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Symmetric numerical ranges of four-by-four matrices

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Numerical ranges of matrices with rotational symmetry are studied. Some cases in which symmetry of the numerical range implies symmetry of the spectrum are described. A parametrized class of 4×4 matrices $K(a)$ such that the numerical range $W(K(a))$ has fourfold symmetry about the origin but the generalized numerical range $W_{K(a)^*}(K(a))$ does not have this symmetry is included. In 2011, Tsai and Wu showed that the numerical ranges of weighted shift matrices, which have rotational symmetry about the origin, are also symmetric about certain axes. We show that any 4×4 matrix whose numerical range has fourfold symmetry about the origin also has the corresponding axis symmetry. The support function used to prove these results is also used to show that the numerical range of a composition operator on Hardy space with automorphic symbol and minimal polynomial $z^4 - 1$ is not a disk.

1. Introduction

Let H be a Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and T a bounded linear operator on H. The *numerical range* of T, denoted by $W(T)$, is the subset of the complex plane $\mathbb{C}$ defined by

$$W(T) = \{\langle Tv, v\rangle \mid v \in H, \|v\| = 1\}.$$

The Toeplitz–Hausdorff theorem states that the numerical range of any bounded linear operator on a Hilbert space is convex [Toeplitz 1918; Hausdorff 1919]. In addition, it follows immediately from the definition that $W(T)$ is unitarily invariant; that is, if R is a linear operator satisfying $R = UTU^*$ for a unitary operator U, then $W(R) = W(T)$. Other well-known results about the numerical range are listed below; these and many other properties of the numerical range appear in [Gustafson

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and Rao 1997; Horn and Johnson 1991]. The set of $n \times n$ complex matrices is denoted by $M_n(\mathbb{C})$.

(I) The numerical range contains the spectrum $\sigma(T)$ of T.

(II) If the Hilbert space H is finite-dimensional, then $W(T)$ is compact.

(III) The numerical range $W(T)$ is bounded by $\|T\|$.

(IV) If A is a Hermitian matrix, then $W(A)$ is a real line segment with endpoints equal to the maximum and minimum eigenvalues of A.

(V) $W(A^*) = \{\bar{z} \mid z \in W(A)\}$.

(VI) If A is a normal matrix then $W(A)$ is the convex hull of the eigenvalues of A.

(VII) If $A \in M_2(\mathbb{C})$ then $W(A)$ is a (possibly degenerate) ellipse with foci equal to the eigenvalues of A.

In this paper, some 4×4 matrices with numerical ranges that have a strong type of symmetry are studied. A parametrized family of matrices $K(a)$ where $W(K(a))$ has fourfold symmetry about the origin but certain generalized numerical ranges of $K(a)$ are not symmetric are described; this class generalizes an example in [Deaett et al. 2013]. The relationship between symmetry of the numerical range and symmetry of the spectrum is discussed. In particular, we show that if an associated algebraic curve to an $n \times n$ matrix is irreducible, then symmetry of the numerical range implies symmetry of the spectrum; when $n = 4$, the irreducibility assumption can be dropped. Applications to symmetry about axes are included. The derivations of these results will use two closely related functions associated with the numerical range of a matrix, namely Kippenhahn's boundary-generating curve and the support function of the numerical range. Finally, we show that the numerical range of a composition operator on the Hardy space of the disk with automorphic symbol and minimal polynomial $q(z) = z^4 - 1$ is not a circular disk.

2. Boundary-generating curve and support function

Kippenhahn [1951; 2008] defined the boundary-generating curve for (the numerical range of) an $n \times n$ matrix A as follows. Let $H = (A + A^*)/2$ and $K = (A - A^*)/(2i)$, and let I_n denote the $n \times n$ identity matrix. The polynomial

$$f_A(x, y, z) = \det(xH + yK + zI_n) \tag{1}$$

is homogeneous of degree n with real coefficients. The domain of f_A is complex projective space $\mathbb{P}_2(\mathbb{C})$, which consists of all equivalence classes of points in $\mathbb{C}^3 \backslash \{(0, 0, 0)\}$ under the equivalence relation $\sim$; this relation is defined by

$$(x, y, z) \sim (x', y', z')$$

if and only if there is a nonzero $\alpha \in \mathbb{C}$ such that $(x, y, z) = \alpha(x', y', z')$. Any point $x + iy$ (with $x, y \in \mathbb{R}$) in the complex plane can be identified with the equivalence class of the point $(x, y, 1)$. The natural setting for the study of algebraic curves is $\mathbb{P}_2(\mathbb{C})$; see [Fischer 2001; Gibson 1998] for an introduction to this subject. However, properties of the numerical range primarily involve the restriction of the domain of f_A to points identified with the complex plane, because Kippenhahn showed that $W(A)$ is the convex hull of the curve C defined in line coordinates by $f_A(x, y, 1) = 0$ with $(x, y) \in \mathbb{R}^2$; that is, C is the real part of the zero set of f_A. Kippenhahn called C "the boundary-generating curve of the matrix A". Since C is defined in terms of line coordinates, the line consisting of all $(u, v) \in \mathbb{R}^2$ such that $ux + vy + 1 = 0$ is tangent to C if and only if $f_A(x, y, 1) = 0$. For convenience, if f is a homogeneous polynomial, we will set

$$V_{\mathbb{R}}(f) = \{(x, y) \in \mathbb{R}^2 \mid f(x, y, 1) = 0\}.$$

The polynomial f_A is *reducible* if there exist nonconstant polynomials g and h with real coefficients such that $f_A = gh$; if this occurs, g and h are necessarily homogeneous. A nonconstant polynomial is *irreducible* if it is not reducible. It suffices to consider irreducibility over the real numbers; if f_A was reducible over $\mathbb{C}$ and irreducible over $\mathbb{R}$, then $f_A = g\bar{g}$, where g is an irreducible polynomial with complex coefficients. The polynomials g, $\bar{g}$, and f_A have the same zero set in the complex plane so any arguments requiring irreducibility could be applied to g.

An $n \times n$ matrix A is *unitarily reducible* if there exist matrices B and C of sizes $r \times r$ and $s \times s$, respectively, where $r + s = n$ and $1 \le r, s \le n - 1$, and a unitary matrix $U \in M_n(\mathbb{C})$ such that

$$U^* A U = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}.$$

The matrix A is called *unitarily irreducible* if A is not unitarily reducible. Determinant properties show that if A is unitarily reducible, then f_A is reducible. However, the converse does not hold because, as shown in [Kippenhahn 1951; 2008], there exist unitarily irreducible matrices A such that f_A is reducible.

In addition to developing properties of the boundary-generating curve, Kippenhahn classified the numerical ranges of 3×3 matrices by showing that the shape of $W(A)$ depends on whether $f(x, y, z)$ is reducible or irreducible. He showed that $W(A)$ is either (1) the convex hull of the eigenvalues of A; (2) the convex hull of an ellipse and a point (reducing to an ellipse when the point is inside the ellipse); (3) a shape with one flat part on the boundary; (4) an ovular shape with no flat part. In [Keeler et al. 1997; Rodman and Spitkovsky 2005], Kippenhahn's classifications are used to derive straightforward tests in terms of the entries of a matrix A that determine the shape. Recently, Chien and Nakazato [2012] classified the numerical ranges of 4×4 matrices using the boundary-generating curve.

The boundary of the numerical range can also be described more directly in terms of its support lines. If S is a closed convex subset of $\mathbb{C}$, then for each point z on the boundary of S, there exists a line L such that $z \in L$ and S lies entirely in one half-plane determined by L. The line L is called a support line for S at z. See [Valentine 1964] for more background on convex sets.

When S is the numerical range of an $n \times n$ matrix A, the rightmost vertical support line $x = \lambda$ of $W(A)$ can be determined directly, because λ is the maximum real part of any complex number in $W(A)$. Straightforward calculations involving inner products produce the following equality:

$$\max\left\{\operatorname{Re}\langle Av, v\rangle \;\middle|\; v \in \mathbb{C}^n,\ \|v\|=1\right\} = \max\left\{\left\langle \tfrac{1}{2}(A + A^*)v, v\right\rangle \;\middle|\; v \in \mathbb{C}^n,\ \|v\|=1\right\}.$$

The set in braces on the right side of the equality is the numerical range of the Hermitian matrix $H = \frac{1}{2}(A + A^*)$, so by (IV) the maximum value in this set is the maximum eigenvalue of H. Hence the rightmost vertical support line of $W(A)$ is the line $x = \lambda$, where λ is the maximum eigenvalue of H.

