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Jason DeVito and Wesley Martin

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Quasipositive curvature on a biquotient of Sp(3)

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Suppose $\phi_3 : \mathrm{Sp}(1) \to \mathrm{Sp}(2)$ denotes the unique irreducible complex 4-dimensional representation of $\mathrm{Sp}(1) = \mathrm{SU}(2)$, and consider the two subgroups $H_i \subseteq \mathrm{Sp}(3)$ with $H_1 = \{\mathrm{diag}(\phi_3(q_1), q_1) : q_1 \in \mathrm{Sp}(1)\}$ and $H_2 = \{\mathrm{diag}(\phi_3(q_2), 1) : q_2 \in \mathrm{Sp}(1)\}$. We show that the biquotient $H_1 \backslash \mathrm{Sp}(3)/H_2$ admits a quasipositively curved Riemannian metric.

1. Introduction

Manifolds of positive sectional curvature have been studied extensively. Despite this, there are very few known examples of positively curved manifolds. In fact, other than spheres and projective spaces, every known compact simply connected manifold admitting a metric of positive curvature is diffeomorphic to an Eschenburg space [Eschenburg 1982; Aloff and Wallach 1975], Eschenburg's inhomogeneous flag manifold, the projectivized tangent bundle of $\mathbb{K}P^2$ with $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ [Wallach 1972], a Bazaikin space [Barden 1965], the Berger space [1961], or a certain cohomogeneity one manifold which is homeomorphic, but not diffeomorphic, to T^1S^4 [Dearricott 2011; Grove et al. 2011].

Because of the difficulty in constructing new examples, attention has turned to the easier problem of finding examples with quasi- or almost positive curvature. Recall that a Riemannian manifold is said to be quasipositively curved if it admits a nonnegatively curved metric with a point p for which the sectional curvatures of all 2-planes at p are positive. A Riemannian manifold is called almost positively curved if the set of points for which all 2-planes are positively curved is dense. Examples of manifolds falling into either of these cases are more abundant. See [DeVito et al. 2014; Dickinson 2004; Eschenburg and Kerin 2008; Gromoll and Meyer 1974; Kerin 2011; 2012; Kerr and Tapp 2014; Petersen and Wilhelm 1999; Tapp 2003; Wilhelm 2001; Wilking 2002].

In [DeVito et al. 2014], the first author, together with DeYeso, Ruddy, and Wesner, proves that there are precisely 15 biquotients of the form $\mathrm{Sp}(3)/\!\!/\,\mathrm{Sp}(1)^2$ and show

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that eight of them admit quasipositively curved metrics. We show that their methods can be adapted to work on a ninth example, called N_9 in [DeVito et al. 2014]. That is, we show N_9 admits a metric of quasipositive curvature as well.

To describe this example, we first set up notation. Let $\phi_3 : \mathrm{Sp}(1) = \mathrm{SU}(2) \to \mathrm{Sp}(2)$ denote the unique irreducible complex 4-dimensional representation of $\mathrm{Sp}(1)$. Further, let $G = \mathrm{Sp}(3)$, and let $H_1 = \{\mathrm{diag}(\phi_3(q_1), q_1) \in G : q_1 \in \mathrm{Sp}(1)$ and $H_2 = \{\mathrm{diag}(\phi_3(q_2), 1) \in G : q_2 \in \mathrm{Sp}(1)\}$. Finally, set $H = H_1 \times H_2 \subseteq G \times G$.

Theorem 1.1. *The biquotient $H_1 \backslash G / H_2$ admits a metric of quasipositive curvature.*

In fact, we show the metric constructed on G in [DeVito et al. 2014] is H invariant and the induced metric on N_9 is quasipositively curved.

Finally, we point out that one of the first steps in the proof, Proposition 2.3, does not hold for any of the remaining inhomogeneous biquotients of the form $\mathrm{Sp}(3) /\!\!/ \mathrm{Sp}(1)^2$. In particular, a new approach is needed to determine whether these other biquotients admit metrics of quasipositive curvature.

The outline of this paper is as follows. Section 2 will cover the necessary background, leading to a system of equations parameterized by $p \in G$, which govern the existence of a zero curvature plane at $[p^{-1}] \in G /\!\!/ H$. In Section 3, we find a particular point $p \in G$ for which there are no nontrivial solutions to the system of equations, establishing Theorem 1.1.

2. Background

We will use the setup of [DeVito et al. 2014]. As the calculations will be done on the Lie algebra level, we now describe all the relevant Lie algebras.

We recall the Lie algebra $\mathfrak{sp}(n)$ consists of all $n \times n$ quaternionic skew-Hermitian matrices with Lie bracket given by the commutator. That is, $\mathfrak{sp}(n) = \{A \in M_n(\mathbb{H}) : A + \bar{A}^t = 0\}$, where $\mathbb{H}$ denotes the skew-field of quaternions, and the Lie bracket is given by $[A, B] = AB - BA$. When $n = 1$, this Lie algebra is simply $\mathrm{Im}\,\mathbb{H}$.

Then the Lie algebra of $G = \mathrm{Sp}(3)$, denoted $\mathfrak{g} = \mathfrak{sp}(3)$, consists of the 3×3 skew-Hermitian matrices over $\mathbb{H}$. Further, we set $K = \mathrm{Sp}(2) \times \mathrm{Sp}(1)$, block diagonally embedded into G via $(A, q) \mapsto \mathrm{diag}(A, q) \in G$. Then one easily sees that $\mathfrak{k} = \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ is embedded into $\mathfrak{g}$ via $(B, r) \mapsto \mathrm{diag}(B, r)$.

We also use the description of ϕ_3 on the Lie algebra level given by [DeVito et al. 2014, Proposition 4.5].

Proposition 2.1. *For $t = t_i + t_j + t_k \in \mathrm{Im}\,\mathbb{H} = \mathfrak{sp}(1)$,*

$$\phi_3(t) = \begin{bmatrix} 3t_i & \sqrt{3}(t_j + t_k) \\ \sqrt{3}(t_j + t_k) & 2(t_k - t_j) - t_i \end{bmatrix}$$

defines the unique irreducible 4-dimensional representation of $\mathfrak{sp}(1) = \mathfrak{su}(2)$.

