Journal of
Mechanics of
Materials and Structures

SEMI-ANALYTICAL SOLUTION FOR A VISCOELASTIC PLANE CONTAINING MULTIPLE CIRCULAR HOLES

Yun Huang, Sofia G. Mogilevskaya
and Steven L. Crouch

Volume 1, № 3

March 2006

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 1, No. 3, 2006

SEMI-ANALYTICAL SOLUTION FOR A VISCOELASTIC PLANE CONTAINING MULTIPLE CIRCULAR HOLES

YUN HUANG, SOFIA G. MOGILEVSKAYA AND STEVEN L. CROUCH

The paper considers the problem of an infinite, homogeneous, isotropic viscoelastic plane containing multiple circular holes. Constant or time-dependent loading is applied at infinity or on the boundaries of the holes. The sizes and locations of the holes are arbitrary provided they do not overlap. The solution of the problem is based on the use of the correspondence principle, and the governing equation in the Laplace domain is a complex hypersingular boundary integral equation written in terms of the unknown transformed displacements at the boundaries of the holes. The main feature of this equation is that the material parameters are only involved as multipliers for the terms other than the integrals of transformed displacements. The unknown transformed displacements are approximated by truncated complex Fourier series with coefficients dependent on the transform parameter. A system of linear algebraic equations is formulated using Taylor series expansion for determining these coefficients. The viscoelastic stresses and displacements are calculated through the viscoelastic analogs of Kolosov–Muskhelishvili potentials, and an inverse Laplace transform is used to provide the time domain solution. All the operations (space integration, Laplace transform and its inversion) are performed analytically. The method described in the paper enables the consideration of a variety of viscoelastic models. For the sake of illustration, examples are given for the cases where the viscoelastic solid responds as (*i*) a Boltzmann model in shear and elastically in dilatation, (*ii*) a Boltzmann model in both shear and dilatation, and (*iii*) a Burgers model in shear and elastically in dilatation. The accuracy and efficiency of the approach are demonstrated by comparing selected results with the solutions obtained by the finite element method (ANSYS) and the time stepping boundary element approach.

1. Introduction

Circular cavities are frequently present in various engineering applications. Time-independent problems involving multiple circular cavities have been extensively studied. A comprehensive review of the literature related to elastic problems can be found in [Crouch and Mogilevskaya 2003]. The solutions of various harmonic

Keywords: viscoelasticity, correspondence principle, boundary integral method, Laplace transform.

and biharmonic problems have been obtained in [Bird and Steele 1991; 1992; Bird 1992]. More recently, [Chen et al. 2006c; 2006a; 2006b] described a null-field integral equation approach for plane, anti-plane shear and torsion problems.

Efficient solutions of time-dependent problems involving a large number of circular cavities have not yet been published. The present paper aims to present such a solution for linear viscoelastic problems.

Traditional methods of solving problems in linear viscoelasticity fall into three categories. Methods in the first category are based on the use of the correspondence principle. For these methods, the reformulated problem in Laplace space is solved analytically or numerically (for example, using finite element or boundary element methods), and the results are inverted into the time domain using numerical Laplace transform inversion [Schapery 1962; Rizzo and Shippy 1971; Kusama and Mitsui 1982; Wang and Crouch 1982; Sun and Hsiao 1985; Carini and Gioda 1986; Lee et al. 1994]. The accuracy of these methods depends on the choice of the appropriate values for the transform parameters [Lee and Kim 1995], which changes with each problem under consideration. This disadvantage limits the application of these methods.

In the second category, a temporal integral equation is formulated using the time-dependent Green's functions, and the time history is divided into a number of discrete steps. By approximating time-dependent unknowns by some functions (for example, polynomials) at each time step and integrating them numerically or analytically, the time convolution is replaced by a sum of integrals for all steps. In general, the computation for one time step requires knowledge of the results from all the previous steps [Lee and Kim 1995; Sim and Kwak 1988]. The amount of computation therefore increases with time. If a linear or constant time interpolation function is used, the influence from all previous steps can be stored in a time-dependent function and updated after each step. In such cases, the computational expense can be decreased to some extent [Shinokawa et al. 1985].

In the third category of solution procedures for problems in linear viscoelasticity, a boundary integral equation involving time derivatives of the principal unknown variables (for example, the displacements) is obtained using the differential constitutive equation for a particular viscoelastic model and a weighted residual technique. A finite difference scheme is adopted to approximate the time derivatives, which results in a time stepping algorithm, and the space integrals are carried out using the boundary element method [Mesquita and Coda 2001; 2002b; 2003]. For the special case in which all of the geometric features are circular, we suggested a time stepping boundary integral method based on a truncated Fourier series approximation for the boundary variables [Huang et al. 2005c; 2005b; 2005a]. All the space integrals can be evaluated analytically in this method. However, such methods are based on the assumption of a constant viscoelastic Poisson's ratio, which is physically unrealistic for practical materials. A volume integral must be included in the analysis if one wishes to consider a time-dependent Poisson's ratio, and this requires the adoption of a finite element type approach, as done for example in [Mesquita and Coda 2002a].

To overcome the disadvantages noted above for traditional methods of solution, a new approach is desired. As an attempt in this direction, we describe here a semi-analytical solution for the problem of an infinite viscoelastic plane containing multiple holes. The time-independent analog of this approach has been presented earlier in the series of papers [Mogilevskaya and Crouch 2001; 2002; Crouch and Mogilevskaya 2003; Wang et al. 2003a; 2003b; Mogilevskaya and Crouch 2004; Legros et al. 2004]. The technique presented in those papers was based on the use of complex or real versions of the two-dimensional Somigliana's formula. The unknown variables on the circular boundaries were approximated by truncated Fourier series. All the space integrals involved were evaluated analytically. In fact, infinite Fourier series provide the analytical solution for this class of problems; apart from round-off error, the only errors introduced in the numerical model are due to truncation of the series.

In the present paper, we extend this technique to the area of linear viscoelasticity. The solution presented in this paper is based on the correspondence principle and the analytical Laplace transform and its inversion, rather than the time stepping scheme used in our previous work [Huang et al. 2005c; 2005b; 2005a]. The governing equation for the problem in the Laplace domain is a complex hypersingular boundary integral equation written in terms of the unknown transformed displacements at the boundaries of the holes. A significant feature of this equation is that the space integrals involving the unknown variables (the transformed boundary displacements) do not include the material properties; the material parameters only appear as multipliers for the terms involving transformed far-field stress and pore pressures. The unknown transformed displacements on the circular boundary are approximated by truncated complex Fourier series with the coefficients dependent on the transform parameters. A system of linear algebraic equations is formed and solved for these Fourier coefficients. The solution for stresses and displacements anywhere in the viscoelastic plane is obtained in both the Laplace and time domains. No specific physical model is involved in the governing complex variable hypersingular integral equation, which means that the method is capable of handling a variety of viscoelastic models.

Several computational examples are given. In these examples, the viscoelastic solid responds as a Boltzmann model in shear and elastically in dilatation; a Boltzmann model in both shear and dilatation; and a Burgers model in shear and elastically in dilatation. Three loading cases are considered:

(i) the viscoelastic plane is subjected to constant far-field stresses;

(ii) the holes are subjected to constant pressure; and

(iii) the holes are subjected to time-dependent pressure.

The accuracy and efficiency of the method are examined by comparison to the numerical solution obtained by commercial finite element software (ANSYS) or by a time stepping boundary element approach [Huang et al. 2005b].

2. Problem formulation

Consider an infinite, isotropic, viscoelastic plane containing an arbitrary number of nonoverlapping circular holes, as shown in Figure 1. A plane strain condition is assumed. The holes are assumed to be either traction-free or subjected to time-dependent uniform normal traction. The viscoelastic plane is subjected to time-dependent far-field stress $\sigma^{\infty}(t)$. Let R_j, z_j and L_j denote the radius, center, and boundary of the jth hole, and let $p_j(t)$ denote the time-dependent uniform normal traction acting on L_j ($p_j < 0$ for compression). Any point of the plane is identified by the complex coordinate $z = x + iy$. The global and local Cartesian coordinate systems are shown in Figure 1. The direction of travel is clockwise for all the boundaries L_j. The unit tangent q points in the direction of travel and the unit outward normal n points to the right of this direction, away from the viscoelastic solid. The evolution of displacements and stresses in the perforated viscoelastic solid is to be determined.

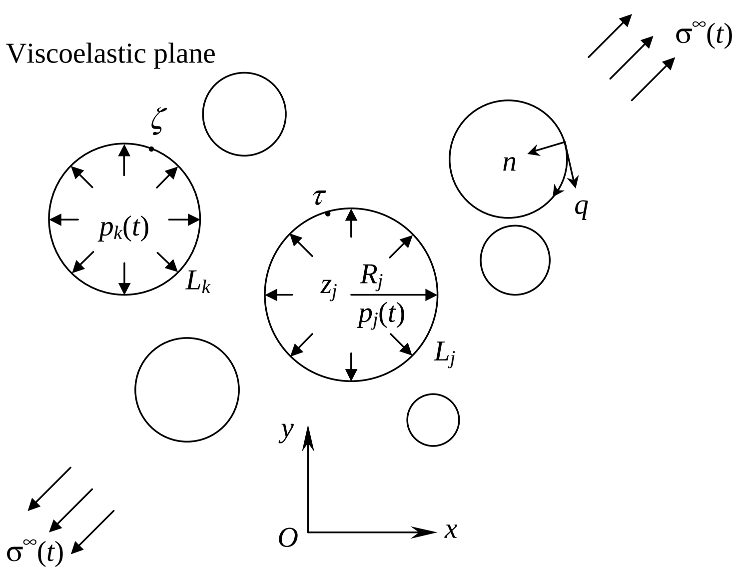

Figure 1. An infinite viscoelastic plane with multiple circular holes.

3. Correspondence principle

The correspondence principle allows the time domain solution for a linear viscoelastic problem to be obtained from the solution of a corresponding elastic problem by employing the following procedure. By using the Laplace transform, the time-dependent parameters for the original viscoelastic problem are removed by replacing them by s-varying analogs of these parameters. The resulting problem is formally equivalent to a linear elastic problem. However, the 'elastic constants' are functions of the transform parameter s, as are the transformed boundary conditions for the problem. By solving the corresponding elastic problem and taking the inverse Laplace transform, the time-dependent solution is found [Lee 1955; Findly et al. 1989].

The Laplace transform of a function $f(t)$ and its inversion are defined as [Haberman 1998]

$$\begin{aligned} f^*(s) &\equiv \Gamma[f(t)] = \int_0^\infty f(t)e^{-st}dt && (\operatorname{Re} s = \varsigma \geq 0), \\ f(t) &\equiv \Gamma^{-1}[f^*(s)] = \frac{1}{2\pi i}\int_{\varsigma - i\infty}^{\varsigma + i\infty} f^*(s)e^{st}ds && (t \geq 0,\ \varsigma \geq 0), \end{aligned} \tag{1}$$

where s is the transform parameter and ς is a vertical contour in the complex plane chosen in such a way that all singularities of $f^*(s)$ are located to the left of it.

The general way to obtain the s-varying analog of the Young's modulus $E^*(s)$ and Poisson's ratio $\nu^*(s)$ from the constitutive equations of a viscoelastic model is explained elsewhere (for example, [Wang and Crouch 1982]). Using the relations among elastic constants, one can easily obtain the s-varying shear modulus $G^*(s)$, bulk modulus $K^*(s)$ and s-varying Kolosov–Muskhelishvili parameter $\kappa^*(s)$ (which equals $3-4\nu^*(s)$ in plane strain and $(3-\nu^*(s))/(1+\nu^*(s))$ in plane stress). In Section 6, $G^*(s)$ and $\kappa^*(s)$ are given for three different viscoelastic models.