Since $W(cA) = cW(A)$ for any complex scalar c, we can derive the support line in every direction by rotating A. The rightmost vertical support line of $W(e^{-i\theta}A)$ will be $x = p_A(\theta)$ where:

Definition 1. $$p_A(\theta) = \max \sigma\left(\tfrac{1}{2}(e^{-i\theta}A + e^{i\theta}A^*)\right).$$

Rotating this line back by an angle θ will yield the support line of the original numerical range $W(A)$ that is orthogonal to the line from 0 to $e^{i\theta}$.

Therefore the *support function* $p_A(\theta)$ completely determines the numerical range of the matrix A since it describes the support lines in every direction. When T is an operator on an infinite-dimensional Hilbert space, the analogously defined support function determines the closure of $W(T)$.

Note that for any real θ, we have $f_A(\cos(\theta), \sin(\theta), -p_A(\theta)) = 0$ because $(x, y, z) = (\cos(\theta), \sin(\theta), -p_A(\theta))$ satisfies $\det(xH + yK + zI) = 0$.

3. *n*-fold symmetry about the origin

As mentioned in the list of properties of $W(A)$ above, the numerical range of any matrix A contains the eigenvalues of A and when A is normal, $W(A)$ is the convex hull of $\sigma(A)$. In many cases, a plot of the eigenvalues of A along with $W(A)$ shows no obvious relationship except containment. However, a special class of generalized permutation matrices have numerical ranges consisting of a “fattened up” convex hull of the eigenvalues of A. These matrices, whose numerical ranges are studied in [Tsai and Wu 2011; Li and Tsing 1991] as discussed later, are weighted shifts. For consistency with some other references we will work with their adjoints, which by property (V) in the Introduction will produce equivalent results.

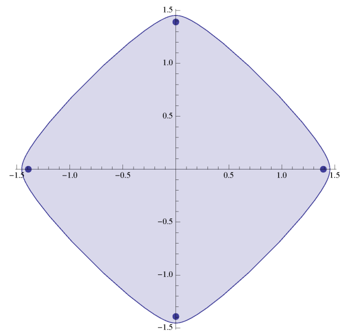

Figure 1. $W(A)$.

Definition 2. A matrix $A \in M_n(\mathbb{C})$ is an AWS (adjoint of weighed shift) matrix if $A = (a_{ij})$ with $a_{ij} = 0$ unless $i = j + 1$ or $i = 1$ and $j = n$.

In the 4×4 case, this yields

$$A = \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ a_{21} & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & a_{43} & 0 \end{pmatrix}. \tag{2}$$

If A is $n \times n$ of class AWS and the entries of A which are not specified to be zero are in fact nonzero, then the eigenvalues of A are given by a common scalar multiple of the n-th roots of unity. It turns out that the numerical range $W(A)$ is symmetric about the origin in a similar manner.

For example, if A is the 4×4 matrix of class AWS given by

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & \frac{5}{4} & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \end{pmatrix},$$

then the eigenvalues of A are $\{c, ci, -c, -ci\}$,where $c = 15^{1/4}/\sqrt{2}$.

The numerical range $W(A)$ and the eigenvalues are shown in Figure 1.

This motivates the following definition.

Definition 3. Let n be a positive integer. A subset S of the complex plane has n-fold symmetry about the origin (n-sato) if $z \in S$ implies $e^{2\pi i/n} z \in S$.

That is, S has n-sato if the set S' obtained by rotating S by $2\pi/n$ radians around the origin is equal to S. Clearly the numerical range and the spectrum in Figure 1 have 4-sato.

A result credited to Anderson, which appears in [Tam and Yang 1999; Wu 2011], provides an immediate result about numerical range symmetry.

Theorem 4 [Tam and Yang 1999; Wu 2011]. *Assume $N \geq 2$ and $A \in M_N(\mathbb{C})$. If $W(A)$ is contained in a circular disk and $\partial W(A)$ meets the boundary of the disk at more than N points, then $W(A)$ is equal to the circular disk.*

Corollary 5. *Assume $n > N \geq 2$. Assume $A \in M_N(\mathbb{C})$ is a nonzero matrix. If $W(A)$ has n-sato, then $W(A)$ is a circular disk centered at the origin.*

Proof. Assuming the hypotheses of the corollary, let z_0 be a point of $\partial W(A)$ where the numerical radius of A is attained. Note that $z_0 \neq 0$ and $W(A)$ is contained in the circular disk D with center at the origin and radius $|z_0|$. Since $W(A)$ has n-sato, the distinct points $e^{2\pi ki/n} z_0$ are on $\partial W(A)$ for $k = 0, \ldots, n-1$. Therefore $W(A)$ meets D in more than N points and hence $W(A) = D$. $\square$

Symmetry results about numerical ranges of block AWS operators are proved in [Li and Tsing 1991]. In fact, they prove that much stronger symmetry results hold for AWS operators because the symmetry extends to certain generalized numerical ranges introduced in [Goldberg and Straus 1977]. This generalization is defined below.

Definition 6. Let A and C be in $M_n(\mathbb{C})$. The *C-numerical range of A* is the subset of $\mathbb{C}$ defined by

$$W_C(A) = \{\mathrm{tr}(CUAU^*) \mid U \in M_n(\mathbb{C}),\ U^*U = I\}.$$

Recall that the standard inner product on $M_n(\mathbb{C})$ is $\langle A, B\rangle = \mathrm{tr}(B^*A)$, so $\mathrm{tr}(B^*A)$ can be considered a scaled projection of A onto B. Hence $W_C(A)$ can be considered the projection of the collection of all matrices unitarily equivalent to A (this collection is called the *unitary orbit of* A) onto the matrix C^*. When $C = E_{11}$, the $n \times n$ matrix with 1 in the first row, first column entry and zeroes elsewhere, the generalized numerical range $W_{E_{11}}$ equals the classical numerical range. Unlike the classical numerical range, the C-numerical range is not convex in general [Westwick 1975] but it is always star-shaped [Cheung and Tsing 1996]. See [Li 1994] for more background and properties of the C-numerical range.

Li and Tsing [1991] showed that the Hilbert space operators for which all the (appropriately generalized) C-numerical ranges have n-sato are exactly those unitarily similar to a block form of the AWS. For convenience, we state below a special case of their results that is directly related to the results in this paper.

Theorem 7 (Li–Tsing, special case). *Let n be a positive integer and $A \in M_n(\mathbb{C})$. The following conditions are equivalent*:

(a) *$W_C(A)$ has n-sato for all $C \in M_n(\mathbb{C})$.*

(b) *$W_{A^*}(A)$ has n-sato.*

(c) *A is unitarily equivalent to an $n \times n$ AWS matrix.*

Thus the only $n \times n$ matrices for which all C-numerical ranges have n-sato are those unitarily equivalent to AWS matrices. Since the classical numerical range is one C-numerical range, it follows that the classical numerical range of any AWS matrix has n-sato. However, based on the Li–Tsing theorem, it is possible that there exists an $n \times n$ matrix A that is not unitarily equivalent to an AWS matrix but where $W(A)$ has n-sato. Of course for such a matrix A, there would exist C (in particular $C = A^*$) such that $W_C(A)$ does not have n-sato.

Results in [Tam and Yang 1999] provide conditions (some of which are in terms of associated graphs) that are necessary and sufficient for classes of matrices with the same zero or ray pattern as a given matrix A to have numerical ranges with n-sato or circular symmetry. In particular, conditions for a single matrix with nonnegative entries and a connected undirected graph to have a numerical range with n-sato are provided.

In the 2×2 case, however, it is straightforward to show that $W(A)$ has 2-sato if and only if the eigenvalues of A have 2-sato if and only if A is unitarily equivalent to a 2×2 AWS matrix. These facts follow from property (VII) in the Introduction, basic facts about ellipses, and unitary equivalence arguments. See [Horn and Johnson 1991].

In [Harris et al. 2011], it is shown that $W(A)$ has 3-sato (and is not a circular disk) if and only if A is unitarily similar to a matrix of class AWS.

In [Deaett et al. 2013], matrices in $M_n(\mathbb{C})$ for $n \geq 4$ such that $W(A)$ has n-sato are studied and the following result is proved.

Theorem 8 [Deaett et al. 2013]. *Assume A is a 4×4 matrix with complex entries whose eigenvalues have 4-fold symmetry about the origin. Assume $W(A)$ is not a circular disk. Then the numerical range $W(A)$ has 4-fold symmetry about the origin if and only if* $\operatorname{tr}(A^2 A^*) = 0$ *and* $\operatorname{tr}(A^3 A^*) = 0$.