It follows that, for $H_1 = \{\operatorname{diag}(\phi_3(q_1), q_1) : q_1 \in \mathrm{Sp}(1)\} \subseteq \mathrm{Sp}(3)$,

$$\mathfrak{h}_1 = \left\{ \begin{bmatrix} 3t_i & \sqrt{3}(t_j+t_k) & \\ \sqrt{3}(t_j+t_k) & 2(t_k-t_j)-t_i & \\ & & t \end{bmatrix} : t \in \operatorname{Im}\mathbb{H} \right\}.$$

Likewise, for $H_2 = \{\operatorname{diag}(\phi_3(q_2), 1) : q_2 \in \mathrm{Sp}(1)\} \subseteq G$, we have

$$\mathfrak{h}_2 = \left\{ \begin{bmatrix} 3s_i & \sqrt{3}(s_j+s_k) & \\ \sqrt{3}(s_j+s_k) & 2(s_k-s_j)-s_i & \\ & & 0 \end{bmatrix} : s \in \operatorname{Im}\mathbb{H} \right\}.$$

The metric we will use is constructed in [DeVito et al. 2014] via a combination of Cheeger deformations [1973] and Wilking's doubling trick [2002]. More specifically, we let g_0 denote the bi-invariant metric on G with $g_0(X, Y) = -\operatorname{Re}\operatorname{Tr}(XY)$ for $X, Y \in \mathfrak{g}$. We let g_1 denote the left G-invariant, right K-invariant metric obtained by Cheeger deforming g_0 in the direction of K. That is, g_1 is the metric induced on G by declaring the canonical submersion $(G\times K, g_0+g_0|_K) \to G$ with $(p, k) \mapsto pk^{-1}$ to be a Riemannian submersion.

We now equip $G\times G$ with the metric g_1+g_1 and consider the isometric action of $G\times H_1\times H_2$ on $G\times G$ given by $(p, h_1, h_2) * (p_1, p_2) = (pp_1h_1^{-1}, pp_2h_2^{-1})$. This action is free and induces a metric on the orbit space $\Delta G\backslash(G\times G)/(H_1\times H_2)$.

Following Eschenburg [1984], the orbit space $\Delta G\backslash(G\times G)/(H_1\times H_2)$ is canonically diffeomorphic to the biquotient $H_1\backslash G/H_2$, which is called N_9 in [DeVito et al. 2014]. To see this, one verifies that the map $G\times G\to G$, sending (p_1, p_2) to $p_1^{-1}p_2$, descends to a diffeomorphism of the orbit spaces. We use this diffeomorphism to transport the submersion metric on $\Delta G\backslash(G\times G)/(H_1\times H_2)$ to $H_1\backslash G/H_2$ and let g_2 denote this metric on $H_1\backslash G/H_2$.

We note that since g_0 is bi-invariant, it is nonnegatively curved. It follows from O'Neill's formula [1966] that g_1 and g_2 are nonnegatively curved as well.

We now describe the points having 0-curvature planes in $(H_1\backslash G/H_2, g_2)$. To do this, we let

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ -\bar{z}_1 & -\bar{z}_2 & 0 \end{bmatrix} : z_1, z_2 \in \mathbb{H} \right\} \subseteq \mathfrak{g}$$

denote the g_0-orthogonal complement of $\mathfrak{k}$: $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$. Then, for $X \in \mathfrak{g}$ we can write it as $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$, where $X_{\mathfrak{k}}$ is the projection of X onto $\mathfrak{k}$, and similarly for $X_{\mathfrak{p}}$. We also let $\mathrm{Ad}_p : \mathfrak{g}\to\mathfrak{g}$ denote the adjoint map $\mathrm{Ad}_p(X) = pXp^{-1}$. Then, as shown in [DeVito et al. 2014, Corollary 2.8], we have the following description of points $[p^{-1}] \in H_1\backslash G/H_2$ containing 0-curvature planes.

Theorem 2.2. *There is a* 0*-curvature plane at* $[p^{-1}] \in (H_1 \backslash G/H_2, g_2)$ *if and only if there are linearly independent vectors* $X, Y \in \mathfrak{g}$ *satisfying the following equations*:

(A) $g_0(X, \mathrm{Ad_p}\,\mathfrak{h}_1) = g_0(X, \mathfrak{h}_2) = g_0(Y, \mathrm{Ad_p}\,\mathfrak{h}_1) = g_0(Y, \mathfrak{h}_2) = 0$,

(B) $[X, Y] = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = 0$,

(C) $[(\mathrm{Ad_{p^{-1}}}\,X)_{\mathfrak{k}}, (\mathrm{Ad_{p^{-1}}}\,Y)_{\mathfrak{k}}] = [(\mathrm{Ad_{p^{-1}}}\,X)_{\mathfrak{p}}, (\mathrm{Ad_{p^{-1}}}\,Y)_{\mathfrak{p}}] = 0$.

It is clear from inspecting these equations that if $\mathrm{span}\{X, Y\} = \mathrm{span}\{X', Y'\}$, then X and Y satisfy all three conditions if and only if X' and Y' do.

We also note that there is some redundancy in these equations because (G, K) is a symmetric pair. Specifically, assuming $[X, Y] = 0$, it follows that $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$ if and only if $[X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = 0$ and also that $[(\mathrm{Ad_{p^{-1}}}\,X)_{\mathfrak{k}}, (\mathrm{Ad_{p^{-1}}}\,Y)_{\mathfrak{k}}] = 0$ if and only if $[(\mathrm{Ad_{p^{-1}}}\,X)_{\mathfrak{p}}, (\mathrm{Ad_{p^{-1}}}\,Y)_{\mathfrak{p}}] = 0$. To see this, we first note that $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$ for a symmetric pair (G, K). Using the fact that $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, we see that $[X, Y]_{\mathfrak{k}} = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] + [X_{\mathfrak{p}}, Y_{\mathfrak{p}}]$. Since condition (B) forces $[X, Y]_{\mathfrak{k}} = 0$, we see that $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$ if and only if $[X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = 0$. To get the result for the vectors $\mathrm{Ad_{p^{-1}}}\,X$ and $\mathrm{Ad_{p^{-1}}}\,Y$, we note that $\mathrm{Ad_{p^{-1}}} : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra isomorphism, so $[X, Y] = 0$ if and only if $[\mathrm{Ad_{p^{-1}}}\,X, \mathrm{Ad_{p^{-1}}}\,Y] = 0$.

We now show that for many $p \in \mathrm{Sp}(3)$, if X and Y satisfy conditions (A) and (B) of Theorem 2.2, then we may replace X and Y with X', Y' having a nice form.