4. Basic equations

4.1. ***Basic hypersingular integral equation in the Laplace domain.*** The governing equation for the problem of a viscoelastic plane with holes (Figure 1) in the Laplace domain is an analog of the complex hypersingular integral equation for the corresponding elastic problems [Linkov and Mogilevskaya 1994; Mogilevskaya and Linkov 1998; Linkov and Mogilevskaya 1998; Linkov 2002]. To state the equation, let N be the number of holes; let i denote $\sqrt{-1}$ and $\bar{z}$ the complex conjugate of z; let $u_j^*(\tau;s) = u_{j,x}^*(\tau;s) + iu_{j,y}^*(\tau;s)$ be the result of the Laplace transform applied to the complex-valued displacement $u(\tau) = u_{j,x}(\tau) + iu_y(\tau)$ in the global coordinate system on the boundary of the j-th hole; let $\sigma_{ij}^{\infty,*}(s)$ (with i,

j representing x or y) be the components of far-field stress in the Laplace domain. The problem is then described by the N equations ($k = 1, 2, \ldots, N$)

$$\begin{aligned}\frac{1}{2\pi i}\sum_{j=1}^{N}\Biggl(2\int_{L_j}\frac{u_j^*(\tau;s)}{(\tau-\zeta)^2}d\tau-\int_{L_j}u_j^*(\tau;s)\frac{\partial^2}{\partial\tau\partial\zeta}K_1(\tau,\zeta)\,d\tau\\ -\int_{L_j}\overline{u_j^*(\tau;s)}\frac{\partial^2}{\partial\bar\tau\partial\zeta}K_2(\tau,\zeta)\,d\bar\tau\\ +\frac{p_j^*}{2G^*}\biggl((1-\kappa^*)\int_{L_j}\frac{d\tau}{\tau-\zeta}-\kappa^*\int_{L_j}\frac{\partial}{\partial\zeta}K_1(\tau,\zeta)\,d\tau+\int_{L_j}\frac{\partial}{\partial\zeta}K_2(\tau,\zeta)\,d\bar\tau\biggr)\Biggr)\\ =\frac{\kappa^*+1}{4G^*}\Bigl(p_k^*-\bigl(\sigma_{xx}^{\infty,*}+\sigma_{yy}^{\infty,*}\bigr)-\frac{d\bar\zeta}{d\zeta}\bigl(\sigma_{yy}^{\infty,*}-\sigma_{xx}^{\infty,*}-2i\sigma_{xy}^{\infty,*}\bigr)\Bigr),\end{aligned}\tag{2}$$

where G^* stands for $G^*(s)$ and likewise p_j^*, p_k^*, κ^*, and the $\sigma_{ij}^{\infty,*}$; $d\bar\zeta/d\zeta = \exp(-2i\beta)$, where β is the angle between the axis Ox and the tangent at the point ζ; $\tau \in L_j$ and $\zeta \in L_k$ for $k = 1, 2, \ldots, N$; and the kernels K_1 and K_2 are

$$K_1(\tau,\zeta)=\ln\frac{\tau-\zeta}{\bar\tau-\bar\zeta},\qquad K_2(\tau,\zeta)=\frac{\tau-\zeta}{\bar\tau-\bar\zeta}.\tag{3}$$

4.2. ***The viscoelastic analog of the Kolosov–Muskhelishvili potentials.*** In the Laplace domain the displacements and stresses at any point of the viscoelastic plane can be calculated using the viscoelastic analogs of the Kolosov–Muskhelishvili potentials [Muskhelishvili 1963]

$$2G^*(s)u^*(z;s)=\kappa^*(s)\varphi^*(z;s)-z\overline{(\partial/\partial z)\varphi^*(z;s)}-\overline{\psi^*(z;s)},\tag{4}$$

$$\sigma_{xx}^*(z;s)+\sigma_{yy}^*(z;s)=4\mathrm{Re}(\partial/\partial z)\varphi^*(z;s),\tag{5}$$

$$\sigma_{yy}^*(z;s)-\sigma_{xx}^*(z;s)+2i\sigma_{xy}^*(z;s)=2\bigl(\bar z(\partial^2/\partial z^2)\varphi^*(z;s)+(\partial/\partial z)\psi^*(z;s)\bigr),\tag{6}$$

where, as in [Wang et al. 2003a],

$$\varphi^*(z;s)=\frac{G^*(s)}{\pi i(\kappa^*(s)+1)}\sum_{j=1}^{N}\int_{L_j}\frac{u_j^*(\tau;s)}{\tau-z}\,d\tau+\varphi^{\infty,*}(z;s)\tag{7}$$

and

$$\begin{aligned}\psi^*(z;s)\\ =\frac{G^*(s)}{\pi i(\kappa^*(s)+1)}\sum_{j=1}^{N}\Biggl(\frac{p_j^*(s)}{2G^*(s)}\biggl(\int_{L_j}\frac{\bar\tau\,d\tau}{\tau-z}+\kappa^*(s)\int_{L_j}\ln(\tau-z)\,d\bar\tau\biggr)\\ +\int_{L_j}\frac{\overline{u_j^*(\tau;s)}}{\tau-z}\,d\tau-\int_{L_j}\frac{u_j^*(\tau;s)}{\tau-z}d\bar\tau+\int_{L_j}\frac{u_j^*(\tau;s)\bar\tau}{(\tau-z)^2}\,d\tau\Biggr)+\psi^{\infty,*}(z;s),\end{aligned}\tag{8}$$

with

$$\varphi^{\infty,*}(z;s)=\frac{\sigma_{xx}^{\infty,*}(s)+\sigma_{yy}^{\infty,*}(s)}{4}z,\qquad (9)$$
$$\psi^{\infty,*}(z;s)=\frac{\sigma_{yy}^{\infty,*}(s)-\sigma_{xx}^{\infty,*}(s)+2i\sigma_{xy}^{\infty,*}(s)}{2}z.$$

After the displacements $u_j^*(\tau;s)$ on boundary L_j $(j=1,\dots,N)$ have been obtained from the solution of Equation (2), the displacements and stresses in the Laplace domain at point z can be calculated using (4)–(6) and (7)–(9), provided that the integrals involved in (7) and (8) can be evaluated.

5. Numerical solution

The Laplace domain equation (2) is similar to the corresponding equation for elasticity (for example, equation (1) in [Wang et al. 2003a]). Thus, Equation (2) can be solved in the same way as its elastic counterpart [Wang et al. 2003a]. The main steps of the solution are outlined below.

5.1. ***Approximation of the boundary variables.*** The unknown displacement on the boundary L_j $(j=1,\dots,N)$ in the Laplace domain is approximated by a truncated complex Fourier series as

$$u_j^*(\tau;s)=\sum_{m=1}^{M_j}D_{-m,j}^*(s)g_j^m(\tau)+\sum_{m=0}^{M_j}D_{m,j}^*(s)g_j^{-m}(\tau),\qquad (10)$$

where the function $g_j(\tau)$ is defined as

$$g_j(\tau)=\frac{R_j}{\tau-z_j}.$$

The unknown complex Fourier coefficients $D_{\pm m,j}^*(s)$ $(m=1,\dots,M_j)$ in (10) are functions of the Laplace transform parameter s. In the following discussion, we will omit the argument s in the expressions for the Fourier coefficients for notational convenience.

With the substitution of the Fourier series representation (10) into Equation (2), the unknown coefficients can be moved outside of the space integrals. The kernel space integrals are the same as those for the elastic problem [Wang et al. 2003a]. Thus, we can use the results of the space integrals provided in that article. In this way we obtain the following system of N complex algebraic equations, one for each hole:

$$\sum_{m=1}^{M_k} m D^*_{-m,k} g_k^{m+1}(\zeta) + \big(D^*_{1,k} + \overline{D^*_{1,k}}\big) + \sum_{m=2}^{M_k} m D^*_{m,k} g_k^{1-m}(\zeta)$$
$$+ \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{R_k}{R_j} \Bigg(\sum_{m=1}^{M_j} m D^*_{-m,j} g_j^{m+1}(\zeta) + \big(D^*_{1,j} + \overline{D^*_{1,j}}\big) g_k^2(\zeta) \overline{g_j^2(\zeta)}$$
$$+ \sum_{m=2}^{M_j} m D^*_{m,j} g_k^2(\zeta) \overline{g_j^{m+1}(\zeta)}$$
$$+ \sum_{m=1}^{M_j} m \overline{D^*_{-m,j}} \Bigg(\overline{g_j^{m+1}(\zeta)} - (m+2) g_k^2(\zeta) \overline{g_j^{m+3}(\zeta)}$$
$$+(m+1)\bigg(\frac{R_k}{R_j} g_k(\zeta) + \frac{g_k^2(\zeta)}{g_j(z_k)}\bigg) \overline{g_j^{m+2}(\zeta)} \Bigg)\Bigg)$$
$$= \frac{\kappa^*(s)+1}{4G^*(s)} R_k \big(\sigma_{xx}^{\infty,*} + \sigma_{yy}^{\infty,*} - g_k^2(\zeta)(\sigma_{yy}^{\infty,*} - \sigma_{xx}^{\infty,*} - 2i\sigma_{xy}^{\infty,*})\big)$$
$$- \frac{p_k^*}{G^*(s)} R_k - \frac{1-\kappa^*(s)}{2G^*(s)} R_k g_k^2(\zeta) \sum_{\substack{j=1\\ j\neq k}}^{N} p_j^* \overline{g_j^2(\zeta)} \tag{11}$$

Similarly, with the substitution of Fourier series approximation (10) into (7) and (8), and using the results of the space integrals provided in [Wang et al. 2003a], the viscoelastic analogs of the Kolosov–Muskhelishvili potentials $\varphi^*(z;s)$ and $\psi^*(z;s)$ are expressed as

$$\varphi^*(z;s) = \frac{2G^*(s)}{\kappa^*(s)+1} \sum_{j=1}^{N} \sum_{m=1}^{M_j} D^*_{-m,j} g_j^m(z) + \frac{\sigma_{xx}^{\infty,*}(s) + \sigma_{yy}^{\infty,*}(s)}{4} z, \tag{12}$$

$$\psi^*(z;s) = \frac{2G^*(s)}{\kappa^*(s)+1} \sum_{j=1}^{N} \Bigg(\bigg(g_j^2(z) + \frac{\overline{z_j}}{z - z_j} \bigg) \sum_{m=1}^{M_j} m D^*_{-m,j} g_j^m(z) \tag{13}$$
$$- \big(D^*_{1,j} + \overline{D^*_{1,j}}\big) g_j(z) - \sum_{m=2}^{M_j} \overline{D^*_{m,j}} g_j^m(z) \Bigg)$$
$$- \frac{1-\kappa^*(s)}{1+\kappa^*(s)} \sum_{j=1}^{N} p_j^*(s) R_j g_j(z) + \frac{\sigma_{yy}^{\infty,*}(s) - \sigma_{xx}^{\infty,*}(s) + 2i\sigma_{xy}^{\infty,*}(s)}{2} z.$$

5.2. *Reduction to a linear algebraic system.* To find the unknown coefficients $D^*_{-m,j}$ $(m = 1, \ldots, M_j)$ and $D^*_{m,j}$ $(m = 1, \ldots, M_j)$ for $j = 1, \ldots, N$, we need to reduce the system (11) to a linear algebraic system. We showed in a previous paper