A natural generalization of the trace condition in Theorem 8 that is sufficient to show $W(A)$ has n-sato for all integers $n \geq 4$ also appears in [Deaett et al. 2013].

The matrix

$$B = \begin{pmatrix} 1 & 1 & \frac{1}{3}(-18-5\sqrt{14}) & 1 \\ 0 & i & 2 & \frac{2}{3}(9+2\sqrt{14}) \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -i \end{pmatrix} \tag{3}$$

was also constructed in [Deaett et al. 2013]. The numerical range $W(B)$ has 4-sato; however, B is not unitarily equivalent to an AWS matrix of the form (2). Hence there exist 4×4 matrices C such that $W_C(A)$ does not have 4-sato. We will now use similar methods to produce a simpler collection of matrices whose numerical ranges have the same properties.

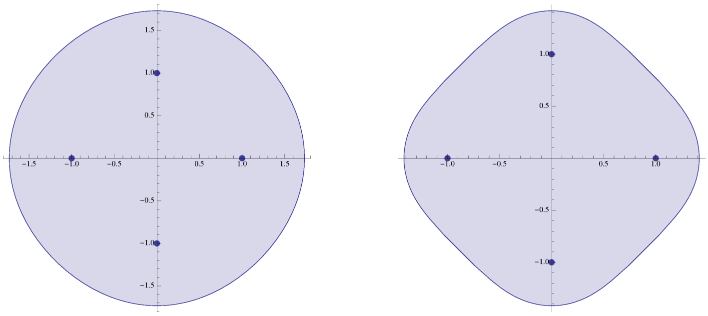

Figure 2. $W(K(a))$ for $a = 1$ (left) and $a = 0.1$ (right)

Let $a \in \mathbb{C}$ with $a \neq 0$ and define

$$K(a) = \begin{pmatrix} 1 & a & \sqrt{2|a|^2+4} & a \\ 0 & i & 0 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \tag{4}$$

Clearly the eigenvalues of $K(a)$ have 4-sato and since there are no repeated eigenvalues, $W(K(a))$ is not a disk; see [Wu 2011]. It is straightforward to check that $\operatorname{tr}(K(a)^j K(a)^*) = 0$ for $j = 2, 3$. Hence $W(K(a))$ has 4-sato. However, $\operatorname{tr}(K(a)^3(K(a)^*)^2) = 4|a|^2 \neq 0$. For any AWS matrix A, we have $\operatorname{tr}(A^3(A^*)^2) = 0$. Since this trace is a unitary invariant, the matrix $K(a)$ is not unitarily equivalent to an AWS matrix. Therefore, there exists $C \in M_4(\mathbb{C})$ such that $W_C(K(a))$ does not have 4-sato. In particular, $W_{K(a)^*}(K(a))$ does not have 4-sato.

More generally, a similar analysis can be done for many matrices of the form

$$K = \begin{pmatrix} 1 & a & b & a \\ 0 & i & f & c \\ 0 & 0 & -1 & f \\ 0 & 0 & 0 & -i \end{pmatrix} \tag{5}$$

by fixing the "keystone" variable a and solving for b, c, and f to obtain the correct trace values.

Both (3) and (4) have the form (5).

A straightforward computation shows that the boundary-generating curve for $K(a)$ is

$$f_{K(a)}(u, v, w) = w^4 - w^2(u^2+v^2)(3+|a|^2) + (2+|a|^2)(u^4+v^4) + (5+2|a|^2)u^2v^2.$$

This polynomial is quadratic in $x = u^2$, $y = v^2$ and $z = w^2$. The Hessian of the resulting polynomial in x, y, and z is $H(f) = 2|a|^2(2+|a|^2) \neq 0$. Therefore this polynomial is irreducible so $f_{K(a)}$ does not factor into two quadratics. One can also show that $f_{K(a)}$ cannot factor into a cubic and a linear factor since the linear factor would correspond to an eigenvalue of $K(a)$ and this leads to a contradiction. Consequently $f_{K(a)}$ is irreducible and thus the matrix $K(a)$ is unitarily irreducible.

We include plots of $W(K(a))$ for $a = 1$ and $a = 0.1$ in Figure 2. In general, the problem of plotting $W_C(A)$ is difficult.

4. Symmetry of the spectrum

Assume A is a 2×2 matrix and therefore $W(A)$ is an ellipse with foci equal to the eigenvalues of A. As mentioned earlier, it clearly follows that $W(A)$ has 2-sato if and only if the spectrum $\sigma(A)$ has 2-sato. In general, the spectrum of A can have n-sato even though $W(A)$ does not have n-sato. However, under an irreducibility condition on the boundary-generating curve, symmetry of $W(A)$ implies that of $\sigma(A)$. Proposition 10 below generalizes the $n = 3$ case that appeared in [Harris et al. 2011]. The following lemma is used in the proofs of Propositions 10 and 11.

Lemma 9. *Let n and N be positive integers and let g be an irreducible homogeneous polynomial of degree N. Then the polynomial $\hat{g}_n$ obtained by rotating each affine point $(x, y) = (x, y, 1)$ on $V_{\mathbb{R}}(g)$ through an angle $-\frac{2\pi}{n}$ about the origin is also irreducible of degree N. Hence if there are infinitely many points on $V_{\mathbb{R}}(g) \cap V_{\mathbb{R}}(\hat{g}_n)$ then g is a nonzero scalar multiple of $\hat{g}_n$.*

Proof. Since the transformation of rotation in the first two coordinates of (x, y, z) is an invertible transformation, it preserves irreducibility and degree of homogeneous polynomials. Therefore if the intersection $V_{\mathbb{R}}(g) \cap V_{\mathbb{R}}(\hat{g}_n)$ is infinite, Bézout's theorem shows that $g = c\hat{g}_n$ for some nonzero scalar c. □

Proposition 10. *Let N and n be integers with $n \geq 2$ and $N \geq 3$. Assume A is an $N \times N$ matrix such that f_A as defined in* (1) *is irreducible. If $W(A)$ has n-sato, then the spectrum $\sigma(A)$ has n-sato.*

Proof. Since f_A is irreducible, it follows that A is unitarily irreducible and hence the boundary of $W(A)$ is smooth [Kippenhahn 1951; Horn and Johnson 1991]. Since there are no corners of $\partial W(A)$, it is not possible that two flat parts on $\partial W(A)$ intersect. There are at most $(N-1)(N-2)/2$ flat parts on the boundary of the numerical range of an $N \times N$ matrix such that f_A is irreducible; see [Gau and Wu 2008]. Any of these finitely many flat parts are separated by a nonflat portion Γ of $\partial W(A)$ consisting of infinitely many points. The numerical range is the convex hull of the boundary-generating curve $C = \{x + iy \in \mathbb{C} \mid f_A(x, y, 1) = 0\}$ and therefore Γ is on a piece of C itself. If there are no flat portions of $\partial W(A)$, then Γ could be any infinite subset of C.

Let $\alpha = \frac{2\pi}{n}$ and $\omega = e^{i\alpha}$. The assumption that $W(A)$ has n-sato is equivalent to the statement that $W(A) = W(\omega A)$. Therefore $\partial W(\omega A)$ also contains Γ and as a nonflat portion of $\partial W(\omega A)$, it must by the argument above be a piece of $V_{\mathbb{R}}(f_{\omega A})$.

The polynomial $f_{\omega A}$ is equal to $(\hat{f}_A)_n$ in the notation of Lemma 9, so $f_A = c f_{\omega A}$ for some scalar c. The coefficient of z^N is 1 in both $f_A(x, y, z)$ and $f_{\omega A}(x, y, z)$; hence $f_A = f_{\omega A}$. Kippenhahn [1951] showed that the eigenvalues of a matrix A are the real foci of the curve f_A. Hence the eigenvalues of A and ωA are equal. Since the eigenvalues of ωA are obtained from those of A by rotating by α about the origin, this proves that $\sigma(A)$ has n-sato. □

The irreducibility condition on f_A, or at least a condition on the size of A, is necessary in Proposition 10. If $A = B \oplus C$, where $W(B)$ has n-sato and C is diagonal with any (i.e., nonsymmetrical) spectrum contained in $W(B)$, then $f_A = f_B f_C$ and the spectrum of A need not have n-sato. However, we can show that if $n = 4$ and A is a 4×4 matrix, noncircular symmetry of the numerical range implies symmetry of the spectrum.