Proposition 2.3. *Let* $\rho : \mathfrak{g} \to \mathrm{Im}\,\mathbb{H}$ *with* $\rho(Z) = Z_{33}$, *the entry of* Z *in the last row and last column. Suppose* $[p^{-1}] \in G/\!\!/H$ *is a point for which* $\rho|_{\mathrm{Ad_p}\,\mathfrak{h}_1}$ *is surjective. If* $X, Y \in \mathfrak{g}$ *satisfy conditions (A) and (B) of Theorem 2.2 at the point* $[p^{-1}]$, *then there are vectors* $X', Y' \in \mathfrak{g}$ *with* $\mathrm{span}\{X, Y\} = \mathrm{span}\{X', Y'\}$ *and* $X'_{\mathfrak{p}} = Y'_{\mathfrak{sp}(2)} = 0$, *where* $Y'_{\mathfrak{sp}(2)}$ *denotes the projection of* Y' *to* $\mathfrak{sp}(2) \oplus 0 \subseteq \mathfrak{k} \subseteq \mathfrak{g}$.

Proof. We start with the equation $[X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = 0$ from condition (B). Since we can identify $\mathfrak{p}$ with $T_{[eK]}G/K$, where $G/K = \mathbb{H}P^2$ has positive sectional curvature, it follows that $[X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = 0$ if and only if $X_{\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ are dependent. Thus, either $X_{\mathfrak{p}} = 0$ and $X = X'$ or $X_{\mathfrak{p}} = \lambda Y_{\mathfrak{p}}$ for some real number λ. Then $X' = \lambda X - Y$ has no $\mathfrak{p}$ part. We may thus assume without loss of generality that X has no $\mathfrak{p}$ part.

Since $\mathrm{Sp}(2) \times \{I\}$ is an ideal in $K = \mathrm{Sp}(2) \times \mathrm{Sp}(1)$, the condition $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$ implies $[X_{\mathfrak{sp}(2)}, Y_{\mathfrak{sp}(2)}] = 0$. By condition (A), we know $g_0(X, \mathfrak{h}_2) = g_0(Y, \mathfrak{h}_2) = 0$, so we may interpret $X_{\mathfrak{sp}(2)}$ and $Y_{\mathfrak{sp}(2)}$ as tangent vectors on $\mathrm{Sp}(2)/\phi_3(\mathrm{Sp}(1))$. But, $\mathrm{Sp}(2)/\phi_3(\mathrm{Sp}(1))$ is the Berger space [1961] and is known to admit a normal homogeneous metric of positive curvature. So we see that $[X_{\mathfrak{sp}(2)}, Y_{\mathfrak{sp}(2)}] = 0$ if and only if $X_{\mathfrak{sp}(2)}$ and $Y_{\mathfrak{sp}(2)}$ are linearly dependent.

If $X_{\mathfrak{sp}(2)} = 0$, then the only nonvanishing entry of X is X_{33}. Since, by assumption, $\rho|_{\mathrm{Ad_p}\,\mathfrak{h}_1}$ is surjective, the condition $g_0(X, \mathrm{Ad_p}\,\mathfrak{h}_1) = 0$ forces $X = 0$, contradicting the fact that $\{X, Y\}$ is linearly independent. Hence, we may assume $X_{\mathfrak{sp}(2)} \neq 0$.

Then, we may subtract an appropriate multiple of X from Y to obtain a new vector Y' with $Y'_{\mathfrak{sp}(2)} = 0$. $\square$

We now work out conditions (A), (B), and (C) of Theorem 2.2 more explicitly at a point of the form

$$p = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

We will always assume $\theta \in \left(0, \frac{1}{4}\pi\right)$. Also, we will often identify $\mathfrak{p}$, consisting of matrices of the form

$$\begin{bmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ -\bar{z}_1 & -\bar{z}_2 & 0 \end{bmatrix},$$

with $\mathbb{H}^2$ via the canonical $\mathbb{R}$-linear isomorphism mapping such a matrix to $\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$.

We note that for points of this form, $\rho|_{\mathrm{Ad}_p \mathfrak{h}_1}$ has an image consisting of all elements of $\operatorname{Im}\mathbb{H}$ of the form $3\sin^2\theta\, t_i + \cos^2\theta\, t$ for $t = t_i + t_j + t_k \in \operatorname{Im}\mathbb{H}$. Since $\cos^2\theta \neq 0$ because $\theta \in \left(0, \frac{1}{4}\pi\right)$, this map has no kernel, so it is surjective. In particular, the conditions of Proposition 2.3 are verified at all such p, and thus, we may assume

$$X = \begin{bmatrix} x_1 & x_2 & 0 \\ -\bar{x}_2 & x_3 & 0 \\ 0 & 0 & x_4 \end{bmatrix}$$

with $x_1, x_3, x_4 \in \operatorname{Im}\mathbb{H}$ and $x_2 \in \mathbb{H}$. Similarly, we may assume

$$Y = \begin{bmatrix} 0 & 0 & y_1 \\ 0 & 0 & y_2 \\ -\bar{y}_1 & -\bar{y}_2 & y_3 \end{bmatrix}$$

with $y_1, y_2 \in \mathbb{H}$ and $y_3 \in \operatorname{Im}\mathbb{H}$

Lemma 2.4. *For a point p of the above form and $X, Y \in \mathfrak{g}$, conditions (A), (B), and (C) of Theorem 2.2 are equivalent to the following list of conditions*:

$$x_1 y_1 + x_2 y_2 - y_1 x_4 = 0, \tag{1}$$

$$-\bar{x}_2 y_1 + x_3 y_2 - y_2 x_4 = 0, \tag{2}$$

$$\{x_4, y_3\} \text{ is linearly dependent over } \mathbb{R}. \tag{3}$$

For

$$v = \begin{bmatrix} \cos\theta \sin\theta (x_1 - x_4) \\ -\sin\theta\, \bar{x}_2 \end{bmatrix} \in \mathbb{H}^2 \cong \mathfrak{p}$$

and

$$w = \begin{bmatrix} \mathrm{Re}(y_1) + (\cos^2\theta - \sin^2\theta)\,\mathrm{Im}(y_1) - \sin\theta\cos\theta\; y_3 \\ \cos\theta\; y_2 \end{bmatrix} \in \mathbb{H}^2 \cong \mathfrak{p},$$

the following hold:

$$\text{the set } \{v, w\} \text{ is linearly dependent over } \mathbb{R}, \tag{4}$$