[Wang et al. 2003a] that, excluding collocation, there are two equivalent methods to obtain the linear algebraic system: (i) the Galerkin weighted residual method and (ii) the Taylor series expansion method. Using a Taylor series expansion technique [Wang et al. 2003a] a linear algebraic equation system is obtained as follows:

$$
\begin{aligned}
&\sum_{m=2}^{M_k} m D^*_{m,k} g_k^{1-m}(\zeta) + \sum_{n=1}^{\infty} \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{R_k}{R_j} \frac{g_k^n(z_j)}{g_k^n(\zeta)} \sum_{m=1}^{M_j} m \binom{m+n}{n} D^*_{-m,j} g_j^{m+1}(z_k)\\
&\quad + 2\,\mathrm{Re}(D^*_{1,k}) + \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{R_k}{R_j} \sum_{m=1}^{M_j} m \Big(D^*_{-m,j} g_j^{m+1}(z_k) + \overline{D^*_{-m,j}}\,\overline{g_j^{m+1}(z_k)} \Big)\\
&\quad + \sum_{m=1}^{M_k} m D^*_{-m,k} g_k^{m+1}(\zeta)\\
&\quad + \sum_{n=1}^{\infty} \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{R_k}{R_j} \overline{g_k^n(z_j)}\, g_k^n(\zeta) \sum_{m=1}^{M_j} m \binom{m+n}{n} \overline{D^*_{-m,j}}\,\overline{g_j^{m+1}(z_k)}\\
&\quad + \sum_{n=0}^{\infty} \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{R_k}{R_j} \overline{g_k^n(z_j)} \Bigg(g_k^{n+2}(\zeta) \Bigg(2(n+1)\,\mathrm{Re}(D^*_{1,j})\, \overline{g_j^2(z_k)}\\
&\qquad\qquad + \sum_{m=2}^{M_j} m \binom{m+n}{n} D^*_{m,j} \overline{g_j^{m+1}(z_k)}\\
&\qquad\qquad - \sum_{m=1}^{M_j} m(m+2) \binom{m+n+2}{n} \overline{D^*_{-m,j}}\,\overline{g_j^{m+3}(z_k)}\\
&\qquad\qquad + \sum_{m=1}^{M_j} m(m+1) \binom{m+n+1}{n} \overline{D^*_{-m,j}}\, \frac{\overline{g_j^{m+2}(z_k)}}{g_j(z_k)} \Bigg)\\
&\qquad - g_k^{n+1}(\zeta) \sum_{m=1}^{M_j} m(m+1) \binom{m+n+1}{n} \overline{D^*_{-m,j}}\,\overline{g_j^{m+1}(z_k)}\,\overline{g_k(z_j)} \Bigg)\\
&= \frac{\kappa^*(s)+1}{4G^*(s)} R_k \big(\sigma_{xx}^{\infty,*} + \sigma_{yy}^{\infty,*} \big) - \frac{p_k^*}{G^*(s)} R_k\\
&\qquad - \frac{\kappa^*(s)+1}{4G^*(s)} R_k g_k^2(\zeta) \big((\sigma_{yy}^{\infty,*} - \sigma_{xx}^{\infty,*} - 2i\sigma_{xy}^{\infty,*}) \big)\\
&\qquad - \frac{1-\kappa^*(s)}{2G^*(s)} R_k \sum_{n=0}^{\infty} (n+1) g_k^{n+2}(\zeta) \sum_{\substack{j=1\\ j\neq k}}^{N} p_j^* \overline{g_j^2(z_k)}\,\overline{g_k^n(z_j)}. \qquad (14)
\end{aligned}
$$

By equating the coefficients of the positive powers $g_k^{l+1}(t)$ ($1 \le l \le M_k$), the constant terms, and the negative powers $g_k^{1-l}(t)$ ($2 \le l \le M_k$) in (14), we obtain a system of $2M_k$ ($k = 1, \dots, N$) linear complex algebraic equations for all the Fourier coefficients. To simplify the notation, we set

$$\mathrm{RHS}_1(s) = -\frac{\kappa^*(s)+1}{4G^*(s)} R_k(\sigma_{yy}^{\infty,*}(s) - \sigma_{xx}^{\infty,*}(s) - 2i\sigma_{xy}^{\infty,*}(s)) - \frac{1-\kappa^*(s)}{2G^*(s)} R_k \sum_{\substack{j=1 \\ j\neq k}}^{N} p_j^*(s) \overline{g_j^2(z_k)}$$

and, for $l = 2, \dots, M_k$,

$$\mathrm{RHS}_l(s) = -\frac{1-\kappa^*(s)}{2G^*(s)} R_k \sum_{\substack{j=1 \\ j\neq k}}^{N} p_j^*(s) \overline{g_j^2(z_k)}\, \overline{g_k^{l-1}(z_j)}.$$

The desired system is

$$D_{-l,k}^* - \sum_{\substack{j=1 \\ j\neq k}}^{N} \overline{g_k^l(z_j)} \Bigg(\sum_{m=1}^{M_j} (l+1) \binom{m+l}{l+1} \overline{D_{-m,j}^*}\, \overline{g_j^m(z_k)} \times \left(\frac{\overline{g_j(z_k)}}{g_j(z_k)} - \frac{m+l+1}{l+1} \overline{g_k^2(z_j)} - \frac{m+l+1}{m+1} \overline{g_j^2(z_k)} \right) + 2\,\mathrm{Re}(D_{1,j}^*) \overline{g_j(z_k)} + \sum_{m=2}^{M_j} \binom{m+l-1}{l} D_{m,j}^* \overline{g_j^m(z_k)} \Bigg) = \mathrm{RHS}_l(s), \tag{15}$$

$$\mathrm{Re}\, D_{1,k}^* + \frac{1}{2} \sum_{\substack{j=1 \\ j\neq k}}^{N} \frac{R_k}{R_j} \sum_{m=1}^{M_j} m \big(D_{-m,j}^* g_j^{m+1}(z_k) + \overline{D_{-m,j}^*}\, \overline{g_j^{m+1}(z_k)} \big) = \frac{\kappa^*(s)+1}{8G^*(s)} R_k \big(\sigma_{xx}^{\infty,*}(s) + \sigma_{yy}^{\infty,*}(s) \big) - \frac{p_k^*(s)}{2G^*(s)} R_k, \tag{16}$$

$$D_{l,k}^* - \sum_{\substack{j=1 \\ j\neq k}}^{N} g_k^l(z_j) \sum_{m=1}^{M_j} \binom{m+l-1}{l} D_{-m,j}^* g_j^m(z_k) = 0 \quad (l = 2, \dots, M_k). \tag{17}$$

The system (15)–(17) can be written in compact form as

$$AD = B. \tag{18}$$

The matrix A is s-independent and can be inverted directly and stored in computer memory. The unknown vector D is s-dependent and is defined as

$$D = \big[\ D_1^*(s)\ \ \dots\ \ D_N^*(s)\ \big]^T,$$

where each subvector, such as $D_j^*(s)$, is a vector of unknown Fourier coefficients for one hole, given by

$$D_j^*(s) = \begin{bmatrix} D_{-M_j,j}^*(s) & \dots & D_{-1,j}^*(s) & \mathrm{Re}(D_{1,j}^*(s)) & D_{2,j}^*(s) & \dots & D_{M_j,j}^*(s) \end{bmatrix}^T.$$

The vector B on the right is composed of loading terms multiplied by certain constants involving the transformed material parameters. The three constants are $\big(\kappa^*(s)+1\big)/\big(4G^*(s)\big)$, $\big(1-\kappa^*(s)\big)/\big(2G^*(s)\big)$, and $1/\big(2G^*(s)\big)$. We decompose B into three parts, each containing only one constant:

$$B = B^{(1)} + B^{(2)} + B^{(3)},$$

where, for $k = 1, \dots, N$, we have set

$$B_{l,k}^{(1)} = \begin{cases} -\dfrac{\kappa^*(s)+1}{4G^*(s)} R_k \big(\sigma_{yy}^{\infty,*}(s) - \sigma_{xx}^{\infty,*}(s) - 2i\sigma_{xy}^{\infty,*}\big)(s) & (l=-1), \\ \dfrac{\kappa^*(s)+1}{4G^*(s)} R_k \dfrac{\sigma_{xx}^{\infty,*}(s) + \sigma_{yy}^{\infty,*}(s)}{2} & (l=1), \\ 0 & (l = \pm 2, \dots, \pm M_k), \end{cases} \tag{19}$$

$$B_{l,k}^{(2)} = \begin{cases} -\dfrac{1-\kappa^*(s)}{2G^*(s)} R_k \sum_{\substack{j=1 \\ j\neq k}}^{N} p_j^*(s) \overline{g_j^2(z_k)} & (l=-1), \\ -\dfrac{1-\kappa^*(s)}{2G^*(s)} R_k \sum_{\substack{j=1 \\ j\neq k}}^{N} p_j^*(s) \overline{g_j^2(z_k) g_k^{-l-1}(z_j)} & (l = -M_k, \dots, -2), \\ 0 & (l = 1, \dots, M_k), \end{cases} \tag{20}$$

$$B_{l,k}^{(3)} = \begin{cases} \dfrac{1}{2G^*(s)} \big(-p_k^*(s) R_k\big) & (l=1), \\ 0 & (l = -1, \pm 2, \dots, \pm M_k). \end{cases} \tag{21}$$

Assume for simplicity that all components of the far-field stress vary in the same time-dependent manner; for example,

$$\sigma_{ij}^{\infty}(t) = \tilde{\sigma}_{ij}^{\infty} \cdot f_\infty(t),$$

where i, j represent x, y. Thus, in the Laplace domain the far-field stress is expressed by

$$\sigma_{ij}^{\infty,*}(s) = \tilde{\sigma}_{ij}^{\infty} \cdot f_\infty^*(s), \tag{22}$$

where the s-dependent function $f_\infty^*(s)$ is given as

$$f_\infty^*(s) = \Gamma\big[f_\infty(t)\big] = \int_0^\infty f_\infty(t) e^{-st}\, dt.$$

Similarly we assume that the tractions on the boundaries of the holes vary as

$$p_j(t) = \tilde{p}_j \cdot f_p(t), \quad j = 1, \ldots, N.$$

Thus, in the Laplace domain the boundary tractions are expressed by

$$p_j^*(s) = \tilde{p}_j \cdot f_p^*(s).$$

Substituting this and (22) into (19)–(21) and separating the s-dependent terms, we get

$$B^{(1)} = \frac{\kappa^*(s)+1}{4G^*(s)} \langle B^{(1)} \rangle f_\infty^*(s),$$

$$B^{(2)} = \frac{1-\kappa^*(s)}{2G^*(s)} \langle B^{(2)} \rangle f_p^*(s),$$

$$B^{(3)} = \frac{1}{2G^*(s)} \langle B^{(3)} \rangle f_p^*(s),$$

where

$$\langle B_{l,k}^{(1)} \rangle = \begin{cases} -R_k\big(\tilde{\sigma}_{yy}^\infty - \tilde{\sigma}_{xx}^\infty - 2i\tilde{\sigma}_{xy}^\infty\big) & (l=-1), \\ \frac{1}{2}R_k(\tilde{\sigma}_{xx}^\infty + \tilde{\sigma}_{yy}^\infty) & (l=1), \\ 0 & (l=\pm 2, \ldots, \pm M_k), \end{cases}$$

$$\langle B_{l,k}^{(2)} \rangle = \begin{cases} -R_k \sum_{\substack{j=1 \\ j\neq k}}^N \tilde{p}_j \overline{g_j^2(z_k)} & (l=-1), \\ -R_k \sum_{\substack{j=1 \\ j\neq k}}^N \tilde{p}_j \overline{g_j^2(z_k) g_k^{-l-1}(z_j)}, & (l=-M_k, \ldots, -2), \\ 0 & (l=1, \ldots, M_k), \end{cases}$$

$$\langle B_{l,k}^{(3)} \rangle = \begin{cases} -\tilde{p}_k R_k & (l=1), \\ 0 & (l=-1, \pm 2, \ldots, \pm M_k). \end{cases}$$

Thus, the solution of equation system (18) can be written compactly as

$$D = \frac{\kappa^*(s)+1}{4G^*(s)} \langle D^{(1)} \rangle f_\infty^*(s) + \frac{1-\kappa^*(s)}{2G^*(s)} \langle D^{(2)} \rangle f_p^*(s) + \frac{1}{2G^*(s)} \langle D^{(3)} \rangle f_p^*(s), \quad (23)$$

where the s-independent vectors $\langle D^{(j)} \rangle$ $(j = 1, \ldots, 3)$ are the solution of the following equation systems

$$\langle D^{(j)} \rangle = A^{-1} \langle B^{(j)} \rangle. \quad (24)$$

We emphasize that A^{-1} is computed only once. The system (24) can also be solved (after separating the real and imaginary parts) using standard numerical methods (Gauss elimination, Gauss–Seidel iteration, etc.).