Proposition 11. *Assume A is a 4×4 matrix and $W(A)$ has 4-sato but is not a circular disk. Then $\sigma(A)$ has 4-sato. Under these hypotheses, if $\sigma(A) = \{0\}$, then A is the zero matrix.*

Proof. Assume A is a 4×4 matrix and $W(A)$ has 4-sato but is not a circular disk. Let f_A be defined as in (1). If f_A is irreducible, then $\sigma(A)$ has 4-sato by Proposition 10. Therefore, assume f_A is reducible with the factorization $f_A(u, v, w) = g(u, v, w)\, h(u, v, w)$, where g is irreducible. In addition, assume the degree m of g is greater than or equal to the degree of every other factor of f_A. So m is either 1, 2, or 3. Note that since the coefficient of w^4 is 1 in the polynomial f_A, we may assume the coefficient of any monomial w^k in any degree-k factor of f_A is also 1.

Case 1: If $m = 1$, then f_A factors into four factors of degree 1; that is, $f_A = h_1 h_2 h_3 h_4$, where $h_j(u, v, w) = a_j u + b_j v + 1w$ and $\lambda_j = a_j + i b_j$ is an eigenvalue of A for $j = 1, 2, 3, 4$. The numerical range $W(A)$ is a polygon (which could reduce to a line or point) which is the convex hull of these four points. In fact, $W(A)$ is the convex hull of its uniquely determined vertices, which could be a priori a proper subset of $\sigma(A) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. If $0 \in \sigma(A)$ and 0 is the only vertex of $W(A)$, then $W(A) = \{0\}$ and consequently $\sigma(A) = \{0\}$, which has 4-sato. In this case, A is the zero matrix. Otherwise, let λ_ℓ be a nonzero element of $\sigma(A)$ which is a vertex of $W(A)$. Then $i\lambda_\ell$, $-\lambda_\ell$ and $-i\lambda_\ell$ are distinct and they are also vertices of $W(A)$ by the 4-sato assumption. This means $\sigma(A) = \{\lambda_\ell, i\lambda_\ell, -\lambda_\ell, -i\lambda_\ell\}$, which has 4-sato.

Case 2: If $m = 2$, then $f_A = gh$, where g is irreducible of degree 2 and the set $V_{\mathbb{R}}(g)$ is an ellipse E_1. If h has two factors of degree 1 then $h(u, v, w) = (a_1 u + b_1 v + w)(a_2 u + b_2 v + w)$, where $\lambda_1 = a_1 + i b_1$ and $\lambda_2 = a_2 + i b_2$ are eigenvalues of A. In this case, $W(A)$ is the convex hull of $E_1 \cup \{\lambda_1\} \cup \{\lambda_2\}$. If both λ_1 and λ_2 are inside the convex hull of E_1, then $W(A)$ is the convex hull of E_1, which does not have 4-sato unless it is a disk; this is precluded by hypothesis. If one or both of λ_1 and λ_2 are outside E_1, then $\partial W(A)$ will have exactly one or two corners where lines intersect, which is impossible if $W(A)$ has 4-sato. Therefore h is also irreducible of degree 2 and hence $V_{\mathbb{R}}(h)$ is an ellipse E_2. If either E_1 or E_2 is contained inside the convex hull of the other, then $\partial W(A)$ is the outer ellipse and we are back at the impossible case where $W(A)$ is a circular disk. Therefore $\partial W(A)$ consists of portions of E_1, E_2, and flat portions connecting the two ellipses. In particular, there is a (nonuniquely determined) arc of E_j (denoted by γ_j) that is contained in $\partial W(A)$ for each $j = 1, 2$.

Notate $g(u, v, w) = a_1u^2 + a_2v^2 + w^2 + a_4uv + a_5uw + a_6vw$ and $h(u, v, w) = b_1u^2 + b_2v^2 + w^2 + b_4uv + b_5uw + b_6vw$. If $(u, v, 1) \in \gamma_1$, then $g(u, v, 1) = 0$. The assumption that $W(A)$ has 4-sato means that the point $(-v, u, 1)$ obtained by rotating $(u, v, 1)$ by $\frac{\pi}{2}$ radians is either on E_1 or E_2. If $(-v, u, 1)$ is in E_1 for infinitely many points on the arc γ_1, then $g(-v, u, 1) = 0$ for those points and the (irreducible) polynomials $g(u, v, w)$ and $g(-v, u, w)$ are the same. Matching coefficients of these polynomials shows that $a_1 = a_2$, $a_4 = -a_4$, $a_5 = a_6$, and $a_6 = -a_5$. Therefore $g(u, v, w) = a_1(u^2 + v^2) + w^2$, and $V_{\mathbb{R}}(g)$ is a circle centered at the origin. A similar analysis applied to points in γ_2 shows that either there are infinitely many points of $i\gamma_2$ on E_1 or else E_2 is also a circle centered at the origin. Since $W(A)$ is the convex hull of $E_1 \cup E_2$, both curves cannot be circles centered at the origin or else $W(A)$ will be the circular disk with the smaller radius.

In fact, neither E_1 nor E_2 can be a circle centered at the origin. To prove this, assume without loss of generality that E_1 is a circle centered at the origin. If infinitely many points of γ_2 rotate to land on E_1, then the rotated curve is a circle centered at the origin, so E_2 is also such a circle. But the argument above shows that if infinitely many points of γ_2 rotate to E_2, then E_2 is also a circle centered at the origin. Thus neither E_1 nor E_2 can be a circle centered at the origin.

So without loss of generality, there must be infinitely many points on the arc γ_1 such that the corresponding rotated points are on E_2. Thus $g(u, v, 1) = 0$ and $h(-v, u, 1) = 0$ for infinitely many (u, v). Therefore since g and h are irreducible, $g(u, v, w) = h(-v, u, w)$. Setting corresponding coefficients equal yields

$$h(u, v, w) = a_2u^2 + a_1v^2 + w^2 - a_4uv - a_6uw + a_5vw.$$

We can rotate the points $(u, v, 1)$ on γ_1 again to obtain that either $h(-u, -v, 1) = 0$ or $g(-u, -v, 1) = 0$ for infinitely many points satisfying $g(u, v, 1) = 0$. The former means that $h(-u, -v, w) = g(u, v, w)$, which leads to $a_4 = a_5 = a_6 = 0$, which results in the circle contradiction. Consequently $g(-u, -v, w) = g(u, v, w)$, which results in $a_5 = a_6 = 0$. Therefore the original ellipse E_1 is centered at the origin and E_2 is described by $h(u, v, w) = g(-v, u, w) = 0$, which is the ellipse E_1 rotated by $\frac{\pi}{2}$. We conclude that the original boundary-generating curve f_A satisfies

$$f_A(u, v, w) = (a_1u^2 + a_2v^2 + w^2 + a_4uv)(a_2u^2 + a_1v^2 + w^2 - a_4uv).$$

Because we know the eigenvalues of A are precisely the values of $-w$ for which $f_A(1, i, w) = 0$, it follows that the eigenvalues of A are solutions to

$$0 = (a_1 - a_2 + ia_4 + w^2)(a_2 - a_1 - ia_4 + w^2) = (w^4 - (a_1 - a_2 + ia_4)^2).$$

Therefore the eigenvalues are the four fourth roots of a fixed complex number and thus have 4-sato. Note that if $\sigma(A) = \{0\}$, then $a_1 = a_2$ and $a_4 = 0$, which again leads to a circular numerical range and is thus impossible in this case by hypothesis.

Case 3: If $m = 3$, then $f_A = gh$, where g is irreducible of degree 3 and h has degree 1. As in Case 1, $h(u, v, w) = au + bv + w$, where $\lambda = a + ib$ is an eigenvalue of A. The numerical range is the convex hull of λ and the real part of the curve $V_{\mathbb{R}}(g)$ in line coordinates. If $\lambda \in \operatorname{conv}(V_{\mathbb{R}}(g))$ then $W(A) = \operatorname{conv}(V_{\mathbb{R}}(g))$. As in the proof of Proposition 10, there must be a nonflat portion of $\partial W(A)$ that consists of a portion γ of $V_{\mathbb{R}}(g)$ with infinitely many points. When $W(A)$ is rotated by $\frac{\pi}{2}$ radians, the rotation of γ is also on $\partial W(A)$. Hence by Lemma 9, $g(u, v, w) = g(-v, u, w)$ for all (u, v, w) in $\mathbb{P}_2(\mathbb{C})$. Since g has degree 3, it must be of the form

$$\begin{aligned} g(u, v, w) = c_1u^3 + c_2v^3 + w^3 + c_4u^2v + c_5u^2w \\ + c_6uvw + c_7uv^2 + c_8uw^2 + c_9v^2w + c_{10}w^2v. \end{aligned}$$

Setting equivalent coefficients of $g(u, v, w)$ and $g(-v, u, w)$ equal yields

$$c_1 = c_2, \quad c_2 = -c_1, \quad c_4 = -c_7, \quad c_5 = c_9, \quad c_6 = -c_6,$$
$$c_7 = c_4, \quad c_8 = c_{10}, \quad c_9 = c_5 \quad \text{and} \quad c_{10} = -c_8.$$

Therefore $g(u, v, w) = w^3 + c_5u^2w + c_5v^2w$. If $c_5 < 0$, then $W(A)$ is a circular disk, contradicting our hypothesis. If $c_5 > 0$, then $V_{\mathbb{R}}(g)$ is empty, contradicting the assumption that $\lambda \in \operatorname{conv}(V_{\mathbb{R}}(g))$. If $c_5 = 0$, then $f_A(u, v, w) = w^4$, which contradicts our assumption that Case 3 holds.