$$3(x_1)_i - (x_3)_i = 0, \tag{5_i}$$

$$\sqrt{3}(x_2)_j - (x_3)_j = 0, \tag{5_j}$$

$$\sqrt{3}(x_2)_k + (x_3)_k = 0, \tag{5_k}$$

$$(x_1)_i(-2\sin^2\theta) + (x_4)_i(1 + 2\sin^2\theta) = 0, \tag{6_i}$$

$$(x_2)_j(\cos\theta - 1)2\sqrt{3} + (x_1)_j\sin^2\theta + (x_4)_j\cos^2\theta = 0, \tag{6_j}$$

$$(x_2)_k(\cos\theta - 1)2\sqrt{3} + (x_1)_k\sin^2\theta + (x_4)_k\cos^2\theta = 0, \tag{6_k}$$

$$-4\sin\theta\cos\theta\,(y_1)_i + (2\sin^2\theta + 1)(y_3)_i = 0, \tag{7_i}$$

$$2\sin\theta\cos\theta\,(y_1)_j - 2\sqrt{3}\sin\theta\,(y_2)_j + \cos^2\theta\,(y_3)_j = 0, \tag{7_j}$$

$$2\sin\theta\cos\theta\,(y_1)_k - 2\sqrt{3}\sin\theta\,(y_2)_k + \cos^2\theta\,(y_3)_k = 0. \tag{7_k}$$

Proof. We first claim that condition (A) is equivalent to (5_i) through (7_k). To begin with, we note that since $Y_{\mathfrak{sp}(2)} = 0$ and $\mathfrak{h}_2 \subseteq \mathfrak{sp}(2) \oplus 0 \subseteq \mathfrak{k}$, the equation $g_0(Y, \mathfrak{h}_2) = 0$ is automatically satisfied.

Now, a calculation shows that for $s = s_i + s_j + s_k \in \mathrm{Im}\,\mathbb{H}$,

$$0 = g_0(X, \mathfrak{h}_2) = 3s_i x_1 + 2\sqrt{3}(s_j + s_k)\,\mathrm{Im}(x_2) + (2(s_k - s_j) - s_i)x_3.$$

Then, using each of $s = i$, $s = j$, and $s = k$ respectively gives (5_i), (5_j), (5_k) which, using linearity, are therefore equivalent to the condition that $g_0(X, \mathfrak{h}_2) = 0$.

Further, with $t = t_i + t_j + t_k \in \mathrm{Im}\,\mathbb{H}$, we compute

$$\mathrm{Ad}_p\,\mathfrak{h}_1 = \left\{ \begin{bmatrix} 3\cos^2\theta\, t_i + \sin^2\theta\, t & \sqrt{3}\cos\theta\,(t_j + t_k) & \cos\theta\sin\theta\,(t - 3t_i) \\ \sqrt{3}\cos\theta\,(t_j + t_k) & 2(t_k - t_j) - t_i & -\sqrt{3}\sin\theta\,(t_j + t_k) \\ \cos\theta\sin\theta\,(t - 3t_i) & -\sqrt{3}\sin\theta\,(t_j + t_k) & 3\sin^2\theta\, t_i + \cos^2\theta\, t \end{bmatrix} \right\}.$$

A calculation now shows that the expression $g_0(X, \mathrm{Ad}_p\,\mathfrak{h}_1)$ is given by the expression

$$(3\cos^2\theta\, t_i + \sin^2\theta\, t)x_1 + 2\sqrt{3}\cos\theta\,(t_j + t_k)\,\mathrm{Im}(x_2) \\ + (2(t_k - t_j) - t_i)x_3 + (3\sin^2\theta\, t_i + \cos^2\theta\, t)x_4.$$

Substituting each of $t = i$, $t = j$, and $t = k$ and using (5_i), (5_j), and (5_k) to eliminate x_3 respectively gives (6_i), (6_j), and (6_k) after using $\sin^2\theta + \cos^2\theta = 1$.

Likewise, the equation $g_0(Y, \mathrm{Ad}_p \mathfrak{h}_1) = 0$ is equivalent to the vanishing of the expression

$$2\cos\theta \sin\theta(-3t_i + t)\,\mathrm{Im}(y_1) - 2\sqrt{3}\sin\theta(t_j + t_k)\,\mathrm{Im}(y_2) + (3\sin^2\theta\, t_i + \cos^2\theta\, t)\,y_3.$$

Substituting each of $t = i$, $t = j$, and $t = k$ respectively gives (7_i), (7_j), and (7_k).

We next claim that (1), (2), and (3) are equivalent to condition (B) of Theorem 2.2. Computing, we see $[X, Y] = 0$ if and only if (1) and (2) are satisfied and $[x_4, y_3] = 0$. But this latter condition is equivalent to (3) since $\mathrm{Sp}(1) = S^3$ has positive sectional curvature. Further, $X_{\mathfrak{p}} = 0$, so $[X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = 0$ and since $Y_{\mathfrak{sp}(2)} = 0$, condition (3) is satisfied if and only if$[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$.

Lastly, we claim that (4) is equivalent to condition (C) of Theorem 2.2. To see this, first recall that it was shown directly following Theorem 2.2 that the conditions $[(\mathrm{Ad}_{p^{-1}} X)_{\mathfrak{k}}, (\mathrm{Ad}_{p^{-1}} Y)_{\mathfrak{k}}] = 0$ and $[(\mathrm{Ad}_{p^{-1}} X)_{\mathfrak{p}}, (\mathrm{Ad}_{p^{-1}} Y)_{\mathfrak{p}}] = 0$ are equivalent, so we may focus on only one of these.

A direct calculation shows that $v = (\mathrm{Ad}_{p^{-1}} X)_{\mathfrak{p}}$ and $w = (\mathrm{Ad}_{p^{-1}} Y)_{\mathfrak{p}}$, so we need only argue that $[v, w] = 0$ if and only if v and w are dependent over $\mathbb{R}$. But we may interpret v, w as elements of $T_{[eK]}G/K$ where $G/K = \mathbb{H}P^2$ has a normal bi-invariant metric of positive sectional curvature. It follows that the bracket of v and w vanishes if and only if v and w are linearly dependent. $\square$

3. Quasipositive curvature

In this section, we prove $N_9 = H_1 \backslash \mathrm{Sp}(3)/H_2$ is quasipositively curved with the metric g_2 constructed in Section 2. As mentioned above, the metric g_2 is nonnegatively curved, so it is sufficient to find a single point for which all 2-planes have nonzero curvature. In fact, we will show the following theorem.

