrary that there is a set P such that $P \in \overleftrightarrow{AC}$ and P forms a Pythagorean triple with A and B with hypotenuse $\overline{AB}$. Let $h(A, P) = s$, where $0 < s < r$. Then since P forms a Pythagorean triple with A and B, $h(B, P) = t = \sqrt{r^2 - s^2}$. First, we consider the case when P is between A and C, implying that $h(P, C) = q - s$ and $q > s$. We note that $P \subseteq (A)_s \cap (B)_t \cap (C)_{q-s}$. It follows that $P \subseteq (A)_s \cap (C)_{q-s}$. Let $W = (A)_s \cap (C)_{q-s}$. Lemma 1.7 shows that W is between A and C. Let $w \in \overline{ac}$ such that $d_E(a, w) = s$. Then $d_E(w, c) = d_E(a, c) - d_E(a, w) = q - s$ and $w \in W$.

Now we will show that $w \in P$. Since $(A)_s$ is a disk of radius s, the only point of $(A)_s$ that lies on $\overline{ac}$ that is a distance s from a is w. Let $a_s \in (A)_s$ such that $a_s \neq w$. Then $d_E(a_s, a) \leq s$ and the triangle inequality shows that

$$d_E(c, a_s) \geq d_E(a, c) - d_E(a_s, a) \geq q - s.$$

Since $a_s \neq w$, we must have $d_E(c, a_s) > q - s$. Now $d(C, P) \geq d(c, P)$ and $d(c, P)$ is achieved at some point $p \in P \subseteq (A)_s$. If $p \neq w$, then

$$q - s = h(C, P) \geq d(C, P) = d_E(c, p) > q - s.$$

We conclude that $p = w$ and $w \in P$.

Since $w \in P$, we must have that $w \in (B)_t = (\{b\})_t$ as well. The points a, b, and w form a right triangle as in Figure 8, with $d_E(b, w) = \sqrt{r^2+s^2}$. Since $w \in (B)_t$ it follows that $d_E(b, w) \leq t$. Recall that $t = \sqrt{r^2 - s^2}$, so $d_E(b, w) \leq \sqrt{r^2 - s^2}$. But this makes $\sqrt{r^2 - s^2} \geq \sqrt{r^2 + s^2}$, which is impossible since $s > 0$. Thus, no set P exists such that P forms a Pythagorean triple with A and B, and P is between A and C.

For the second case, suppose that such a P exists such that A is between P and C and $h(A, P) = s$. In this case we have $h(C, P) = q + s$ and $P \subseteq (A)_s \cap (B)_t \cap (C)_{q+s}$. Again, we begin by examining the characteristics of $(A)_s \cap (C)_{q+s}$.

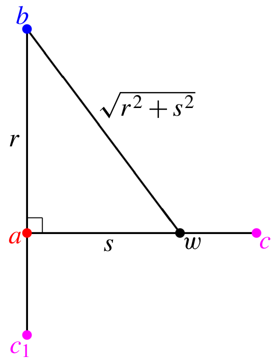

Figure 8. A right triangle with vertices a, b, and w.

Let $a_s \in (A)_s$. Then $d_E(a, a_s) \le s$. Since $d_E(c, a) = q$, by the triangle inequality we have

$$d_E(c, a_s) \le d_E(c, a) + d_E(a, a_s) \le q + s.$$

It follows that $a_s \in (C)_{q+s}$ and $(A)_s \subseteq (C)_{q+s}$, so $P \subseteq (A)_s \cap (B)_t$.

Let $w \in \overleftrightarrow{ac}$ with wac and $d_E(a, w) = s$. Now we will show that $w \in P$. Note that $d(A, C) = d_E(a, c_1) < d_E(a, c) = q$. Let $w' \in P$. Since $P \subseteq (A)_s$ we know that $w' \in (A)_s$ and we must have $d_E(w', a) \le s$. By the triangle inequality,

$$d_E(w', c_1) \le d_E(a, c_1) + d_E(a, w') < q + s.$$

Thus, $d(w', C) < q + s$ and $d(P, C) < q + s$. Now let us turn to $d(C, P)$.

The above argument shows that $d_E(c_1, w') < q + s$, from which it follows that $d(c_1, P) < q + s$. From the triangle inequality we can see that

$$d_E(c, w') \le d_E(c, a) + d_E(a, w') \le q + s.$$

Note that if $d_E(c, w') = q + s$ then $w' \in \overleftrightarrow{ac}$, $d_E(a, w') = s$, and $w'ac$. This forces $w' = w$. So in order to have $h(P, C) = q + s$ we must have $w \in P$.