If the eigenvalue λ is not in $\operatorname{conv}(V_{\mathbb{R}}(g))$ then $\partial W(A)$ has a vertex at λ where two flat portions of the boundary must meet. By assumption, $\partial W(A)$ also has vertices at $i\lambda$, $-\lambda$, and $-i\lambda$. Note that these points are distinct; the assumption that $W(A)$ has 4-sato means that the origin is either the only point in $W(A)$ (precluded by this case) or in the interior of $W(A)$. Since any vertex on $\partial W(A)$ is an eigenvalue of A, this would immediately show that $\sigma(A)$ has 4-sato. However, this case will not even occur because the convex hull of the real part of the irreducible cubic g will not contain four vertices. □

5. Support function and symmetry about axes

If A is a 4×4 matrix such that $W(A)$ has 4-sato, we will use the support function for $W(A)$ to derive the numerical radius of A and we will provide an estimate that measures how far $W(A)$ is from a circular disk. We will also prove that $W(A)$ has a particular type of axis symmetry. We will assume that $W(A)$ is noncircular; clearly if $W(A)$ is a circular disk centered at the origin, then $W(A)$ is also symmetric about every line through the origin and the numerical radius is the radius of the circle. Determining the support function involves a lot of calculation which was done in a special case in [Deaett et al. 2013], so we will use the support function from that special case to obtain the general case.

Accordingly, assume that A is a nonzero 4×4 matrix such that $W(A)$ has 4-sato but is not a circular disk. By Proposition 11, the eigenvalues of A have 4-sato. By Theorem 8, $\operatorname{tr}(A^2A^*)=\operatorname{tr}(A^3A^*)=0$. Now rename this matrix B and assume we are in the special case where the eigenvalues of B are $1, i, -1, -i$. Then the proof of Theorem 3.1 in [Deaett et al. 2013] shows that the characteristic polynomial for $\operatorname{Re}(e^{-i\theta}B)$ is

$$q_\theta(z)=z^4-\tfrac{1}{4}\operatorname{tr}(BB^*)z^2-\tfrac{1}{4}\operatorname{tr}\Bigl(\tfrac{1}{16}(e^{-4i\theta}B^4+4(B^*)^2B^2+2(B^*B)^2+e^{4i\theta}(B^*)^4)\Bigr)+\tfrac{1}{32}(\operatorname{tr}(BB^*))^2.$$

Since $\operatorname{Re}(e^{-i\theta}B)$ is Hermitian, all of the roots of q_θ are real. The support function is the maximum root of q_θ, so the formula for $p_B(\theta)$ follows directly from the equation above and each expression under a root is real and nonnegative for all θ.

$$p_B(\theta)=\frac{\sqrt{\operatorname{tr}(BB^*)+\sqrt{8\cos(4\theta)+4\operatorname{tr}(B^{*2}B^2)+2\operatorname{tr}(B^*BB^*B)-(\operatorname{tr}(BB^*))^2}}}{2\sqrt{2}}. \tag{6}$$

Now assume the general case where A is a nonzero 4×4 matrix such that $W(A)$ is noncircular and has 4-sato. By Proposition 11 the eigenvalues of A are $a, ai, -a, -ai$ for some nonzero $a\in\mathbb{C}$. Thus $A=aB$ for some B with eigenvalues $1, i, -1, -i$. Let $\alpha=\arg a$. It is straightforward to compute that $p_A(\theta)=|a|p_B(\theta-\alpha)$. Therefore, by (6), we obtain

$$\begin{aligned} p_A(\theta) &\\ &=\frac{|a|\sqrt{\operatorname{tr}(BB^*)+\sqrt{8\cos(4\theta-4\alpha)+4\operatorname{tr}(B^{*2}B^2)+2\operatorname{tr}(B^*BB^*B)-(\operatorname{tr}(BB^*))^2}}}{2\sqrt{2}}\\ &=\frac{\sqrt{\operatorname{tr}(AA^*)+\sqrt{8|a|^4\cos(4\theta-4\alpha)+4\operatorname{tr}(A^{*2}A^2)+2\operatorname{tr}(A^*AA^*A)-(\operatorname{tr}(AA^*))^2}}}{2\sqrt{2}}. \end{aligned} \tag{7}$$

The numerical radius of A is the maximum value of the support function. The previous discussion leads to the following result.

Proposition 12. *Assume A is a 4×4 matrix such that $W(A)$ has 4-sato and is not a circular disk. Assume $\sigma(A)=\{a, ai, -a, -ai\}$ for some nonzero complex* *number a. Then the numerical radius of A is*

$$\omega(A) = p_A(\alpha) = \frac{\sqrt{\operatorname{tr}(AA^*)+\sqrt{8|a|^4+4\operatorname{tr}(A^{*2}A^2)+2\operatorname{tr}(A^*AA^*A)-(\operatorname{tr}(AA^*))^2}}}{2\sqrt{2}},$$

the minimum value of the support function for A is

$$p_A\big(\alpha+\tfrac{\pi}{4}\big) = \frac{\sqrt{\operatorname{tr}(AA^*)+\sqrt{-8|a|^4+4\operatorname{tr}(A^{*2}A^2)+2\operatorname{tr}(A^*AA^*A)-(\operatorname{tr}(AA^*))^2}}}{2\sqrt{2}},$$

and

$$-8|a|^2+4\operatorname{tr}(A^{*2}A^2)+2\operatorname{tr}(A^*AA^*A)-(\operatorname{tr}(AA^*))^2 \geq 0. \tag{8}$$

We will now derive an expression that measures the "noncircularity" of the numerical range of a 4×4 matrix A (where $W(A)$ has 4-sato) in terms of $\operatorname{tr}(A^*A)$. Let g_A denote the difference between the maximum and minimum values of the support function of A. That is, $g_A = p_A(\alpha) - p_A\big(\alpha+\frac{\pi}{4}\big)$ as defined above. The quantity g_A measures the gap between the points on the boundary of $W(A)$ that are farthest from, and closest to, the origin.

To prove the following lower bound for g_A, we will produce some inequalities involving $\operatorname{tr}(A^*A)$ and traces of more complicated words in A and A^*. This proposition will be used in the next section to prove that the numerical range of a certain composition operator is not circular.

Proposition 13. *Assume A is a 4×4 matrix such that $W(A)$ has 4-sato and $\sigma(A) = \{1,-1,i,-i\}$. Let $\alpha = \operatorname{tr}(A^*A)$. Then*

$$g_A \geq \frac{8}{\sqrt{2}\big(\sqrt{\alpha+\sqrt{5\alpha^2+8}}+\sqrt{\alpha+\sqrt{5\alpha^2-8}}\big)\big(\sqrt{5\alpha^2+8}+\sqrt{5\alpha^2-8}\big)}.$$

Proof. Recall that

$$\langle A, B\rangle = \operatorname{tr}(B^*A)$$

defines an inner product on $M_n(\mathbb{C})$ and in particular on $M_4(\mathbb{C})$. Thus $\langle A, A\rangle = \|A\|_{\mathrm{tr}}^2 = \operatorname{tr}(A^*A)$.