Theorem 3.1. *With respect to the metric g_2, N_9 is positively curved at points of the form $[p^{-1}] \in H_1\backslash G/H_2 \cong N_9$, where*

$$p = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

with $\theta \in \left(0, \tfrac{1}{6}\pi\right)$.

We will always work with points p of the above form.

Assume $[p^{-1}] \in H_1\backslash G/H_2$ is a point for which there is a 0-curvature plane. Then, using Theorem 2.2 and Proposition 2.3, it follows that there are linearly independent $X, Y \in \mathfrak{g} = \mathfrak{sp}(3)$ with

$$X = \begin{bmatrix} x_1 & x_2 & 0 \\ -\bar{x}_2 & x_3 & 0 \\ 0 & 0 & x_4 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & 0 & y_1 \\ 0 & 0 & y_2 \\ -\bar{y}_1 & -\bar{y}_2 & y_3 \end{bmatrix}$$

and which satisfy all of the conditions given by Lemma 2.4. By repeatedly applying the conditions of Lemma 2.4, we will constrain the forms of X and Y until we finally find that no such X and Y exist. This contradiction will establish that there are no zero curvature planes at $[p]^{-1}$, and hence, that N_9 is positively curved at these points.

Proposition 3.2. *If $\theta \in \left(0, \frac{1}{6}\pi\right)$, the two vectors*

$$v = \begin{bmatrix} \cos\theta \sin\theta\,(x_1 - x_4) \\ -\sin\theta\,\bar{x}_2 \end{bmatrix}, \; w = \begin{bmatrix} \mathrm{Re}(y_1) + (\cos^2\theta - \sin^2\theta)\,\mathrm{Im}(y_1) - \sin\theta\cos\theta\, y_3 \\ \cos\theta\, y_2 \end{bmatrix}$$

are both nonzero.

Proof. Suppose for a contradiction that $v = 0$. Since $0 < \theta < \frac{1}{6}\pi$, we have that $v = 0$ implies $x_2 = 0$ and $x_1 = x_4$. Then (6_i), (6_j), and (6_k) imply $x_1 = x_4 = 0$. Then (5_i), (5_j), and (5_k) imply that x_3 vanishes as well. Thus, in this case, $X = 0$, contradicting the fact that X and Y are linearly independent. Thus, $v \neq 0$.

Now, suppose $w = 0$, so $y_2 = 0$, $\mathrm{Re}(y_1) = 0$ and

$$\mathrm{Im}(y_1) = y_1 = \frac{\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} y_3 = \tfrac{1}{2}\tan(2\theta) y_3. \tag{8}$$

This equation implies that the i, j, and k components of y_1 and y_3 are positive multiples of each other. However, (7_j) and (7_k) imply that the j and k components of y_1 and y_3 are negative multiples of each other. Thus, we must have $(y_1)_j = (y_1)_k = (y_3)_j = (y_3)_k = 0$.

Solving (7_i) for $y_1 = (y_1)_i$ and combining with (8), we see that either $y_1 = y_3 = 0$, or θ must satisfy the equation

$$\frac{\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{2\sin^2\theta + 1}{4\sin\theta\cos\theta}.$$

Clearing denominators and simplifying gives $2\sin^2\theta\cos^2\theta + 2\sin^4\theta + \sin^2\theta = \cos^2\theta$. Factoring $\sin^2\theta$ out of the expression $2\sin^2\theta\cos^2\theta + 2\sin^4\theta$, we see this expression simplifies to $2\sin^2\theta$. Substituting this back in gives the equation $3\sin^2\theta = \cos^2\theta$, which has no solutions in $\left(0, \frac{1}{6}\pi\right)$.

Thus, for $\theta \in \left(0, \frac{1}{6}\pi\right)$, we conclude $y_1 = y_3 = 0$, which implies $Y = 0$, again contradicting the fact that X and Y are linearly independent. □

Using (4), it follows that by rescaling X, we may thus assume $v = w$. Further, the first component of v is purely imaginary, and hence $\mathrm{Re}(y_1) = 0$, that is, $y_1 = \mathrm{Im}\, y_1$. Thus, (4) is equivalent to the following two equations:

$$\cos\theta\sin\theta(x_1 - x_4) = (\cos^2\theta - \sin^2\theta)y_1 - \sin\theta\cos\theta\, y_3, \tag{9}$$

$$y_2 = -\tan\theta\,\bar{x}_2. \tag{10}$$

Proposition 3.3. *For any $\theta \in \left(0, \frac{1}{6}\pi\right)$, x_2, y_1, and y_2 are all nonzero.*

Proof. Assume for a contradiction that $y_2 = 0$. Note that, because all the coefficients in (7_i), (7_j), (7_k) are nonzero, it follows that $y_1 = 0$ if and only if $y_3 = 0$. Because $Y \neq 0$, it follows that $y_1 \neq 0$.

Rearranging (1) gives $x_1 y_1 = y_1 x_4$. Taking lengths, we see that $|x_1| = |x_4|$. We now compare the i, j, and k components of x_1 and x_4.

For the i component, we rearrange (6_i) to obtain

$$(x_1)_i = \frac{1 + 2\sin^2\theta}{2\sin^2\theta}(x_4)_i = \left(1 + \frac{1}{2\sin^2\theta}\right)(x_4)_i.$$

Since the sum in the parentheses is positive, we conclude that $|(x_1)_i| \geq |(x_4)_i|$, with equality if and only if $(x_1)_i = (x_4)_i = 0$.

For the j component, we first remark that (10) shows that $x_2 = 0$ because $y_2 = 0$. Then, rearranging (6_j) gives

$$(x_1)_j = -\frac{\cos^2\theta}{\sin^2\theta}(x_4)_j.$$

Thus, since $0 < \theta < \frac{1}{6}\pi$, we conclude that $|(x_1)_j| \geq |(x_4)_j|$ with equality if and only if $(x_1)_j = (x_4)_j = 0$. The same argument shows $|(x_1)_k| \geq |(x_4)_k|$ with equality if and only if $(x_1)_k = (x_4)_k = 0$.

Thus, each component of x_1 is at least as large, in magnitude, as the corresponding component of x_4. Hence, since $|x_1| = |x_4|$, it follows that each of these inequalities must be equalities, so $x_1 = x_4 = 0$. Since we have already shown $x_2 = 0$, equations (5_i), (5_j), and (5_k) force $x_3 = 0$ as well. That is, $X = 0$, a contradiction. Thus, $y_2 \neq 0$.