5.3. *Solution in the Laplace domain.* With the substitution of the solution for Fourier coefficients (23) into the expressions for the potentials (12), followed by substitution of those potentials and their derivatives into (4)–(6), one obtains the solution for the displacements and stresses at any point z in the Laplace domain:

$$u_x(z;s)+iu_y(z;s)=\sum_{k=1}^{3}\big(f_k^*(s)\Phi_k-zf_{k+3}^*(s)\overline{\Phi_k'}-f_{k+3}^*(s)\overline{\Psi_k}\big)+f_5^*(s)\overline{\Psi^{(p)}}\\ +f_7^*(s)\frac{\tilde\sigma_{xx}^\infty+\tilde\sigma_{yy}^\infty}{2}z-f_4^*(s)\big(\tilde\sigma_{yy}^\infty-\tilde\sigma_{xx}^\infty-2i\tilde\sigma_{xy}^\infty\big)\bar z, \quad (25)$$

$$\sigma_{xx}(z;s)+\sigma_{yy}(z;s)=4\,\mathrm{Re}\big(\tfrac12 f_\infty^*(s)\Phi_1'+f_8^*(s)\Phi_2'+f_9^*(s)\Phi_3'\big)\\ +\big(\tilde\sigma_{xx}^\infty+\tilde\sigma_{yy}^\infty\big)f_\infty^*(s), \quad (26)$$

$$\sigma_{yy}(z;s)-\sigma_{xx}(z;s)+2i\sigma_{xy}(z;s)\\ =2\Big(\bar z\big(\tfrac12 f_\infty^*(s)\Phi_1''+f_8^*(s)\Phi_2''+f_9^*(s)\Phi_3''\big)\\ +\big(\tfrac12 f_\infty^*(s)\Psi_1'+f_8^*(s)\Psi_2'+f_9^*(s)\Psi_3'\big)+f_8^*(s)s^{-1}\Psi^{(p)\prime}\Big)\\ +\big(\tilde\sigma_{yy}^\infty-\tilde\sigma_{xx}^\infty+2i\tilde\sigma_{xy}^\infty\big)f_\infty^*(s), \quad (27)$$

where $\Phi^{(k)}$, $\Psi^{(k)}$ $(k=1,\dots,3)$ and $\Psi^{(p)}$ are given by

$$\Phi^{(k)}=\sum_{j=1}^{N}\sum_{m=1}^{M_j}\big\langle D_{-m,j}^{(k)}\big\rangle g_j^m(z),$$

$$\Psi^{(k)}=\sum_{j=1}^{N}\Bigg(\Big(g_j^2(z)+\frac{\overline{z_j}}{z-z_j}\Big)\sum_{m=1}^{M_j}m\big\langle D_{-m,j}^{(k)}\big\rangle g_j^m(z)\\ -\Big(\big\langle D_{1,j}^{(k)}\big\rangle+\overline{\big\langle D_{1,j}^{(k)}\big\rangle}\Big)g_j(z)-\sum_{m=2}^{M_j}\overline{\big\langle D_{m,j}^{(k)}\big\rangle}g_j^m(z)\Bigg),$$

$$\Psi^{(p)}=\sum_{j=1}^{N}\tilde p_jR_jg_j(z).$$

Note that $\Phi^{(k)}$, $\Psi^{(k)}$ and $\Psi^{(p)}$ are independent of the transform parameter s.

In these equations the s-independent Fourier coefficients $\big\langle D_{m,j}^{(k)}\big\rangle$, for $m=\pm1$, $\dots,\pm M_j$, are the components of the vectors $\big\langle D^{(k)}\big\rangle$ $(k=1,\dots,3)$ obtained from the equation systems (24).

The s-dependent functions involved in Equations (25)–(27) are written as

$$f_1^*(s)=\frac{\kappa^*(s)}{4G^*(s)}f_\infty^*(s),\qquad f_2^*(s)=\frac{1}{2G^*(s)}\frac{\kappa^*(s)\big(1-\kappa^*(s)\big)}{\kappa^*(s)+1}f_p^*(s),$$

$$f_3^*(s) = \frac{1}{2G^*(s)} \frac{\kappa^*(s)}{\kappa^*(s)+1} f_p^*(s), \qquad f_4^*(s) = \frac{1}{4G^*(s)} f_\infty^*(s),$$

$$f_5^*(s) = \frac{1}{2G^*(s)} \frac{1-\kappa^*(s)}{\kappa^*(s)+1} f_p^*(s), \qquad f_6^*(s) = \frac{1}{2G^*(s)} \frac{1}{\kappa^*(s)+1} f_p^*(s),$$

$$f_7^*(s) = \frac{\kappa^*(s)-1}{4G^*(s)} f_\infty^*(s), \qquad f_8^*(s) = \frac{1-\kappa^*(s)}{\kappa^*(s)+1} f_p^*(s),$$

$$f_9^*(s) = \frac{1}{\kappa^*(s)+1} f_p^*(s).$$

Note again that the space functions $\Phi^{(k)}$, $\Psi^{(k)}$ $(k=1,2,3)$ and $\Psi^{(p)}$ are independent of the viscoelastic model and the time-dependent behavior for the loadings: $f_\infty(t)$ and $f_p(t)$. Thus the procedure is universal for any viscoelastic model and any loading situation.

5.4. ***Solution in the time domain.*** Upon application of the analytical inverse Laplace transform, equations (25)–(27) become

$$u_x(z;t)+iu_y(z;t) = \sum_{k=1}^{3}\bigl(f_k(t)\Phi_k - zf_{k+3}(t)\overline{\Phi_k'} - f_{k+3}(t)\overline{\Psi_k}\bigr) + f_5(t)\overline{\Psi^{(p)}} + f_7(t)\frac{\tilde\sigma_{xx}^\infty+\tilde\sigma_{yy}^\infty}{2}z - f_4(t)\bigl(\tilde\sigma_{yy}^\infty-\tilde\sigma_{xx}^\infty-2i\tilde\sigma_{xy}^\infty\bigr)\bar z, \tag{28}$$

$$\sigma_{xx}(z;t)+\sigma_{yy}(z;t) = 4\,\mathrm{Re}\bigl(\tfrac12 f_\infty(t)\Phi_1' + f_8(t)\Phi_2' + f_9(t)\Phi_3'\bigr) + \sigma_{xx}^\infty(t)+\sigma_{yy}^\infty(t), \tag{29}$$

$$\begin{aligned}\sigma_{yy}(z;t)-\sigma_{xx}(z;t)+2i\sigma_{xy}(z;t) &= 2\Bigl(\bar z\bigl(\tfrac12 f_\infty(t)\Phi_1''+f_8(t)\Phi_2''+f_9(t)\Phi_3''\bigr) \\ &\quad + \bigl(\tfrac12 f_\infty(t)\Psi_1'+f_8(t)\Psi_2'+f_9(t)\Psi_3'\bigr)+f_8(t)s^{-1}\Psi^{(p)\prime}\Bigr) \\ &\quad + \sigma_{yy}^\infty(t)-\sigma_{xx}^\infty(t)+2i\sigma_{xy}^\infty(t),\end{aligned} \tag{30}$$

where $f_j(t)$ $(j=1,\dots,9)$ are the analytical inverse Laplace transforms (1) of the s-functions, that is,

$$f_j(t) = \Gamma^{-1}\bigl[f_j^*(s)\bigr]. \tag{31}$$

It is observed from Equations (28) and (30) that to compute the displacements and stresses at multiple time instants, one need compute the potentials $\Phi^{(k)}$, $\Psi^{(k)}$, $\Psi^{(p)}$ and their derivatives only once (following the procedure described in Sections 5.1–5.3), and then successively multiply them by the time functions $f_j(t)$ $(j=1,\dots,9)$ for each time instant. This procedure dramatically reduces the computational costs, as compared with time stepping approaches that use a nonconstant time step size.

Since the time dependence of the solutions is simply determined by the time functions $f_j(t)$ ($j = 1, \ldots, 9$), the present approach provides the capability to adopt a variety of physical models and loading conditions. It is more flexible than the traditional time-stepping approach, in which the constitutive equation for the physical model is involved in the governing equations; see [Huang et al. 2005c; 2005b; 2005a].

In the solution procedure, the Fourier coefficients are not computed explicitly. The accuracy of the solution is nevertheless still dependent on the number of Fourier terms, as can be seen from the expressions of the potentials (12). We will perform the computation for given values M_k ($k = 1$ to N), and then increase the values of M_k until a specified degree of accuracy is achieved. Details about determining the number of terms in the Fourier expansion and the error estimation are given by [Mogilevskaya and Crouch 2001].

6. Examples

It is well known that for the class of problems considered in this paper the viscoelastic stresses are time-independent and are exactly same as the stresses in the corresponding elastic problems [Timoshenko and Goodier 1970]. In our approach, this conclusion can be rigorously proved for the case of one hole. For the case of multiple holes it has been verified numerically for all the examples in this paper. This fact provides the means to verify the solution for the stresses obtained with our approach. To do so we performed the computation and compared the results for the stresses to those for the elastic problems given in [Wang et al. 2003a] (the latter results have been verified with the benchmark results obtained earlier by [Ling 1948] and [Haddon 1967]). We achieved the same accuracy as reported in [Wang et al. 2003a].

Thus, below we only present the results for displacements. To demonstrate the versatility of our approach we present the examples for three different viscoelastic models.

6.1. ***Examples for viscoelastic model I.*** In this series of examples we assume that the viscoelastic material responds as a Boltzmann model in shear and elastically in dilatation (Figure 2). The constitutive equations for shear and dilatation are

$$\frac{G_1 + G_2}{G_1} s_{ij} + \frac{\eta}{G_1} \dot{s}_{ij} = 2G_2 \epsilon_{ij} + 2\eta \dot{\epsilon}_{ij}, \qquad \sigma_{kk} = 3K\varepsilon_{kk},$$

where the meanings of the elastic and viscous parameters G_1, G_2 and η are explained in Figure 2. The numbers s_{ij} (σ_{kk}) and ϵ_{ij} (ε_{kk}) are the deviatoric (volumetric) components of the stress and strain tensors σ_{ij} and ε_{ij}:

$$\sigma_{ij} = s_{ij} + \tfrac{1}{3}\delta_{ij}\sigma_{kk}, \qquad \varepsilon_{ij} = \epsilon_{ij} + \tfrac{1}{3}\delta_{ij}\varepsilon_{kk}.$$

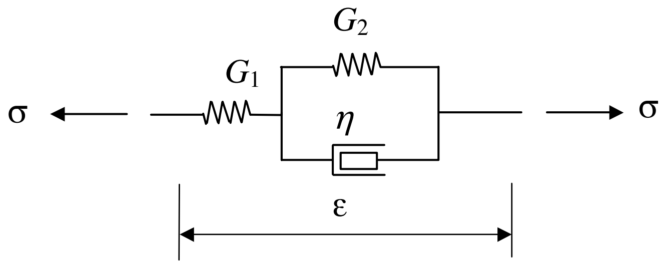

Figure 2. Boltzmann model.