Since $w \in P$ and $P \subseteq (B)_t$, we have that $w \in (B)_t$ as well. The points a, b, w form a right triangle as in Figure 8. We can see that $d_E(b, w) = \sqrt{r^2 + s^2}$, but $d_E(b, w) \le t = \sqrt{r^2 - s^2}$. This forces $\sqrt{r^2 - s^2} \ge \sqrt{r^2 + s^2}$, which is impossible since $s > 0$. Therefore, there does not exist a set P that forms a Pythagorean triple with A and B such that A is between P and C.

The remaining case where P satisfies ACP is similar and is left to the reader.

Therefore, there exists no set $P \in \overleftrightarrow{AC}$ that forms a Pythagorean triple with A and B such that $\overline{AB}$ is a hypotenuse.

Example 4.1 illustrates the extreme case that there is no set P that lies on $\overleftrightarrow{AB}$ that makes a Pythagorean triple with A and B with $\overline{AB}$ as hypotenuse. We now

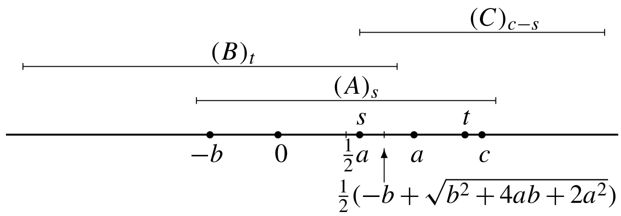

Figure 9. Infinitely many sets P.

show that the other extreme is possible — that there can be infinitely many sets P that satisfy both the betweenness and Pythagorean triple conditions.

Example 4.2. Let $A = \{0, a\}$, $B = \{-b\}$, and $C = \{c\}$ in $\mathcal{H}(\mathbb{R})$, where $a, b, c > 0$ and $c > a$. First we note that $h(A, B) = d(A, B) = a+b$ and $h(A, C) = d(A, C) = c$. Since $a, b > 0$, a little algebra shows that

$$\tfrac{1}{2}a < \tfrac{1}{2}(-b + \sqrt{b^2 + 4ab + 2a^2}).$$

So for any $a, b > 0$ the interval between $\frac{1}{2}a$ and $\frac{1}{2}(-b + \sqrt{b^2 + 4ab + 2a^2})$ is not empty. Let

$$\tfrac{1}{2}a \le s \le \tfrac{1}{2}(-b + \sqrt{b^2 + 4ab + 2a^2})$$

and assume in addition that $c \ge s + \frac{1}{2}a$. Let $t = \sqrt{h(A, B)^2 - s^2} = \sqrt{(a+b)^2 - s^2}$. It follows that

$$(A)_s = [-s, a+s], \quad (B)_t = [-b-t, t-b], \quad \text{and} \quad (C)_{c-s} = [s, 2c-s].$$

See Figure 9 for an illustration. Now let us determine $(A_s) \cap (C)_{c-s}$. The fact that $c \ge s + \frac{1}{2}a$ implies that $2c - s \ge s + a$, and so

$$(A)_s \cap (C)_{c-s} = [s, a+s].$$

Now let $P_s = (A)_s \cap (B)_t \cap (C)_{c-s}$. We will show that $s \le t - b \le a + s$, which means that $P_s = [s, t-b]$.

Let $y = 2s^2 + 2bs - (2ab + a^2)$. Using the quadratic formula and the fact that $\frac{1}{2}(-b - \sqrt{b^2 + 4ab + 2a^2}) < 0 < \frac{1}{2}a$, we have $\frac{1}{2}a \le s \le \frac{1}{2}(-b + \sqrt{b^2 + 4ab + 2a^2})$ and

$$s \le t - b \tag{1}$$

when $\frac{1}{2}a \le s \le \frac{1}{2}(-b + \sqrt{b^2 + 4ab + 2a^2})$. Since $s \le t - b$ it follows that $[s, t-b]$ is not empty.

The fact that $t = \sqrt{(a+b)^2 - s^2} \le a + b$ implies that

$$t - b \le a, \tag{2}$$

and since $s > 0$, we have $t - b < a + s$. Thus we can conclude that $P_s = [s, t - b]$ when $\frac{1}{2}a \leq s \leq \frac{1}{2}(-b + \sqrt{b^2 + 4ab + 2a^2})$.