The Cauchy–Schwarz inequality on this space shows that

$$|\operatorname{tr}(B^*A)| \leq \|A\|_{\mathrm{tr}}\,\|B\|_{\mathrm{tr}} = \sqrt{\operatorname{tr}(A^*A)\operatorname{tr}(B^*B)}. \tag{9}$$

The trace norm induced by this inner product is a matrix norm [Horn and Johnson 1991], so $\|AB\|_{\mathrm{tr}} \leq \|A\|_{\mathrm{tr}}\,\|B\|_{\mathrm{tr}}$, and therefore

$$\operatorname{tr}(A^*AA^*A) = \|A^*A\|_{\mathrm{tr}}^2 \leq \|A^*\|_{\mathrm{tr}}^2\,\|A\|_{\mathrm{tr}}^2 = \|A\|_{\mathrm{tr}}^4 = (\langle A, A\rangle)^2 = \alpha^2. \tag{10}$$

Also,

$$\operatorname{tr}((A^*)^2A^2) = \|A^2\|_{\mathrm{tr}}^2 \leq \|A\|_{\mathrm{tr}}^4 = (\langle A, A\rangle)^2 = \alpha^2. \tag{11}$$

Define

$$d(A, A^*) = 4\,\mathrm{tr}((A^*)^2 A^2) + 2\,\mathrm{tr}(A^* A A^* A) - (\mathrm{tr}(A^* A))^2. \tag{12}$$

Combining (8), (10), and (11) yields

$$8 \le d(A, A^*) \le 4\alpha^2 + 2\alpha^2 - \alpha^2 = 5\alpha^2. \tag{13}$$

The assumptions on the spectrum of A imply that $\alpha^2 \ge 4$. The maximum value of the support function can be written in terms of $d(A, A^*)$ and α as

$$p_A(0) = \frac{\sqrt{\alpha + \sqrt{8 + d(A, A^*)}}}{2\sqrt{2}},$$

while the minimum value is

$$p_A\left(\tfrac{\pi}{4}\right) = \frac{\sqrt{\alpha + \sqrt{-8 + d(A, A^*)}}}{2\sqrt{2}}.$$

Therefore, the distance between the maximum value of the support function and the minimum value of the support function is

$$g_A = \frac{\left(\sqrt{\alpha + \sqrt{d(A, A^*) + 8}} - \sqrt{\alpha + \sqrt{d(A, A^*) - 8}}\right)}{2\sqrt{2}}.$$

We want to find a lower bound for g_A in terms of α.

By multiplying g_A by its algebraic conjugate in the numerator and denominator, we obtain

$$g_A = \frac{\sqrt{d(A, A^*) + 8} - \sqrt{d(A, A^*) - 8}}{2\sqrt{2}\left(\sqrt{\alpha + \sqrt{d(A, A^*) + 8}} + \sqrt{\alpha + \sqrt{d(A, A^*) - 8}}\right)}.$$

Now multiply numerator and denominator by the conjugate of the numerator to see that

$$g_A = \frac{8}{\sqrt{2}\left(\sqrt{\alpha + \sqrt{d(A, A^*) + 8}} + \sqrt{\alpha + \sqrt{d(A, A^*) - 8}}\right)\left(\sqrt{d(A, A^*) + 8} + \sqrt{d(A, A^*) - 8}\right)}.$$

Each term of each factor in the denominator of g_A is a positive increasing function of $d(A, A^*)$. Therefore, (13) implies that

$$g_A \ge \frac{8}{\sqrt{2}\left(\sqrt{\alpha + \sqrt{5\alpha^2 + 8}} + \sqrt{\alpha + \sqrt{5\alpha^2 - 8}}\right)\left(\sqrt{5\alpha^2 + 8} + \sqrt{5\alpha^2 - 8}\right)}. \qquad \square$$

Tsai and Wu [2011] proved a number of results about numerical ranges of weighted shift matrices. In particular, they show that the numerical range of any $n \times n$ weighted shift matrix A (and thus any AWS matrix) is symmetric about each of n lines through the origin that are determined by the entries of A. The angle between each pair of adjacent lines is $\frac{\pi}{n}$. We will show that if the numerical range of a 4×4 matrix A has 4-sato, then $W(A)$ is similarly symmetric about four lines through the origin even if A is not unitarily equivalent to an AWS matrix.

Property (V) from the Introduction shows that for any $n \times n$ matrix A, the set $W(A^*)$ is the reflection of $W(A)$ about the real axis. For any angle θ, the matrix Re $e^{-i\theta}A$ is the same as Re $e^{i\theta}A^*$. Since the support function $p_A(\theta)$ is the maximum eigenvalue of Re $e^{-i\theta}A$, it is also true that $p_A(-\theta) = p_{A^*}(\theta)$ for all real θ.

Proposition 14. *Let A be an $n \times n$ matrix. The numerical range $W(A)$ is symmetric about the real axis if and only if the support function p_A is an even function.*

Proof. Assume $W(A)$ is symmetric about the real axis and consequently $W(A) = W(A^*)$. Hence $p_A(\theta) = p_{A^*}(\theta) = p_A(-\theta)$ for all real θ.

Now assume $p_A(\theta) = p_A(-\theta)$ for all θ. This implies that $p_A = p_{A^*}$ and consequently the numerical ranges $W(A)$ and $W(A^*)$ are equal, so $W(A)$ is symmetric about the real axis. □

Corollary 15. *Let A be an $n \times n$ matrix such that the origin is in the numerical range $W(A)$. Let $\delta \in \mathbb{R}$. The numerical range is symmetric about the line ℓ through the origin and $e^{i\delta}$ if and only if the support function p_A for $W(A)$ satisfies $p_A(\theta + \delta) = p_A(-\theta + \delta)$.*

Proof. $W(A)$ is symmetric about ℓ if and only if the rotated set $e^{-i\delta}W(A)$ is symmetric about the real axis. The latter statement is equivalent to the numerical range of $e^{-i\delta}A$ having symmetry about the real axis. Since the definition of the support function shows that $p_{e^{-i\delta}A}(\theta) = p_A(\theta + \delta)$, the corollary follows from Proposition 14. □

The strong connection between n-fold symmetry about the origin and axis symmetry depends on the size of the matrix relative to n, as the following example due to Spitkovsky (personal communication, 2012) shows:

Example 16. Let A be the 8×8 diagonal matrix with diagonal entries 1, i, -1, $-i$, $0.9e^{\pi i/6}$, $0.9e^{2\pi i/3}$, $0.9e^{7\pi i/6}$, $0.9e^{5\pi i/3}$. Thus $W(A)$ is a polygon with vertices at the eight diagonal entries as shown in Figure 3. Clearly $W(A)$ has 4-sato, as does $\sigma(A)$, but $W(A)$ is not symmetric about any axis.

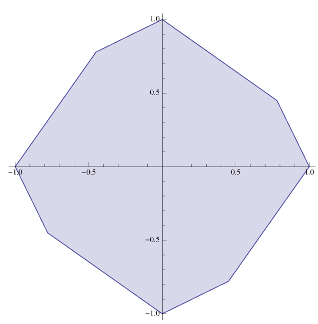

Figure 3. 4-sato but no axis symmetry.

Theorem 17. *Assume A is a 4×4 nonzero matrix such that $W(A)$ has 4-sato and is noncircular. The eigenvalues of A are a, ia, $-a$, and $-ia$ for $a \in \mathbb{C}$ with $a \neq 0$ and $\alpha = \arg a$. Let $\delta_n = \alpha + \frac{n\pi}{4}$ for $n = 0, 1, 2, 3$. The numerical range $W(A)$ is symmetric about the lines through 0 and $e^{i\delta_n}$ for $n = 0, 1, 2, 3$.*

Proof. Assume A is a 4×4 nonzero matrix satisfying the theorem hypotheses. Note that the form of $\sigma(A)$ follows from Proposition 11. The support function $p_A(\theta)$ for $W(A)$ only depends on θ through the term $\cos(4\theta - 4\alpha)$, as seen in (7). For each integer n with $0 \leq n < 4$ and each real θ,

$$\begin{aligned}\cos(4(\theta + \delta_n) - 4\alpha) &= \cos(4\theta + n\pi)\\ &= \cos(-4\theta - n\pi + 2n\pi)\\ &= \cos(-4\theta + n\pi) = \cos(4(-\theta + \delta_n) - 4\alpha).\end{aligned}$$

Therefore $p_A(\theta + \delta_n) = p_A(-\theta + \delta_n)$ for all θ and $n = 0, 1, 2, 3$, which means $W(A)$ has the stated symmetry by Corollary 15. □

After submission of this paper, the authors learned of the preprint [Lentzos and Pasley 2017], one result in which provides an alternate proof of Theorem 17 by showing that any boundary-generating curve for a numerical range with n-sato can be associated with an AWS matrix, even if the original matrix is not unitarily equivalent to an AWS matrix.