Following the procedure described in [Wang and Crouch 1982], one can find the s-varying constants for this model as follows:

$$G^*(s) = \frac{G_1(G_2+\eta s)}{G_1+G_2+\eta s},$$
$$\kappa^*(s) = 1 + \frac{6G_1(G_2+\eta s)}{G_1(G_2+\eta s)+3(G_1+G_2+\eta s)K}. \tag{32}$$

Assume that the stresses at infinity and the tractions on the boundaries of the holes are suddenly applied at $t=0$ and remain constant. Thus,

$$f_\infty(t) = 1 \quad \text{and} \quad f_p(t) = 1.$$

By Laplace transformation, this yields

$$f^*_\infty(s) = \frac{1}{s} \quad \text{and} \quad f^*_p(s) = \frac{1}{s}.$$

With the substitution of these equations and (32) into the expressions for f_i^* (see pages 483–484), and after an analytic inverse Laplace transformation, one obtains for the time functions the expressions

$$\begin{aligned} &f_1(t) = \tfrac{1}{4}\chi_1(t), && f_2(t) = -\tfrac{1}{2}\chi_1(t)+\chi_2(t)-\chi_3(t),\\ &f_3(t) = \tfrac{1}{2}\big(\chi_2(t)-\chi_3(t)\big), && f_4(t) = \tfrac{1}{4}\chi_2(t),\\ &f_5(t) = -\tfrac{1}{2}\chi_2(t)+\chi_3(t), && \mathrm{f}_6(t) = \tfrac{1}{2}\chi_3(t),\\ &f_7(t) = \tfrac{1}{4}\big(\chi_1(t)-\chi_2(t)\big) && f_8(t) = -1+2\chi_4(t),\\ &&f_9(t) = \chi_4(t), \end{aligned} \tag{33}$$

with

$$\chi_1(t) = \frac{(G_1+G_2)(C_2+3G_1G_2)}{G_1G_2C_1} - \frac{1}{G_2}e^{-\alpha t} - \frac{6G_1^2 e^{-\beta t}}{(3K+G_1)C_1},$$
$$\chi_2(t) = \frac{1}{G_1} + \frac{1-e^{-\alpha t}}{G_2},$$
$$\chi_3(t) = \frac{(G_1+G_2)C_1}{2G_1G_2C_2} - \frac{1}{2G_2}e^{-\alpha t} + \frac{6G_1^2 e^{-\gamma t}}{(3K+4G_1)C_2},$$
$$\chi_4(t) = \frac{1}{2} - \frac{3G_1G_2}{2C_2} - \frac{9KG_1^2 e^{-\gamma t}}{2(3K+4G_1)C_2},$$

where the following abbreviations have been introduced:

$$\alpha = \frac{G_2}{\eta}, \tag{34}$$
$$\beta = \frac{1}{\eta}\Big(\frac{3KG_1}{3K+G_1} + G_2\Big), \qquad \gamma = \frac{1}{\eta}\Big(\frac{3KG_1}{3K+4G_1} + G_2\Big),$$
$$C_1 = 3KG_2 + G_1(3K+G_2), \qquad C_2 = 3KG_2 + G_1(3K+4G_2).$$

6.1.1. *An example with constant far-field stresses.* Consider two traction-free circular holes of different sizes in an infinite plane subjected to far-field stresses σ_{xx}^{∞}, σ_{yy}^{∞} and σ_{xy}^{∞}. As shown in Figure 3, two holes L_1 and L_2 with radii R_1 and R_2 are aligned with the x- axis and separated by a distance d.

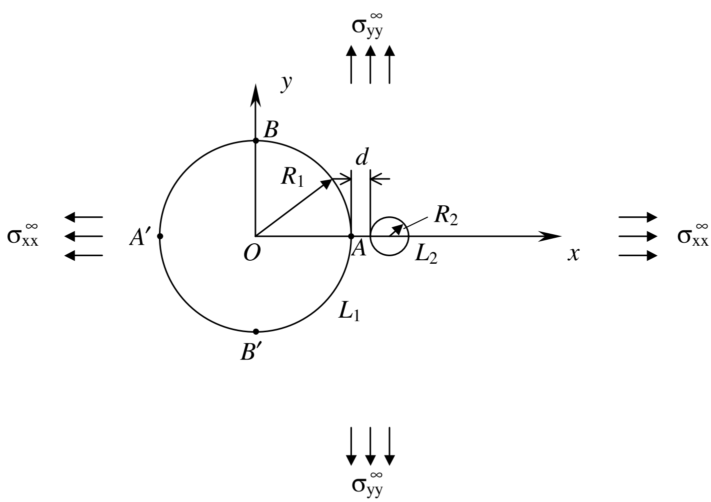

Figure 3. Two circular holes in an infinite plane.

The elastic problem with the same geometrical configuration under three loading conditions (longitudinal tension: $\sigma_{xx}^{\infty} = \sigma_0$, $\sigma_{yy}^{\infty} = \sigma_{xy}^{\infty} = 0$; transverse tension: $\sigma_{yy}^{\infty} = \sigma_0$, $\sigma_{xx}^{\infty} = \sigma_{xy}^{\infty} = 0$; and pure shear: $\sigma_{xy}^{\infty} = \sigma_0$, $\sigma_{xx}^{\infty} = \sigma_{yy}^{\infty} = 0$) was considered in [Wang et al. 2003a]. By using the method described in the present paper, we obtained the results for the corresponding viscoelastic problems. The parameters were adopted for the viscoelastic material were

$$G_1 = 8 \times 10^3 \sigma_0, \quad G_2 = 2 \times 10^3 \sigma_0, \quad \eta = 5 \times 10^3 \sigma_0 \cdot \text{sec}, \quad K = 17333.3\sigma_0,$$

To obtain the dimensionless time we used the viscosity coefficient $\gamma = \eta / G_2 = 2.5$ second.

To examine the numerical results for displacements, the relative elongations of the diameters of hole L_1 in the x and y directions

$$\delta_x = \frac{u_x(A) - u_x(A')}{2R_1} \quad \text{and} \quad \delta_y = \frac{u_y(B) - u_y(B')}{2R_1} \tag{35}$$

are computed for the case $R_1/R_2 = 5$; $d/R_2 = 1$ and $\sigma_{xx}^{\infty} = \sigma_0$, $\sigma_{yy}^{\infty} = 0.5\sigma_0$, $\sigma_{xy}^{\infty} = 0$ and the results were compared with those obtained with the commercial finite element software-ANSYS. Since ANSYS cannot directly model an infinite area, the infinite viscoelastic plane was modeled as a large plate ($200R_2 \times 200R_2$). Prony series were adopted to approximate the relaxation functions of the shear and bulk moduli and a time stepping algorithm was used to obtain the time domain solution in ANSYS. With ANSYS, 4839 finite elements were used and the computation took 2 hours 16 minutes on an IBM SP workstation (500 time steps). With the present approach, only 36 terms in the Fourier series were used to represent the boundary displacements for the two holes and the computation just took 19 seconds with a 900 MHz PC (500 time instants). It is seen from Figure 4 that the results given by the two approaches match very well.

6.1.2. *An example with constant pressure.* Consider the case of three holes shown in Figure 5. The boundaries of two smaller holes with the radii $R_2 = R_3 = R$ are assumed to be traction-free. The central hole with the radius R_1 ($R_1/R = 5$) is subjected to constant pressure $p_1 = -\sigma_0$. The three holes are separated by a distance $d = R$. The material properties for the viscoelastic plane are the same as those in the previous example.

The relative elongations of the diameters of the central hole L_1 in the x and y directions, given by (35), are computed and compared with the results provided by ANSYS. It is seen from Figure 6 that the results given by the two approaches agree very well. Due to the existence of holes L_2 and L_3 along the x axis, the change of diameter of the central hole in the x direction is larger than that in the y direction.

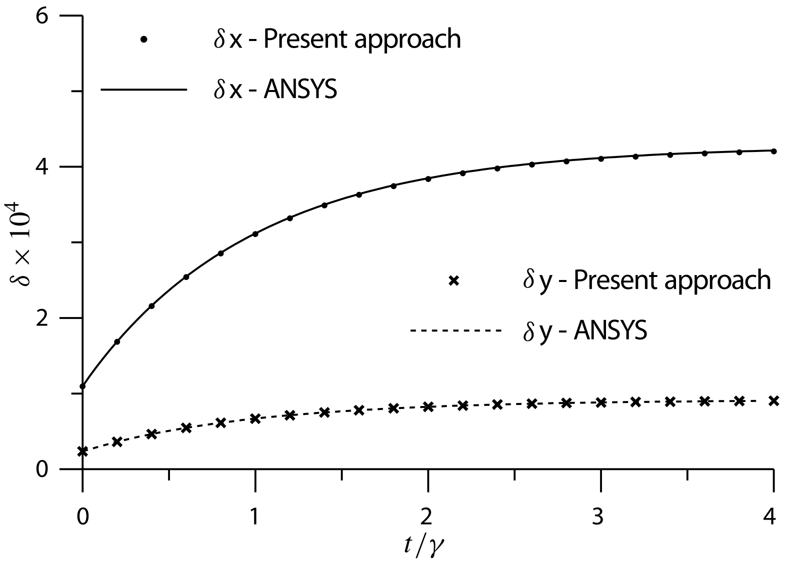

Figure 4. Change of diameter of the hole L_1 due to far-field stresses.

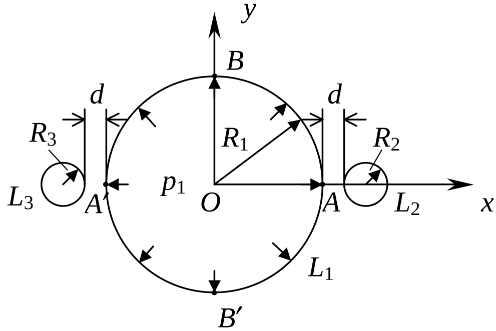

Figure 5. Three holes in an infinite plane with the central hole subjected to constant pore pressure.

6.2. ***Examples for viscoelastic model II.*** In this series of examples we assume that the viscoelastic material responds as a Boltzmann model in both shear and dilatation, and the Poisson's ratio ν is constant. As the result of these assumptions the viscoelastic properties of the material can be represented by the constants G_1, G_2, η (Figure 2) and ν.

Following the procedure described in [Wang and Crouch 1982], one can find the s-varying constants for this model using the equations

$$G^*(s) = \frac{G_1\,(G_2 + \eta s)}{G_1 + G_2 + \eta s}, \quad \kappa^*(s) = 3 - 4\nu.$$

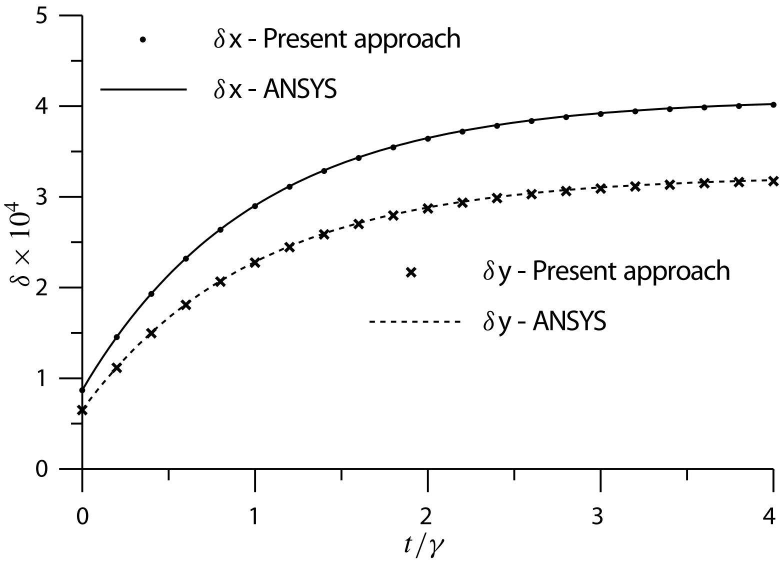

Figure 6. Change of diameter of the central hole due to a constant pressure.

In case that the stresses at infinity and the tractions on the boundaries of the holes are kept constant, the time functions $f_j(t)$ $(j = 1, \dots, 9)$ can again be expressed by Equations (33) with the functions $\chi_k(t)$ $(k = 1, \dots, 4)$ given by

$$\chi_1(t) = (3-4\nu)\left(\frac{1}{G_1} + \frac{1-e^{-\alpha t}}{G_2}\right), \qquad \chi_2(t) = \frac{1}{G_1} + \frac{1-e^{-\alpha t}}{G_2},$$

$$\chi_3(t) = \frac{1}{4-4\nu}\left(\frac{1}{G_1} + \frac{1-e^{-\alpha t}}{G_2}\right), \qquad \chi_4(t) = \frac{1}{4-4\nu},$$

where α is defined in Equation (34).