Now we check that the set P_s satisfies $h(A, B)^2 = h(B, P_s)^2 + h(A, P_s)^2$ and $h(A, P_s) + h(P_s, C) = h(A, C)$.

First, we calculate $h(A, P_s)$. We know $P_s \subseteq (A)_s$, which implies $d(P_s, A) \leq s$. Inequalities (1) and (2) show that $d(A, P_s) = \max\{d_E(0, s), d_E(a, t - b)\}$. We know $d_E(0, s) = s$ and $d_E(a, t - b) = a - (t - b)$. Inequality (1) implies

$$s \geq a - s \geq a - (t - b),$$

and so $d_E(a, t - b) \leq s = d_E(0, s)$. Thus, $h(A, P_s) = s$.

Now we determine $h(P_s, B)$. Note that since $t - b \geq s$ we have $d(B, P_s) = d_E(-b, s) \leq d(P_s, B) = d_E(t - b, -b) = t$. Therefore $h(B, P_s) = t$.

Next, we find $h(C, P_s)$. Here we have $d(C, P_s) = d_E(c, t - b) \leq d(P_s, C) = d_E(s, c) = c - s$. Therefore, $h(C, P_s) = c - s$.

Because $h(A, P_s) + h(C, P_s) = s + (c - s) = c = h(A, C)$, we have AP_sC. Additionally, notice that $h(A, P_s)^2 + h(B, P_s)^2 = s^2 + t^2 = s^2 + (a + b)^2 - s^2 = (a+b)^2 = h(A, B)^2$. Therefore, $P_s \subseteq (A)_s \cap (B)_t \cap (C)_{c-s}$ lies between A and C and forms a Pythagorean triple with A and B for $\frac{1}{2}a \leq s \leq \frac{1}{2}(-b + \sqrt{b^2 + 4ab + 2a^2})$.

Example 4.2 shows us that there are infinitely many sets A, B, C such that there exists an infinite number of sets P_s that lie between A and C and form a Pythagorean triple with A and B. It turns out that each of the sets P_s from Example 4.2 lies on the same ray $\overrightarrow{AC}$ (the proof is left to the reader).

The previous examples demonstrate that the behavior of Pythagorean triples in $\mathcal{H}(\mathbb{R}^n)$ is quite different than from those in $\mathbb{R}^n$. The situation in $\mathcal{H}(\mathbb{R}^n)$ is even stranger than we have already seen, as our final example illustrates.

Example 4.3. Let $A = \{0\}$, $B = \{-b\}$, $C = \{c\}$, for some $b, c > 0$. It is straightforward to see that BAC. For $0 < s < b$, let $t = \sqrt{b^2 - s^2}$ such that $(A)_s = [-s, s]$, $(B)_t = [-b - t, -b + t]$, and $(C)_{c+s} = [-s, 2c + s]$. Calculations similar to those in Example 4.2 show that $P_s = (A)_s \cap (B)_t \cap (C)_{c+s} = [-s, -b + t]$, that P_s satisfies P_sAC, and that P_s forms a Pythagorean triple with A and B. Unlike in Example 4.2 where P_s was defined only in a restricted subinterval of $(0, h(A, B))$, in this situation we can use any value for s from 0 up to $h(A, B)$. Thus not only can we form a projection from a line to itself, we can do so for every value in the interval $[0, h(A, B)]$.

5. Conclusions

While Euclidean geometry is embedded in the Hausdorff metric geometry as single point sets, the Hausdorff metric geometry is quite different. As we have seen, there can be infinitely many different sets at the same location that form a Pythagorean triple with given sets A and B and hypotenuse $\overline{AB}$. Unfortunately, our attempt to measure angles in $\mathcal{H}(\mathbb{R}^n)$ using Pythagorean orthogonality to determine spread does not work in general since we cannot always project a given set onto a given ray (even though we can do this in infinitely many different ways in other cases). It may be that a different notion of orthogonality will allow us to proceed. For example, in Euclidean geometry the segment $\overline{ab}$ is orthogonal to the line ℓ that contains b if the distance from a to any point on ℓ is a minimum. Defining orthogonality in $\mathcal{H}(\mathbb{R}^n)$ in terms of minimum distances might provide different results.

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pallavi054@yahoo.com *California Institute of Technology, Pasadena, CA, United States*

schlicks@gvsu.edu *Department of Mathematics, Grand Valley State University, Allendale, MI, United States*

rdswartzentruber@gmail.com *Eastern Mennonite University, Harrisonburg, VA, United States*

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