Finally, it follows from (10) that $x_2 \neq 0$. From (1) and the fact that $x_2 y_2 \neq 0$, we see that $y_1 \neq 0$. □

Proposition 3.4. *For every $\theta \in \left(0, \frac{1}{6}\pi\right)$, $x_1 \neq x_4$.*

Proof. Suppose for a contradiction that $x_1 = x_4$. Then (1) takes the form

$$0 = x_1 y_1 - y_1 x_1 - \tan\theta\,|x_2|^2 = [x_1, y_1] - \tan\theta\,|x_2|^2.$$

Since $x_1, y_1 \in \operatorname{Im}\mathbb{H}$, we know $[x_1, y_1] \in \operatorname{Im}\mathbb{H}$ as well, so we conclude that $\tan\theta\,|x_2|^2 = 0$. Since $0 < \theta < \frac{1}{6}\pi$, it follows that $x_2 = 0$, a contradiction. □

Our next goal is to demonstrate the following proposition.

Proposition 3.5. *For every $\theta \in \left(0, \frac{1}{6}\pi\right)$, $\dim_{\mathbb{R}} \operatorname{span}_{\mathbb{R}}\{x_1, x_4, y_1, y_3\} = 1$.*

Proof. Since, by Proposition 3.3, $y_1 \neq 0$, the dimension of this span is at least 1, so we need only show it is at most one.

We deal first with the case $x_4 = 0$. Then (1) takes the form $x_1 y_1 - \tan\theta\,|x_2|^2 = 0$. In particular, $x_1 y_1 \in \mathbb{R}$. Since x_1 and y_1 are purely imaginary, this implies $\{x_1, y_1\}$

is linearly dependent over $\mathbb{R}$. Now (10) implies that $y_3 = -x_1 + 2y_1/\tan(2\theta)$, so $\{x_1, y_1, y_3\}$ is linearly dependent. Since x_4 vanishes, $\mathrm{span}_{\mathbb{R}}\{x_1, x_4, y_1, y_3\}$ is 1-dimensional.

We now investigate the case where $x_4 \neq 0$. By (3), we may write $y_3 = \lambda x_4$ for some real number λ. Solving (9) for y_1 and substituting into (1) gives

$$0 = \tfrac{1}{2}\tan(2\theta)\big(x_1(x_1 + (\lambda-1)x_4) - (x_1 + (\lambda-1)x_4)x_4\big) - \tan\theta\,|x_2|^2. \tag{11}$$

Recalling that the square of a purely imaginary number is real, the imaginary part of (11) simplifies to

$$0 = \tfrac{1}{2}\tan(2\theta)(\lambda-2)\,\mathrm{Im}(x_1x_4).$$

If $\lambda \neq 2$, this implies that $\mathrm{Im}(x_1x_4) = 0$, that is, $\{x_1, x_4\}$ must be linearly dependent. Recalling $y_3 = \lambda x_4$ and $y_1 = \cos\theta\sin\theta(x_1 + (\lambda-1)x_4)$, we see that if $\lambda \neq 2$, then $\dim_{\mathbb{R}}\mathrm{span}_{\mathbb{R}}\{x_1, x_4, y_3, y_1\} = 1$.

We now show $\lambda = 2$ cannot occur. Assume for a contradiction that $\lambda = 2$. We first show this implies that the j and k components of x_2 and y_2 must vanish. We carry out the proof for the j component, as the proof for the k component is identical.

Given x_2 and x_4, Equation (6_j) determines the j component of x_1:

$$(x_1)_j = -\frac{\cos^2\theta\,(x_4)_j + 2\sqrt{3}(\cos\theta - 1)(x_2)_j}{\sin^2\theta}.$$

Substituting this into (9) and rearranging gives

$$(y_1)_j = -\frac{\cos\theta}{\sin\theta}(x_4)_j - \frac{2\sqrt{3}\cos\theta(\cos\theta-1)}{\sin\theta(\cos^2\theta - \sin^2\theta)}(x_2)_j.$$

Then substituting this into (7_j), we determine

$$(y_2)_j = -\frac{-2\cos^2\theta(\cos\theta-1)}{\sin\theta(\cos^2\theta-\sin^2\theta)}(x_2)_j.$$

On the other hand, from (10), $y_2 = -\tan\theta\,\bar{x}_2$, the j component of y_2 is determined in a different way by x_2. Thus, either $(x_2)_j = (y_2)_j$ or

$$-\frac{-2\cos^2\theta(\cos\theta-1)}{\sin\theta(\cos^2\theta-\sin^2\theta)} = \frac{\sin\theta}{\cos\theta}. \tag{12}$$

By clearing denominators and replacing $\sin^2\theta$ with $1-\cos^2\theta$ everywhere, (12) is equivalent to $2\cos^3\theta - 3\cos^2\theta + 1 = 0$, which factors as

$$(\cos\theta - 1)^2(2\cos\theta + 1) = 0.$$

But this has no solutions $\theta \in \big(0, \frac{1}{6}\pi\big)$, since $0 < \cos\theta < 1$ on that interval. It follows that if $\lambda = 2$, then the j and k components of x_2 and y_2 vanish.

Because the j and k components of x_2 vanish, the proof of Proposition 3.3 shows that $|x_1| \geq |x_4|$ with equality only if $|x_1| = |x_4| = 0$.

Now, (9) gives $y_1 = \frac{1}{2}\tan(2\theta)(x_1 + x_4)$. Substituting this into (1), we get

$$\tfrac{1}{2}\tan(2\theta)(x_1^2 - x_4^2) = \tan\theta\, |x_2|^2.$$

Because x_1 is purely imaginary, $x_1^2 = -|x_1|^2$ and similarly for x_4, so this equation is equivalent to

$$\tfrac{1}{2}\tan(2\theta)(|x_4|^2 - |x_1|^2) = \tan\theta |x_2|^2. \tag{13}$$

For $\theta \in \left(0, \frac{1}{4}\pi\right)$, both tangents are positive, and so, by Proposition 3.3, the right side of (13) is positive.

On the other hand, since $|x_1| \geq |x_4|$, the left side is nonpositive. This contradiction implies $\lambda = 2$ cannot occur for any $\theta \in \left(0, \frac{1}{6}\pi\right)$. □

Using Proposition 3.5 and the fact that $y_1 \neq 0$, we see that x_1, x_4, and y_3 are real multiples of y_1.