6.2.1. *An example with constant far-field stresses.* The geometry of this example is the same as that depicted in Figure 5. The boundaries of all three holes are assumed to be traction-free and the infinite plane is subjected to biaxial far-field stresses $\sigma_{xx}^{\infty} = \sigma_0$ and $\sigma_{yy}^{\infty} = 0.5\sigma_0$. The material properties adopted in computation were

$$G_1 = 8 \times 10^3 \sigma_0, \quad G_2 = 2 \times 10^3 \sigma_0, \quad \eta = 5 \times 10^3 \sigma_0 \cdot \text{sec}, \quad \nu = 0.25.$$

The viscosity coefficient $\gamma = \eta/G_2 = 2.5$ second was again employed. The relative elongations of the diameters of the central hole L_1 in the x and y directions, given by Equation (35), are computed and compared with the results provided by the time stepping approach described in [Huang et al. 2005b]. It is seen from Figure 7 that the results given by the two approaches are practically identical. To accomplish the computation of the same number (500) of time instants (or steps)

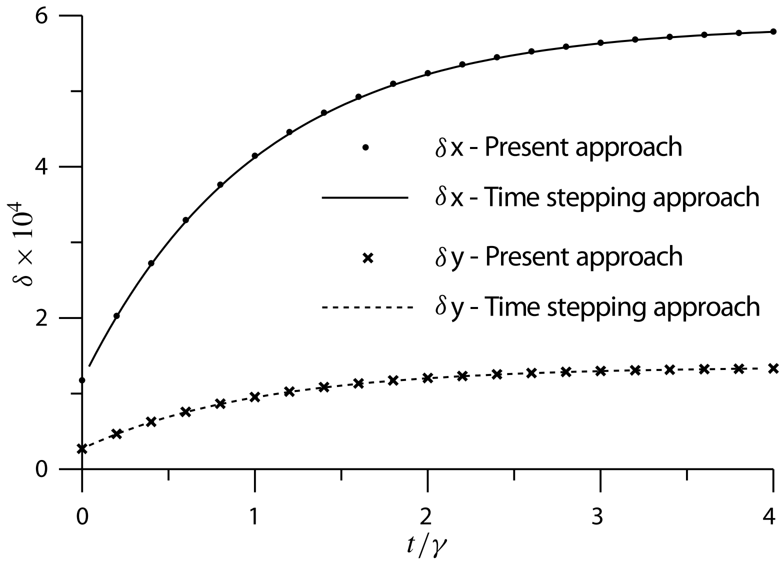

Figure 7. Change of diameter of the central hole due to constant far-field stresses.

and reach the same accuracy, the present approach used 35 terms of the Fourier series for each of the three circular holes and the computation took 42 seconds, while the time stepping approach used 45 terms of the Fourier series for the central hole and 12 terms of the Fourier series for the two smaller holes and the computation took about twice as long, 1 minute 38 seconds. The difference can be explained as follows: with the current approach, the algebraic equation systems are formulated and solved only once and the same potentials (and their derivatives) are used for the computation at every time instant; only the time functions $f_j(t)$ $(j = 1, \ldots, 9)$ need to be recomputed to obtain the stresses and displacements at different time instants. In the time stepping approach, since the solution for a typical time step relies on the results for the previous step, the system of algebraic equations needs to be solved for each time step [Huang et al. 2005b].

6.2.2. *An example with time-dependent pressure.* Now we modify the loading conditions in the previous example (Section 6.2.1) and assume that $\sigma_{xx}^{\infty} = \sigma_{yy}^{\infty} = \sigma_{xy}^{\infty} = 0$ and the central hole is subjected to time-dependent pressure given in sinusoidal form as

$$p_1(t) = -\sigma_0(1 + c \sin \omega t).$$

The boundaries of the other two holes are traction-free.

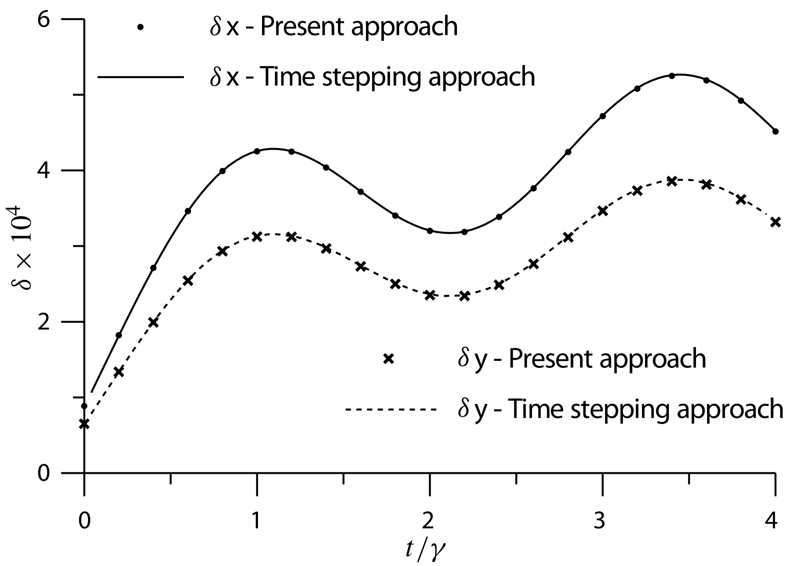

Figure 8. Change of diameter of the central hole due to a time-dependent pressure.

The time functions $f_j(t)$ $(j = 1, \ldots, 7, 9)$ can again be expressed by the corresponding equations in (33), with the functions $\chi_k(t)$ $(k = 1, \ldots, 4)$ given by

$$\chi_1(t) = \frac{3-4\nu}{G_1 G_2(\eta^2\omega^2 + G_2^2)}\Big(-G_1(\eta^2\omega^2 - c\eta\omega G_2 + G_2^2)e^{-(G_2/\eta)t} \\ (1 + c\sin\omega t)G_2(\eta^2\omega^2 + G_2^2 + G_1 G_2) + G_1\eta\omega(\eta\omega - c\cos\omega t\, G_2)\Big),$$

$$\chi_2(t) = \frac{1}{3-4\nu}\chi_1(t), \quad \chi_3(t) = \frac{1}{(3-4\nu)(4-4\nu)}\chi_1(t), \quad \chi_4(t) = \frac{1}{4-4\nu}(1+c\sin\omega t),$$

while

$$f_8(t) = -(1 + c\sin\omega t) + 2\chi_4(t).$$

The material properties for this example are the same as those in the previous subsection. In the computation, the following values for the pressure are adopted: $c = 0.5$ and $\omega = 1\,\text{sec}^{-1}$. The relative elongations of the diameters of the central hole L_1 in the x and y directions, given by Equation (35), are computed and compared with the results provided by the time stepping approach in Figure 8. It is seen that the results given by the two approaches are practically identical. The behaviors of δx and δy are characterized by the combination of exponential and sinusoidal functions.

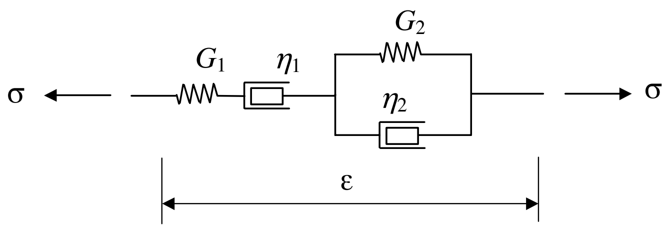

Figure 9. Burgers model.

6.3. ***Examples for viscoelastic model III.*** Here we take

$$\ddot{s}_{ij}+\Big(\frac{G_1}{\eta_1}+\frac{G_1}{\eta_2}+\frac{G_2}{\eta_2}\Big)\dot{s}_{ij}+\frac{G_1G_2}{\eta_1\eta_2}s_{ij}=2G_1\ddot{\epsilon}_{ij}+2\frac{G_1G_2}{\eta_2}\dot{\epsilon}_{ij},\qquad \sigma_{kk}=3K\varepsilon_{kk},$$

where the meanings of the elastic and viscous parameters G_1, G_2, η_1 and η_2 are explained in Figure 9.

Following the procedure described in [Wang and Crouch 1982], one can find the s-varying constants using the equations

$$G^*(s)=\frac{G_1s^2+\dfrac{G_1G_2}{\eta_2}s}{s^2+\Big(\dfrac{G_1}{\eta_1}+\dfrac{G_1}{\eta_2}+\dfrac{G_2}{\eta_2}\Big)s+\dfrac{G_1G_2}{\eta_1\eta_2}},$$

$$\kappa^*(s)=1+\frac{6G_1s^2+6\dfrac{G_1G_2}{\eta_2}s}{G_1s^2+\dfrac{G_1G_2}{\eta_2}s+3\bigg(s^2+\Big(\dfrac{G_1}{\eta_1}+\dfrac{G_1}{\eta_2}+\dfrac{G_2}{\eta_2}\Big)s+\dfrac{G_1G_2}{\eta_1\eta_2}\bigg)K}.$$

If the far-field stresses and the tractions on the boundaries of holes are both constant, one can obtain the time functions $f_j(t)$ $(j=1,\dots,9)$ expressed by (33), with the functions $\chi_k(t)$ $(k=1,\dots,4)$ given as

$$\chi_1(t)=\frac{1}{KG_1}\Bigg(K-\frac{G_1^2}{3K+G_1}e^{-\alpha t}(1+e^{\beta t})+G_1\Big(2+\frac{K}{G_2}(1-e^{-\varrho t})+\frac{K}{\eta_1}t\Big)$$
$$+\frac{e^{-\alpha t}(-1+e^{\beta t})G_1^2\big((3KG_1-(3K+G_1)G_2)\eta_1+3KG_1\eta_2\big)}{(3K+G_1)C_3}\Bigg),$$

$$\chi_2(t)=\frac{1}{G_1}+\frac{1-e^{-\varrho t}}{G_2}+\frac{1}{\eta_1}t,$$

$$\chi_3(t)=\frac{1}{2KG_1}\left(K+\frac{G_1^2}{3K+4G_1}e^{-\alpha' t}(1+e^{\beta' t})+G_1\left(-1+\frac{K}{G_2}(1-e^{-\varrho t})+\frac{K}{\eta_1}t\right)\right.$$
$$\left.-\frac{2e^{-\alpha' t}(-1+e^{\beta' t})G_1^2\big(-3KG_2\eta_1+G_1((3K-4G_2)\eta_1+3K\eta_2)\big)}{(3K+4G_1)C_3'}\right),$$

$$\chi_4(t)=\frac{1}{2}-\frac{3G_1}{4(3K+4G_1)C_3'}\left(e^{\alpha' t}\Big(3KG_2\eta_1+G_1\big((-3K+4G_2)\eta_1-3K\eta_2\big)+C_3'\Big)\right.$$
$$\left.+e^{-\alpha'' t}\big(-3KG_2\eta_1+G_1((3K-4G_2)\eta_1+3K\eta_2)+C_3'\big)\right),$$

where the constants that occur are

$$C_1=(3K+G_1)\eta_1\eta_2,$$
$$C_2=3KG_2\eta_1+G_1\big((3K+G_2)\eta_1+3K\eta_2\big),$$
$$C_3=\sqrt{-12KG_1G_2C_1+C_2^2},$$
$$C_1'=(3K+4G_1)\eta_1\eta_2,$$
$$C_2'=3KG_2\eta_1+G_1\big((3K+4G_2)\eta_1+3K\eta_2\big),$$
$$C_3'=\sqrt{-12KG_1G_2C_1'+C_2'^2},$$
$$\alpha=\frac{C_2+C_3}{2C_1},\quad \beta=\frac{C_3}{C_1},\quad \varrho=\frac{G_2}{\eta_2},\quad \alpha'=\frac{-C_2'+C_3'}{2C_1'},\quad \alpha''=\frac{C_2'+C_3'}{2C_1'}.$$

6.3.1. *An example with constant far-field stresses.* Consider the same example described in Section 6.1.1 (Figure 3). The geometric parameters are taken as follows: $R_1/R_2=5$, $d/R_2=1$. The holes are traction-free and the stresses at infinity are given as $\sigma_{xx}^\infty=\sigma_0$, $\sigma_{xx}^\infty=\sigma_{xy}^\infty=0$. The material properties adopted for the computations are

$$G_1=8\times10^3\sigma_0,\quad G_2=2\times10^3\sigma_0,\quad \eta_1=8\times10^3\sigma_0\cdot\text{sec},$$
$$\eta_2=5\times10^3\sigma_0\cdot\text{sec},\quad K=17333.3\sigma_0,$$

In this example the displacement u_x along the straight line between the two points $(5R_2,0)$ and $(6R_2,0)$ is computed for three time instants: $t=0$ sec, $t=1$ sec, and $t=10$ sec (Figure 10). The left end point $(5R_2,0)$ is fixed. It can be observed that the deformation keeps increasing with time. This can be explained by the linear term in t in the time-dependent expressions for χ_1, χ_2, χ_3, χ_4 starting at the bottom of the previous page.