6. Application to numerical range of composition operator

The Hardy–Hilbert space $H^2 = H^2(\mathbb{D})$ is the set of all analytic functions f on the unit disk $\mathbb{D}$ such that

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

If φ is an analytic self-map of $\mathbb{D}$, the associated composition operator C_φ is defined for $f \in H^2$ by $C_\varphi f = f \circ \varphi$. On H^2, it can be shown that the operator C_φ is bounded for all analytic mappings φ from $\mathbb{D}$ to itself. See [Cowen and MacCluer 1995] for this and other properties of composition operators.

If φ is an automorphism of the disk $\mathbb{D}$, then there exist $\eta \in \partial\mathbb{D}$ and $p \in \mathbb{D}$ such that

$$\varphi(z) = \eta \frac{p - z}{1 - \bar{p}z}.$$

The automorphism φ can be classified as elliptical, hyperbolic, or parabolic depending on the locations of the fixed points of φ. If φ has one of its fixed points in the interior of $\mathbb{D}$ then it is elliptical. Bourdon and Shapiro [2000] determined the shape of the numerical range for many composition operators on $H^2(\mathbb{D})$ with automorphic symbols. In many cases the numerical range was a circular disk and it was also determined whether the numerical range was open, closed, or neither. However, Bourdon and Shapiro noted that when the automorphic symbol satisfies $\varphi \circ \varphi(z) = z$, and hence C_φ^2 is the identity operator I, the numerical range is a noncircular ellipse. This fact holds more generally for all quadratic operators, as shown in [Tso and Wu 1999].

Bourdon and Shapiro conjectured that any composition operator on H^2 with automorphic symbol satisfying $\varphi^{(n)}(z) = z$ (where n is a positive integer and $\varphi^{(n)}$ denotes composition of φ with itself n times) has a noncircular numerical range. Unlike the case for quadratic operators, this fact does not generalize; for example, there exists an operator T on a Hilbert space such that $T^3 = I$ and $W(T)$ is a circular disk [Harris et al. 2011]. The third author showed that Bourdon and Shapiro's conjecture is true for $n = 3$. That is, a composition operator satisfying $C_\varphi^3 = I$ does not have a circular disk as its numerical range [Patton 2013]. The result follows because any composition operator C_φ with automorphic symbol satisfying $\varphi^{(n)}(z) = z$ is unitarily equivalent to a block Toeplitz matrix with Toeplitz symbol equal to an $n \times n$ matrix-valued polynomial of degree 1. That is, the symbol has the form $A(z) = A_0 + A_1 z$ and there is an orthonormal basis with respect to which C_φ has the matrix

$$\mathcal{M}(C_\varphi) = \begin{pmatrix} A_0 & 0 & 0 & \cdots \\ A_1 & A_0 & 0 & \cdots \\ 0 & A_1 & A_0 & \\ 0 & 0 & A_1 & \ddots \\ \vdots & \vdots & & \ddots \end{pmatrix}. \tag{14}$$

Bebiano and Spitkovsky [2012] showed that in general, the closure of the numerical range of a block Toeplitz matrix with matrix-valued symbol a is the convex hull of the set $\{W(A) : A \in R(a)\}$, where $R(a)$ is the essential range of the symbol on $\partial\mathbb{D}$. In the composition operator case above, this reduces to the following theorem.

Theorem 18 [Patton 2013]. *Let $\eta \in \partial\mathbb{D}$ and let $p \in \mathbb{D}$. Define the disk automorphism*

$$\varphi = \eta \frac{p - z}{1 - \bar{p}z}$$

and assume $\varphi^{(n)}(z) = z$. The numerical range of the composition operator C_φ satisfies

$$\operatorname{clos} W(C_\varphi) = \operatorname{conv}\{W(A_0 + A_1 z) \mid z \in \partial\mathbb{D}\},$$

where A_0 and A_1 are $n \times n$ matrices whose entries depend on η and p.

Formulas for the entries of A_0 and A_1 appear in [Patton 2013]. The matrix A_1 is a particularly simple rank-1 matrix. In the case where $n = 4$, the entries of A_0 and

A_1 are shown below:

$$A_0 = \begin{pmatrix} 1 & \frac{p\eta}{P_-} & 0 & 0 \\ 0 & -\eta & \frac{p(1+\eta)}{P_+} & 0 \\ 0 & -\frac{\eta\bar{p}}{P_+} & \frac{-1+\eta}{1+\bar{\eta}} & \frac{p}{P_+} \\ 0 & -\frac{(1+\eta)\bar{p}^2}{(P_+)^2} & \frac{(-1+\eta)\bar{p}}{P_+} & \frac{1-\bar{\eta}}{1+\bar{\eta}} \end{pmatrix}.$$

$$A_1 = -\frac{\bar{\eta}\,\bar{p}\,(1-\bar{\eta})}{1-\bar{\eta}|p|^2} \begin{pmatrix} 0 & \frac{\bar{p}^2}{P_-} & -\frac{\eta\,\bar{p}(1-\bar{\eta}|p|^2)}{P_+P_-} & \frac{P_-}{\eta-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $P_- = \sqrt{1-|p|^2}$ and $P_+ = \sqrt{1+|p|^2}$. We will use the numerical radius estimates in the previous section to show that the conjecture of Bourdon and Shapiro holds for $n = 4$.

The assumption that $\varphi^{(4)}(z) = z$ implies that the parameters η and p satisfy the identity

$$2|p|^2 = \eta + \bar{\eta}, \tag{15}$$

and this immediately yields

$$|1-\bar{\eta}|^2 = 2(1-|p|^2), \quad |1+\bar{\eta}|^2 = 2(1+|p|^2), \quad \left|1-\eta|p|^2\right|^2 = 1-|p|^4, \tag{16}$$

and

$$4(1+|p|^2)^2 = (\eta+1)(\eta+3) + (\bar{\eta}+1)(\bar{\eta}+3). \tag{17}$$

These identities can be used to rewrite the entries of A_0 and A_1 with only real quantities in the denominator, and we obtain

$$A(z) = \begin{pmatrix} 1 & \frac{p\eta}{P_-} - z\frac{(\bar{\eta}+1)\bar{p}^3}{(P_+)^2P_-} & z\frac{\bar{p}^2(1-\bar{\eta})}{P_+P_-} & z\frac{\bar{p}(1-\bar{\eta}|p|^2)}{(P_+)^2P_-} \\ 0 & -\eta & \frac{p(1+\eta)}{P_+} & 0 \\ 0 & -\frac{\eta\bar{p}}{P_+} & \frac{\eta|p|^2-1}{(P_+)^2} & \frac{p}{P_+} \\ 0 & -\frac{(1+\eta)\bar{p}^2}{(P_+)^2} & \frac{(-1+\eta)\bar{p}}{P_+} & \frac{\eta-\bar{\eta}}{2(P_+)^2} \end{pmatrix}.$$

Proposition 13 will be applied to $A(z)$ in order to show there is a fixed gap between the maximum and minimum value of the support function of $W(C_\varphi)$; this suffices to prove $W(C_\varphi)$ is not a circular disk. In order to apply the proposition, we must show that $W(A(z))$ has 4-sato for all z on the unit circle. By Theorem 8, it suffices to show that $\sigma(A(z)) = \{1, i, -1, -i\}$ and the traces of $A(z)^2A(z)^*$ and $A(z)^3A(z)^*$ are zero.

The condition $\varphi^{(4)}(z) = z$ shows that $C_\varphi^4 = I$, and thus $\mathcal{M}(C_\varphi)^4 = I$. The latter and (14) imply that $A_0^4 = I$, and the form of the matrix $A(z)$ guarantees that $\sigma(A(z)) = \sigma(A_0) = \{1, i, -1, -i\}$ for all $z \in \partial\mathbb{D}$.

Some tedious calculations that lead to the trace requirements are done next.