Proposition 3.6. *Suppose $\theta \in \left(0, \frac{1}{6}\pi\right)$. Then the i components of x_1, x_4, y_1, y_3 and x_3 are all zero.*

Proof. If $(y_1)_i = 0$, it follows from Proposition 3.5, together with the fact that $y_1 \neq 0$ (Proposition 3.3), that the i component of x_1, x_4, and y_3 are all 0 as well. Then (5_i) shows $(x_3)_i = 0$ as well. So, we need only show $(y_1)_i = 0$ when $\theta \in \left(0, \frac{1}{6}\pi\right)$.

So, assume for a contradiction that $(y_1)_i \neq 0$. Solving for y_3 in (7_i) and substituting into (9), we see

$$\cos\theta \sin\theta (x_1 - x_4) = \left(\cos^2\theta - \sin^2\theta - \cos\theta \sin\theta \frac{4\cos\theta \sin\theta}{2\sin^2\theta + 1}\right) y_1.$$

Since $\theta \in \left(0, \frac{1}{6}\pi\right)$, the coefficient on the right is positive. It follows that $x_1 - x_4$ is a positive multiple of y_1.

Now, note that (1), rearranged, takes the form $(x_1 - x_4)y_1 = \tan\theta\, |x_2|^2$. Since $\theta \in \left(0, \frac{1}{6}\pi\right)$, the right-hand side is positive. But since $x_1 - x_4$ is a positive multiple of y_1, the left-hand side is a positive multiple of y_1^2. The square of any purely imaginary number is nonpositive, so we have a contradiction. □

We now show that x_3 must be nonzero. Suppose for a contradiction that $x_3 = 0$. By (5_j) and (5_k), x_2 has no j or k component. Since $y_2 = -\tan\theta\, \bar{x}_2$, the j and k components of y_2 vanish as well.

Now, (7_j) and (7_k) give $y_3 = -2\tan\theta\, y_1$. In particular, y_3 is a negative multiple of y_1. From (9), we now see $\cos\theta \sin\theta\, (x_1 - x_4)$ is a positive multiple of y_1. Then, just as in the proof of Proposition 3.6, this contradicts (1).

We also find that the j and k components of x_2 and y_2 are constrained.

Proposition 3.7. *Let x_2', y_2' denote the projection of x_2 and y_2 into the jk-plane. Then* $\dim_{\mathbb{R}} \operatorname{span}_{\mathbb{R}}\{x_1, x_4, y_1, y_3, x_2', y_2'\} = 1$.

Proof. Recalling that x_1 and x_4 have no i component by Proposition 3.6, we see that multiplying (6_j) by j and (6_k) by k and adding gives the equation

$$(x_2')(\cos\theta - 1)2\sqrt{3} + x_1 \sin^2\theta + x_4 \cos^2\theta = 0.$$

Thus, x_2' is dependent on x_1 and x_4. Since $y_2 = -\tan\theta\,\bar{x}_2$, we find that $y_2' = \tan\theta\, x_2'$ is also dependent on x_2'. The result follows. □

Proposition 3.8. *Either $(x_2)_j = 0$ or $(x_2)_k = 0$, but not both.*

Proof. If both are zero, then (5_j) and (5_k) give $x_3 = 0$, which is not possible. We now show at least one vanishes.

We begin by rearranging (2) into the form

$$\bar{x}_2(\tan\theta\, x_4 - y_1) = \tan\theta\, x_3\bar{x}_2.$$

We write $x_2 = x_2'' + x_2'$ as a decomposition into the complex components, together with the j and k components. That is, $x_2'' \in \mathbb{C}$ while $x_2' \in \operatorname{span}\{j, k\}$, as before. Then, the left-hand side can be expanded as $x_2''(\tan\theta\, x_4 - y_1) + x_2'(\tan\theta\, x_4 - y_1)$. Recalling that the i component of x_4 and y_1 vanishes by Proposition 3.6, $x_2''(\tan\theta\, x_4 - y_1) \in \operatorname{span}\{j, k\}$.

Further, we see $x_2'(\tan\theta\, x_4 - y_1) \in \mathbb{R}$ because x_2' is dependent on both x_4 and y_1 by Proposition 3.7. It follows that $x_2(\tan\theta\, x_4 - y_1)$ has no i component.

Hence, the i component of the right-hand side, $\tan\theta\, x_3\bar{x}_2$, must vanish as well. Since $(x_3)_i = 0$ by Proposition 3.6, the i component of $x_3\bar{x}_2$ is given by

$$0 = (x_3\bar{x}_2)_i i = (x_3)_j j(\bar{x}_2)_k k + (x_3)_k k(\bar{x}_2)_j j = (-(x_3)_j(x_2)_k + (x_3)_k(x_2)_j)i.$$

Now, using (5_j) and (5_k), we see $(x_3)_j = \sqrt{3}(x_2)_j$ and $(x_3)_k = -\sqrt{3}(x_2)_k$. Substituting yields $0 = -2\sqrt{3}(x_2)_j(x_2)_k$, so at least one of $(x_2)_j$ and $(x_2)_k$ vanishes. □

As we have already shown $\dim_{\mathbb{R}} \operatorname{span}\{x_1, x_4, y_1, y_3, x_2', y_2'\} = 1$ (Proposition 3.7), it follows that either they all only have a k component, or they all only have a j component. Equations (5_j) and (5_k) show that x_3 is also in the span of $\{x_1, x_4, y_1, y_3, x_2', y_2'\}$.

Our next proposition will show that all the variables must commute.

Proposition 3.9. $$(x_2)_i = (y_2)_i = 0.$$

Proof. Since $y_2 = -\tan\theta\,\bar{x}_2$, it is enough to show that $(x_2)_i = 0$.

Equation (2) can be rearranged into the form

$$\tan\theta\, x_4 - y_1 = \frac{\tan\theta}{|x_2|^2} x_2 x_3 \bar{x}_2.$$

By Propositions 3.6, 3.7, and 3.8, we see that the left-hand side, x_3, and x_2 are all either a real multiple of j or a real multiple of k. For the remainder of the proof, we assume they are all multiples of j; the case where they are multiples of k is identical.

The right side is, up to multiple, given by conjugating x_3 by the unit quaternion $x_2/|x_2|$. Recall that a unit quaternion can be written as $q = (\cos\phi)q_0 + (\sin\phi)q_1$, where q_0 is real and q_1 is purely imaginary and $|q_0| = |q_1| = 1$. Then conjugation by q, viewed as a map from $\mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$ to itself, is a rotation with axis given by q_1 and with rotation angle given by 2ϕ.