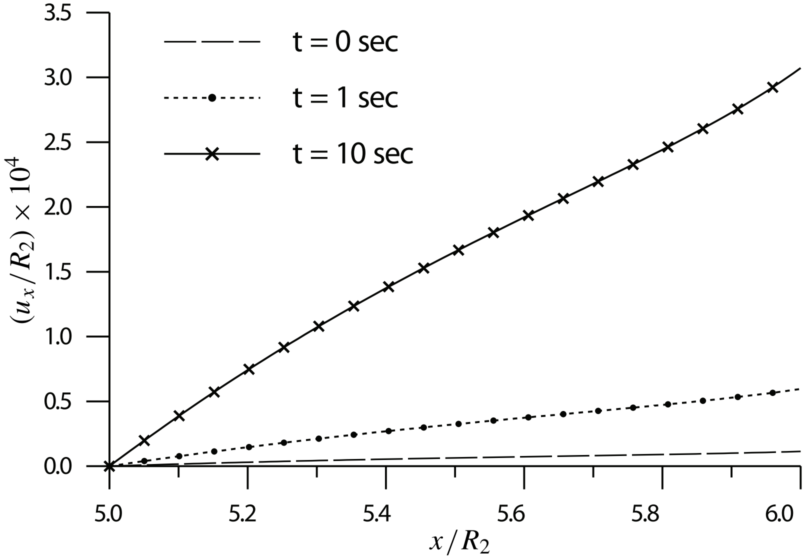

Figure 10. u_x along the line between the two holes.

6.3.2. *An example with constant far-field stresses and pressure.* In this example, we demonstrate the use of our method for solving problems involving multiple randomly distributed holes. The same material properties are adopted as in the previous subsection. The viscoelastic plane is subjected to constant far-field stresses $\sigma_{xx}^{\infty} = \sigma_0$, $\sigma_{xx}^{\infty} = \sigma_{xy}^{\infty} = 0$, and constant uniform pressure $p = -\sigma_0$ is applied to one of the holes. Figures 11–13 show contours of u_x in the plane at three time instants: $t = 0$ sec, $t = 0.5$ sec and $t = 10$ sec. It is shown that the plane is stretched and that the displacements are increasing with time. The solution to this problem took approximately 26 minutes on a 900 MHz PC.

Even though this problem only involves 12 holes, our approach can be used to solve more complicated problems involving a larger number of holes of arbitrary sizes and locations as long as none of the holes overlap, and with more complicated loading conditions.

6.4. ***Special case of one hole.*** For the particular case of a single hole in an elastic plane, the displacements on the boundary of the hole are exactly represented by a two-term complex Fourier series [Muskhelishvili 1963]. This fact is retained for the viscoelastic plane and the only nonzero ‘coefficients’ are

$$\begin{aligned} \langle D_{-1}^{(1)} \rangle &= -R(\tilde{\sigma}_{yy}^{\infty} - \tilde{\sigma}_{xx}^{\infty} - 2i\tilde{\sigma}_{xy}^{\infty}), \\ \langle D_{1}^{(1)} \rangle &= \tfrac{1}{2} R(\tilde{\sigma}_{xx}^{\infty,*} + \tilde{\sigma}_{yy}^{\infty}), \\ \langle D_{1}^{(3)} \rangle &= -\tilde{p} R. \end{aligned} \tag{36}$$

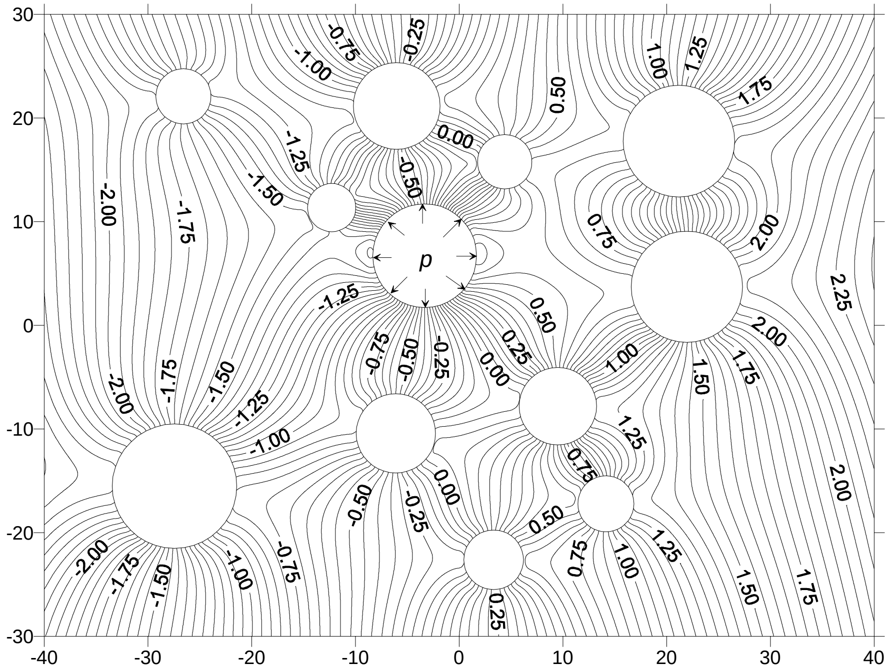

Figure 11. Contour of u_x at $t = 0$ sec.

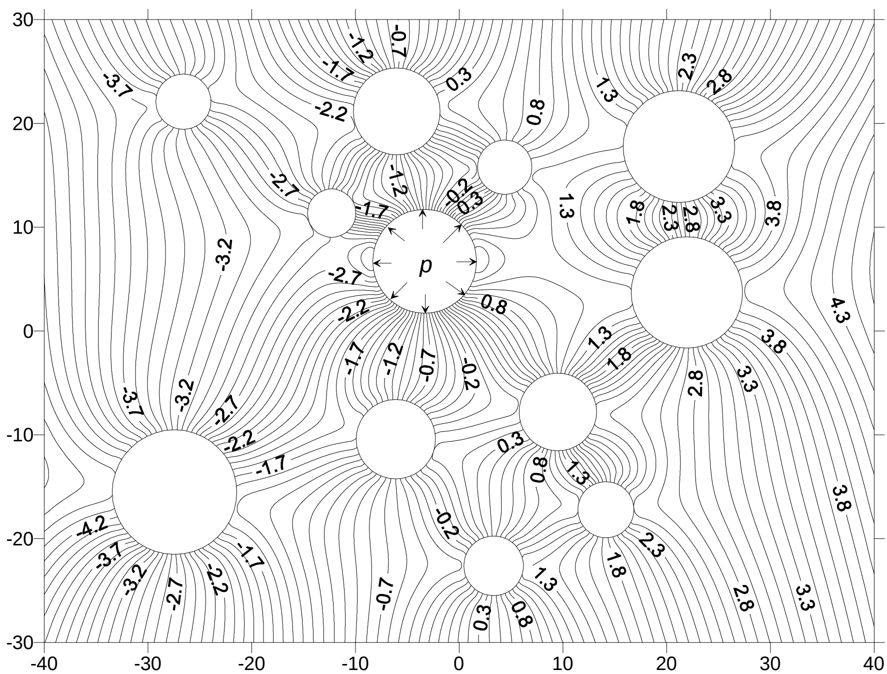

Figure 12. Contour of u_x at $t = 0.5$ sec.

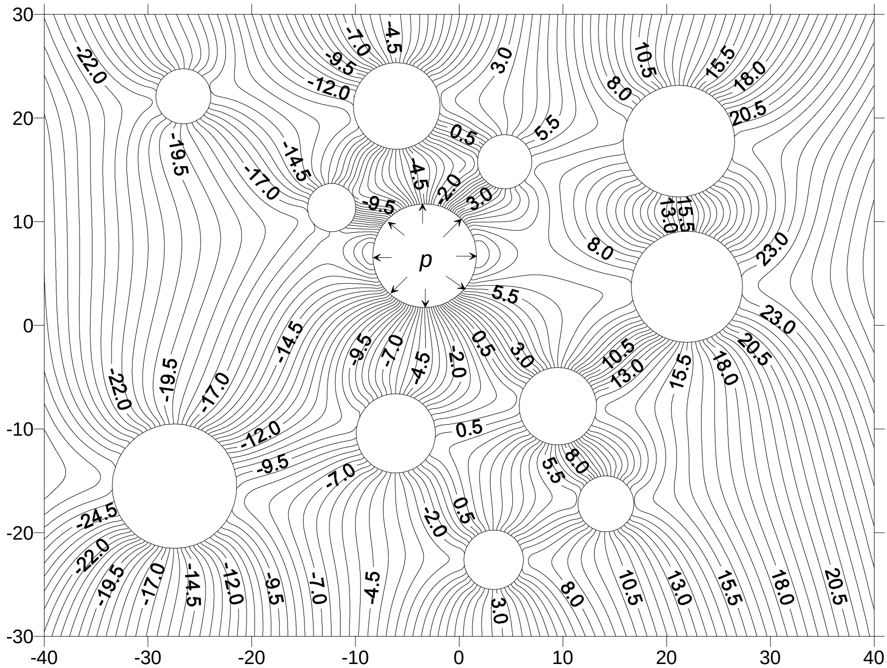

Figure 13. Contour of u_x at $t = 10$ sec.

The time functions $f_j(t)$ $(j = 1, \ldots, 9)$ can be obtained for the specific viscoelastic model and loading condition, as explained in the examples above. Using (36) and the time functions $f_j(t)$ and performing some algebraic manipulations one can obtain the analytical solution for the special case of a single circular hole within an infinite viscoelastic plane.

7. Concluding remarks

A complex variable boundary integral method combined with analytical Laplace transform and its inversion is presented to obtain a semi-analytical solution for the problem of an infinite viscoelastic plane containing multiple circular holes. The method is based on the use of the correspondence principle and a complex variable hypersingular integral equation for a corresponding elastic problem. A significant feature of the governing integral equation is that the transformed material parameters are not involved in the integral terms for the transformed boundary displacements.

The main features of the solution for an analogous elastic problem are preserved in the current method [Wang et al. 2003a]. The unknown displacements on the circular boundaries of the holes in Laplace domain are approximated by truncated complex Fourier series with the coefficients dependent on the transform parameter. A system of linear algebraic equations is formed by using the Taylor series expansion. Solutions of stresses and displacements in Laplace domain are written in terms of viscoelastic analogs of Kolosov–Muskhelishvili potentials, which are defined through integrals of displacements on the boundaries of the holes. The time domain solution for stresses and displacements are obtained using the analytical inverse Laplace transform.

The present method has the following advantages:

(1) The time dependence of the viscoelastic solution is expressed through several simple time functions. Thus, the method can easily incorporate a variety of physical models and loading conditions.

(2) All the mathematical operations (space integration, direct and inverse Laplace transforms) are performed analytically. The accuracy of the problem is only dependent on the number of terms in the complex Fourier series and the only error (apart from round-off) comes from the truncation of the Fourier series. This method provides an analytical solution for the problem involving only one hole, where the boundary displacements can be exactly expressed through finite terms in the Fourier series.

(3) The matrix of the resulting system of linear algebraic equations is inverted only once and the results are used for the calculation of the viscoelastic responses at any time instants. Thus, the method produces significant computational savings as compared with the numerical methods based on time stepping. The latter methods permit the use of a one-time inversion of the matrix only if the time step size is constant. Our method has no such limitation.