First, we obtain

$$A(z)A(z)^* =$$
$$\begin{pmatrix} \left(\frac{P_+}{P_-}\right)^2 + \frac{2\operatorname{Re}(-z\bar{\eta}(1+\bar{\eta})\bar{p}^4)}{(P_+P_-)^2} & -\frac{p}{P_-} + \frac{z(1+\bar{\eta})\bar{p}^3}{(P_+)^2P_-} & \frac{-p^2+z\bar{p}^2\bar{\eta}}{P_+P_-} & \frac{(-p^3(\eta+1)+z\bar{p}(1+\bar{\eta})^2/2)}{(P_+)^2P_-} \\ -\frac{\bar{p}}{P_-} + \frac{\bar{z}(1+\eta)p^3}{(P_+)^2P_-} & 1+2|p|^2 & \frac{\bar{\eta}p}{P_+} & \frac{p^2(1+\bar{\eta})}{(P_+)^2} \\ \frac{-\bar{p}^2+\bar{z}p^2\eta}{P_+P_-} & \frac{\eta\bar{p}}{P_+} & 1 & \frac{p}{P_+} \\ \frac{(-\bar{p}^3(\bar{\eta}+1)+\bar{z}p(1+\eta)^2/2)}{(P_+)^2P_-} & \frac{(1+\eta)\bar{p}^2}{(P_+)^2} & \frac{\bar{p}}{P_+} & 1 \end{pmatrix}.$$

Next we compute that

$$A(z)^2A(z)^* =$$
$$\begin{pmatrix} 1+2|p|^2 & \frac{p\eta(\eta+1)}{P_-} - z\left(\frac{\bar{p}^3(1+\bar{\eta})^2}{(P_+)^2P_-}\right) & \frac{z\bar{p}^2(1-\bar{\eta}^2)}{P_+P_-} & \frac{z\bar{p}(1+\bar{\eta})(1-\bar{\eta}^2)}{2(P_+)^2P_-} \\ \frac{\bar{p}(\eta-|p|^2)}{(P_+)^2P_-} & \frac{\eta(-1-2|p|^2-2|p|^4+\eta|p|^2)}{(P_+)^2} & \frac{\eta p}{P_+} & 0 \\ \frac{\bar{p}^2(1-\bar{\eta})}{P_+P_-} & -\frac{\bar{p}(1+\eta)^2}{(P_+)^3} & \frac{\eta|p|^2-1}{(P_+)^2} & 0 \\ \frac{\bar{p}^3(1+\bar{\eta})}{(P_+)^2P_-} - \bar{z}\left(\frac{p}{P_-}\right) & \frac{\bar{p}^2(-3-2\eta-\bar{\eta})}{(P_+)^2} & \frac{\bar{p}(\eta^2-\bar{\eta}^2+\eta-3\bar{\eta}-2)}{2(P_+)^3} & -\frac{(\bar{\eta}+1)^2}{2(P_+)^2} \end{pmatrix}.$$

Straightforward calculations can be used to simplify $\operatorname{tr} A(z)^2A(z)^*$ as follows:

$$\begin{aligned} &\operatorname{tr} A(z)^2A(z)^* \\ &\quad = 1+2|p|^2 + \frac{\eta(-1-2|p|^2-2|p|^4+\eta|p|^2)}{1+|p|^2} + \frac{\eta|p|^2-1}{1+|p|^2} - \frac{(\bar{\eta}+1)^2}{2(1+|p|^2)} \\ &\quad = \frac{\left(1+3|p|^2+2|p|^4-\eta-2\eta|p|^2-2\eta|p|^4+\eta^2|p|^2+\eta|p|^2-1-\frac{1}{2}\bar{\eta}^2-\bar{\eta}-\frac{1}{2}\right)}{1+|p|^2}. \end{aligned}$$

Using the identities $2|p|^2 = \eta + \bar{\eta}$ and $4|p|^4 = \eta^2 + 2 + \bar{\eta}^2$ and grouping like powers of η and $\bar{\eta}$ proves that

$$\operatorname{tr} A(z)^2A(z)^* = 0. \tag{18}$$

The value $\operatorname{tr} A(z)^3A(z)^*$ has a constant term, to which all four diagonal terms contribute, and a z-term, which only occurs in the $(1, 1)$ entry of $A(z)^3A(z)^*$. This

z-term has coefficient

$$\frac{-\bar{p}^4(\eta-|p|^2)(\bar{\eta}+1)}{(1-|p|^2)(1+|p|^2)^2}+\frac{\bar{p}^4(1-\bar{\eta})^2}{(1-|p|^2)(1+|p|^2)}+\frac{\bar{p}^4(1+\bar{\eta})(1-\bar{\eta}|p|^2)}{(1-|p|^2)(1+|p|^2)^2}.$$

Factoring out $\bar{p}^4/(1-|p|^4)$ yields

$$\frac{\bar{p}^4}{(1-|p|^4)}\biggl(-\frac{(\bar{\eta}+1)(\eta-|p|^2)}{1+|p|^2}+\frac{(1-\bar{\eta}|p|^2)(1+\bar{\eta})}{1+|p|^2}+(1-\bar{\eta})^2\biggr),$$

and after forming a common denominator for the terms inside the square brackets and rewriting everything in terms of η and $\bar{\eta}$ using $2|p|^2=\eta+\bar{\eta}$, we obtain that the coefficient of z in the trace of $A(z)^3A(z)^*$ is zero.

The constant term is more difficult to simplify; we work separately with each diagonal entry.

The $(2,2)$ entry of $A(z)^3A(z)^*$ simplifies to

$$\frac{\eta^3+2\eta^2-2-\bar{\eta}}{4(1+|p|^2)^2}.$$

The $(3,3)$ entry of $A(z)^3A(z)^*$ simplifies to

$$\frac{-(\eta+1)^3-(\bar{\eta}+1)^3}{4(1+|p|^2)^2}.$$

The $(4,4)$ entry of $A(z)^3A(z)^*$ is

$$\frac{-(\bar{\eta}+1)^2(\eta-\bar{\eta})}{4(1+|p|^2)^2}.$$

The constant term of the $(1,1)$ entry of $A(z)^3A(z)^*$ is

$$1+2|p|^2+\frac{|p|^2(\eta-|p|^2)\eta}{1-|p|^4}-\frac{|p|^2(1-\bar{\eta}|p|^2)}{1-|p|^4}.$$

The numerator of the latter expression over the common denominator $(1-|p|^4)$ can be expressed as

$$1+|p|^2\bigl(1+\eta^2+|p|^2(-1-\eta-\bar{\eta})-2|p|^4\bigr),$$

and this simplifies to $1-|p|^4$ using (15) and (16). Consequently the constant part of the $(1,1)$ entry of $A(z)^3A(z)^*$ is 1.

The simplified sum of the $(2,2)$, $(3,3)$, and $(4,4)$ entries of the constant term is

$$\frac{-\bar{\eta}^2-4\bar{\eta}-6-4\eta-\eta^2}{4(1+|p|^2)^2}=-1,$$

where the equality follows from (17). Therefore, we obtain

$$\operatorname{tr} A(z)^3 A(z)^* = 0. \tag{19}$$

Equations (18) and (19) hold for all z on the unit circle; we also saw that the spectrum of $A(z)$ is $\{1, i, -1, -i\}$ for all such z. Therefore Theorem 8 shows that the numerical range of the matrix $A(z)$ has 4-sato for all $z \in \partial\mathbb{D}$.

The lemma below follows immediately from the calculated entries of $A(z)A(z)^*$.

Lemma 19. *If $A(z)$ is the 4×4 block matrix defined above at any value z on the unit circle, then*

$$\operatorname{tr}(A(z)A(z)^*) \le \frac{4+4|p|^2+2|p|^4}{1-|p|^4}.$$

Theorem 20. *If φ is an automorphism of the disk such that C_φ has minimal polynomial z^4-1, then $W(C_\varphi)$ is not a disk.*

Proof. The value $p=\varphi^{-1}(0)$ is in the open unit disk. Let α_p denote the upper bound for $\operatorname{tr}(A(z)A(z)^*)$ from Lemma 19. Combining this value with Proposition 13 shows that there is a uniform lower bound

$$g_p = \frac{8}{\sqrt{2}\big(\sqrt{\alpha_p+\sqrt{5\alpha_p^2+8}}+\sqrt{\alpha_p+\sqrt{5\alpha_p^2-8}}\big)\big(\sqrt{5\alpha_p^2+8}+\sqrt{5\alpha_p^2-8}\big)}$$

for the difference between the maximum and minimum values of the support functions of $A(z)$ because $g_{A(z)} \ge g_p$ for each z on the unit circle. Furthermore, Proposition 12 shows that for each z the maximum value of $p_{A(z)}(\theta)$ is attained at $\theta=0$, while the minimum is attained at $\theta=\frac{\pi}{4}$. Since the numerical range of C_φ is the convex hull of all of these matrix numerical ranges as z ranges over the unit circle, it follows that the difference between the maximum and minimum values of the support function of C_φ is bounded below by g_p. Hence $W(C_\varphi)$ is not a circular disk. □

Recently, Heydari and Abdollahi [2015] showed there is a large class of finite-order elliptic composition operators such that $W(C_\varphi)$ is not a circular disk.

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