Since the j-axis is invariant under conjugation by x_2, we see one of two things happen. Either the j-axis is fixed point-wise, in which case $\mathrm{Im}(x_2)$ has only a j component, or the orientation of it is reversed. We now show the latter case cannot occur.

If the orientation is reversed, the rotation axis $\mathrm{Im}(x_2)$ must be perpendicular to j, so $\mathrm{Im}(x_2) \in \mathrm{span}\{i, k\}$. Because x_2' has no k part, so it follows that $x_2' = 0$. But then, using (5_j) and (6_j), we see that $x_3 = 0$, which is not possible. □

It follows that $\mathrm{Im}(x_2) = x_2'$. Summarizing, we have now shown that at a point containing a 0-curvature plane with $\theta \in \left(0, \frac{1}{6}\pi\right)$ that $x_2' = \mathrm{Im}(x_2)$, $y_2' = \mathrm{Im}(y_2)$, $\dim_{\mathbb{R}} \mathrm{span}\{x_1, x_3, x_4, y_1, y_3, x_2', y_2'\} = 1$ and further, that each element in this set has vanishing i and j components or vanishing i and k components. In particular, the variables x_1, x_2, x_3, x_4, y_1, y_2, and y_3 all commute. Thus, we may replace (2) with the linear equation $\tan\theta\, x_4 - \tan\theta\, x_3 - y_1 = 0$ by substituting $y = -\tan\theta\, \bar{x}_2$ and canceling all occurrences of $\bar{x}_2$. We let $\ell \in \{j, k\}$ and set $\epsilon = 1$ if $\ell = j$ and $\epsilon = -1$ if $\ell = k$. Then, (2)–(7_k) are equivalent to the homogeneous system of linear equations

$$\begin{aligned}
-\tan\theta\,(x_3)_\ell + \tan\theta\,(x_4)_\ell - (y_1)_\ell &= 0,\\
\cos\theta\sin\theta\,(x_1)_\ell - \cos\theta\sin\theta\,(x_4)_\ell + (\sin^2\theta - \cos^2\theta)\,(y_1)_\ell + \cos\theta\sin\theta\,(y_3)_\ell &= 0,\\
\tan\theta\,(x_2)_\ell - (y_2)_\ell &= 0,\\
\sqrt{3}(x_2)_\ell + \epsilon(x_3)_\ell &= 0,\\
\sin^2\theta\,(x_1)_\ell + 2\sqrt{3}(\cos\theta - 1)(x_2)_\ell + \cos^2\theta\,(x_4)_\ell &= 0,\\
2\sin\theta\cos\theta\,(y_1)_\ell - 2\sqrt{3}\sin\theta\,(y_2)_\ell + \cos^2\theta\,(y_3)_\ell &= 0.
\end{aligned}$$

Then one can easily compute that all solutions are given as real multiples of

$$\begin{bmatrix} (x_1)_\ell \\ (x_2)_\ell \\ (x_3)_\ell \\ (x_4)_\ell \\ (y_1)_\ell \\ (y_2)_\ell \\ (y_3)_\ell \end{bmatrix} = \begin{bmatrix} -3\cos\theta((2+\epsilon)\cos^2\theta - 4\cos\theta + 2) \\ -\sqrt{3}\cos\theta \\ 3\epsilon\cos\theta \\ -3(\cos\theta - 1)((2+\epsilon)\cos^2\theta + (\epsilon - 2)\cos\theta - 2) \\ -3\tan\theta((2+\epsilon)\cos^3\theta - 4\cos^2\theta + 2) \\ -\sqrt{3}\sin\theta \\ 6\tan^2\theta((2+\epsilon)\cos^3\theta - 4\cos^2\theta + 1) \end{bmatrix}. \tag{14}$$

We now note that (1) is equivalent to $y_1(x_1 - x_4) = \tan\theta\,|x_2|^2$. In particular, (1) implies that $y_1(x_1 - x_4) > 0$. Thus, if we can show that for $\theta \in \big(0, \frac{1}{6}\pi\big)$, Equation (14) implies $y_1(x_1 - x_4) < 0$, we will have reached our final contradiction, showing N_9 is positively curved at points with $\theta \in \big(0, \frac{1}{6}\pi\big)$.

Proposition 3.10. *For* $\theta \in \big(0, \frac{1}{6}\pi\big)$, $y_1(x_1 - x_4) < 0$.

Proof. We first note that a simple calculation shows

$$(x_1)_\ell - (x_4)_\ell = 6 - (6 + 3\epsilon)\cos\theta.$$

We first prove $y_1(x_1 - x_4) < 0$ when $\ell = j$, that is, $\epsilon = 1$. In this case, $(x_1 - x_4)_j = 6 - 9\cos\theta$ and this is negative so long as $\cos\theta > \frac{2}{3}$. Of course, since $\cos\big(\frac{1}{6}\pi\big) > \frac{2}{3}$, we know that $(x_1 - x_4)_j < 0$ on $\big(0, \frac{1}{6}\pi\big)$.

Further, $(y_1)_j = -3\tan\theta\,(3\cos^3\theta - 4\cos^2\theta + 2)$. The polynomial $3x^3 - 4x^2 + 2$ is clearly positive on the interval $(\sqrt{3}/2, 1)$, so $(y_1)_j < 0$.

It follows that $y_1(x_1 - x_4) = (y_1)_j(x_1 - x_4)_j j^2 = -(y_1)_j(x_1 - x_4)_j < 0$.

Finally, we prove $y_1(x_1 - x_4) < 0$ when $\ell = k$, that is, $\epsilon = -1$. Then it is easy to see that $(y_1)_k$ is positive since the polynomial $x^3 - 4x^2 + 1$ is negative on the interval $(\sqrt{3}/2, 1)$. Further, if $\epsilon = -1$, then $(x_1)_k - (x_4)_k = 6 - 3\cos\theta > 0$.

Thus, $y_1(x_1 - x_4) = (y_1)_k(x_1 - x_4)_k k^2 = -(y_1)_k(x_1 - x_4)_k < 0$, as claimed. □

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jdevito@ut.utm.edu *Department of Mathematics and Statistics, University of Tennessee Martin, Martin, TN 38238, United States*

wesmart@utm.edu *Department of Education, University of Tennessee Martin, Martin, TN 38238, United States*

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