(4) The number of degrees of freedom is much less than in finite element-based methods.

The present approach allows a straightforward extension to the case where the displacements are prescribed on the boundaries of the holes if the Poisson's ratio of the viscoelastic matrix is constant. The case with a time-dependent Poisson's ratio for the matrix and the displacements prescribed at the boundaries is more complicated and needs more investigation.

Future developments of the approach might include the extension to problems of multiple circular holes within a finite circular viscoelastic domain as well as the extension to problems involving a half-plane containing multiple holes (by using the viscoelastic analog of the equation (21) in [Mogilevskaya 2000]). Another possibility is to extend the approach to problems of multiple holes of arbitrary shape. The governing Equation (2) remains valid for this case. Thus, the problem can be solved using the boundary element technique where the boundary of each hole is divided into elements and the unknowns on each of the elements are approximated by piecewise polynomials with time-dependent coefficients. Problems involving multiple curvilinear cracks could be also considered. This class of problems is governed by Equation (2), with unknown displacement discontinuities rather than displacements [Linkov and Mogilevskaya 1994].

Acknowledgments

The authors would like to acknowledge the University of Minnesota Supercomputing Institute for providing access to the finite element software ANSYS.

References

[Bird 1992] M. D. Bird, *A Fourier series method for determining the interaction of circular inhomogeneities in harmonic and biharmonic problems*, Ph.D. thesis, Stanford University, 1992.

[Bird and Steele 1991] M. D. Bird and C. R. Steele, "A separated solution procedure for bending of circular plates with circular holes", *Appl. Mech. Rev.* **44**:11, part 2 (1991), 27–35.

[Bird and Steele 1992] M. D. Bird and C. R. Steele, "Solution procedure for Laplace's equation on multiply connected circular domains", *J. Appl. Mech.* (*ASME*) **59** (1992), 398–404.

[Carini and Gioda 1986] A. Carini and G. Gioda, "A boundary integral equation technique for viscoelastic stress analysis", *Int. J. Numer. Anal. Methods Geomech.* **10**:6 (1986), 585–608.

[Chen et al. 2006a] J. T. Chen, C. C. Hsiao, and S. Y. Leu, "Null-field integral equation approach for plate problems with circular holes", *J. Appl. Mech.* (*ASME*) **73**:4 (2006), 679–693.

[Chen et al. 2006b] J. T. Chen, W. C. Shen, and P. Y. Chen, "Analysis of circular torsion bar with circular holes using null-field approach", *Comput. Model. Eng. Sci.* **12**:2 (2006), 109–119.

[Chen et al. 2006c] J. T. Chen, W. C. Shen, and A. C. Wu, "Null-field integral equations for stress field around circular holes under antiplane shear", *Eng. Anal. Bound. Elem.* **30**:3 (2006), 205–217.

[Crouch and Mogilevskaya 2003] S. L. Crouch and S. G. Mogilevskaya, "On the use of Somigliana's formula and Fourier series for elasticity problems with circular boundaries", *Int. J. Numer. Methods Eng.* **58**:4 (2003), 537–578.

[Findly et al. 1989] W. N. Findly, J. S. Lai, and K. Onaran, *Creep and relaxation of nonlinear viscoelastic materials*, Dover, New York, 1989.

[Haberman 1998] R. Haberman, *Elementary applied partial differential equations with Fourier series and boundary value problems*, 3rd ed., Prentice-Hall, Upper Saddle River, NJ, 1998. MR 84a:35001

[Haddon 1967] R. A. Haddon, "Stresses in an infinite plate with two unequal circular holes", *Q. J. Mech. Appl. Math.* **20**:3 (1967), 277–291.

[Huang et al. 2005a] Y. Huang, S. L. Crouch, and S. G. Mogilevskaya, "A direct boundary integral method for viscoelastic-elastic composite materials", pp. 264–267 in *Computational fluid and solid mechanics 2005*, edited by K. J. Bathe, Elsevier, Amsterdam, 2005.

[Huang et al. 2005b] Y. Huang, S. L. Crouch, and S. G. Mogilevskaya, "Direct boundary integral procedure for a Boltzmann viscoelastic plane with circular holes and elastic inclusions", *Comput. Mech.* **37**:1 (2005), 110–118.

[Huang et al. 2005c] Y. Huang, S. L. Crouch, and S. G. Mogilevskaya, "A time domain direct boundary integral method for a viscoelastic plane with circular holes and inclusions", *Eng. Anal. Bound. Elem.* **29**:7 (2005), 725–737.

[Kusama and Mitsui 1982] T. Kusama and Y. Mitsui, "Boundary element method applied to linear viscoelastic analysis", *Appl. Math. Model.* **6**:4 (1982), 285–290.

[Lee 1955] E. H. Lee, "Stress analysis in visco-elastic bodies", *Quart. Appl. Math.* **13** (1955), 183–190. MR 16,1073e

[Lee and Kim 1995] S. S. Lee and Y. J. Kim, "Time-domain boundary element analysis of cracked linear viscoelastic solids", *Eng. Frac. Mech.* **51**:4 (1995), 585–590.

[Lee et al. 1994] S. S. Lee, Y. S. Sohn, and S. H. Park, "On fundamental solutions in time-domain boundary element analysis of linear viscoelasticity", *Eng. Anal. Bound. Elem.* **13**:3 (1994), 211–217.

[Legros et al. 2004] B. Legros, S. G. Mogilevskaya, and S. L. Crouch, "A boundary integral method for multiple circular inclusions in an elastic half-plane", *Eng. Anal. Bound. Elem.* **28**:9 (2004), 1083–1098.

[Ling 1948] C. B. Ling, "On the stresses in a plate containing two circular holes", *J. Appl. Phys.* **19**:1 (1948), 77–82. MR 9,395k

[Linkov 2002] A. M. Linkov, *Boundary integral equations in elasticity theory*, Kluwer, Dordrecht, 2002.

[Linkov and Mogilevskaya 1994] A. M. Linkov and S. G. Mogilevskaya, "Complex hypersingular integrals and integral equations in plane elasticity", *Acta Mech.* **105**:1–4 (1994), 189–205. MR 95c:73096

[Linkov and Mogilevskaya 1998] A. M. Linkov and S. G. Mogilevskaya, "Complex hypersingular BEM in plane elasticity problems", pp. 299–364 in *Singular integrals in boundary element methods*, edited by V. Sladek and J. Sladek, Computational Mechanics Publications, Southampton, 1998.

[Mesquita and Coda 2001] A. D. Mesquita and H. B. Coda, "An alternative time integration procedure for Boltzmann viscoelasticity: a BEM approach", *Comput. Struct.* **79**:16 (2001), 1487–1496. MR 2002f:74079

[Mesquita and Coda 2002a] A. D. Mesquita and H. B. Coda, "Alternative Kelvin viscoelastic procedure for finite elements", *Appl. Math. Model.* **26**:4 (2002), 501–516.

[Mesquita and Coda 2002b] A. D. Mesquita and H. B. Coda, "Boundary integral equation method for general viscoelastic analysis", *Int. J. Solids Struct.* **39**:9 (2002), 2643–2664.

[Mesquita and Coda 2003] A. D. Mesquita and H. B. Coda, "New methodology for the treatment of two dimensional viscoelastic coupling problems", *Comput. Methods Appl. Mech. Eng.* **192**:16–18 (2003), 1911–1927.

[Mogilevskaya 2000] S. G. Mogilevskaya, "Complex hypersingular integral equation for the piecewise homogeneous half-plane with cracks", *Intl. J. Frac.* **102**:2 (2000), 177–204.

[Mogilevskaya and Crouch 2001] S. G. Mogilevskaya and S. L. Crouch, "A Galerkin boundary integral method for multiple circular elastic inclusions", *Int. J. Numer. Methods Eng.* **52**:10 (2001), 1069–1106.

[Mogilevskaya and Crouch 2002] S. G. Mogilevskaya and S. L. Crouch, "A Galerkin boundary integral method for multiple circular elastic inclusions with homogeneously imperfect interfaces", *Int. J. Solids Struct.* **39**:18 (2002), 4723–4746. Erratum in *ibid.* **40**, 1335.

[Mogilevskaya and Crouch 2004] S. G. Mogilevskaya and S. L. Crouch, "A Galerkin boundary integral method for multiple circular elastic inclusions with uniform interphase layers", *Int. J. Solids Struct.* **41**:5–6 (2004), 1285–1311.

[Mogilevskaya and Linkov 1998] S. G. Mogilevskaya and A. M. Linkov, "Complex fundamental solutions and complex variables boundary element method in elasticity", *Comput. Mech.* **22**:1 (1998), 88–92. MR 99f:73089

[Muskhelishvili 1963] N. I. Muskhelishvili, *Some basic problems of the mathematical theory of elasticity*, Noordhoff, Groningen, 1963.

[Rizzo and Shippy 1971] F. J. Rizzo and D. J. Shippy, "An application of the correspondence principle of linear viscoelasticity theory", *SIAM J. Appl. Math.* **21**:2 (1971), 321–330. Zbl 0211.28303

[Schapery 1962] R. A. Schapery, "Approximate methods of transform inversion for viscoelastic stress analysis", pp. 1075–1085 in *Proc. 4th U.S. Nat. Congr. Appl. Mech.* (Berkeley), vol. 2, Amer. Soc. Mech. Engrs., New York, 1962. MR 27 #3144

[Shinokawa et al. 1985] T. Shinokawa, N. Kaneko, and N. Yoshida, "Application of viscoelastic combined finite and boundary element analysis to geotechnical engineering", Chapter 10, pp. 37–46 in *Boundary elements VII* (Lake Como, Italy, 1985), vol. 2, edited by C. A. Brebbia and G. Maier, Springer, Berlin, 1985.

[Sim and Kwak 1988] W. J. Sim and B. M. Kwak, "Linear viscoelastic analysis in time domain by boundary element method", *Comput. Struct.* **29**:4 (1988), 531–539.

[Sun and Hsiao 1985] B. N. Sun and C. C. Hsiao, "Viscoelastic boundary element method for analyzing polymer crazing as quasifracture", Chapter 3, pp. 69–86 in *Boundary elements VII* (Lake Como, Italy, 1985), vol. 1, edited by C. A. Brebbia and G. Maier, Springer, Berlin, 1985.

[Timoshenko and Goodier 1970] S. P. Timoshenko and J. N. Goodier, *Theory of elasticity*, 3rd ed., McGraw-Hill, New York, 1970.

[Wang and Crouch 1982] Y. Wang and S. L. Crouch, "Boundary element methods for viscoelastic media", pp. 705–711 in *Issues in rock mechanics* (Berkeley), edited by R. E. Goodman and F. E. Heuze, The Society of Mining Engineers of the American Institute of Mining, Metallurgical and Petroleum Engineers, New York, 1982.

[Wang et al. 2003a] J. Wang, S. L. Crouch, and S. G. Mogilevskaya, "A complex boundary integral method for multiple circular holes in an infinite plane", *Eng. Anal. Bound. Elem.* **27**:8 (2003), 789–802.

[Wang et al. 2003b] J. Wang, S. G. Mogilevskaya, and S. L. Crouch, "A numerical procedure for multiple circular holes and elastic inclusions in a finite domain with a circular boundary", *Comput. Mech.* **32**:4–6 (2003), 250–258.

Received 11 Nov 2005.

YUN HUANG: huan0238@umn.edu
Department of Civil Engineering, University of Minnesota, 500 Pillsbury Drive SE, Minneapolis, MN 55455, United States

SOFIA G. MOGILEVSKAYA: mogil003@umn.edu
Department of Civil Engineering, University of Minnesota, 500 Pillsbury Drive SE, Minneapolis, MN 55455, United States
http://www.ce.umn.edu/people/faculty/mogilevs/

STEVEN L. CROUCH: crouch@itdean.umn.edu
Department of Civil Engineering, University of Minnesota, 500 Pillsbury Drive SE, Minneapolis, MN 55455, United States
http://www.ce.umn.edu/people/faculty/crouch/