Journal of
Mechanics of
Materials and Structures

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Volume 1, Nº 3 March 2006

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 1, No. 3, 2006

A NEW CLASS OF EQUILIBRATED STRESS FIELDS FOR NO-TENSION BODIES

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We study the equilibrium problem for bodies made of a no-tension material, subjected to distributed or concentrated loads on their boundary. Admissible and equilibrated stress fields are interpreted as tensor-valued measures with distributional divergence represented by a vector-valued measure, as outlined in two 2005 papers by Lucchesi et al. Such stress fields are generalizations of ordinary functions, which allows us to consider stress concentrations on surfaces and lines. The general framework for this approach is presented first and then illustrated on examples of two-dimensional panels under different loads. In the general framework we determine weak divergences and the surface tractions of several stress field measures via the (surface) divergence theorem. Combinations of these stress fields are shown to give the solutions for the panels, which we assume to be clamped at the bottom, and subjected to various loads on the top and possibly on the sides of the panel. The shapes of the singular lines and stresses are explicitly determined in these cases.

1. Introduction

In studying the equilibrium problem of bodies made of a no-tension (or masonry-like) material [Del Piero 1989; Di Pasquale 1984], it is very hard to find an explicit solution of the corresponding boundary value problem [Lucchesi and Zani 2003a]. Therefore, in applications we often limit ourselves to looking for stress fields that are equilibrated with the applied loads and compatible with the incapability of the material to withstand traction. These admissible equilibrium stress fields can be used in the context of limit analysis [Del Piero 1998] to determine the collapse load, or at least some of its lower bounds. The solution to this problem is considerably simplified by allowing the stress to be singular in some regions of the body. In [Lucchesi and Zani 2002; 2003b], solutions for two-dimensional panels are examined which are regular except on a finite number of singularity curves where the stress field is unbounded. The method of solving the equilibrium equations is based on the fact that if horizontal and vertical loads are distributed only on the panel's top and the stress determinant is null, the equilibrium equations constitute a system of

Keywords: masonry panels, equilibrium, divergence measures.

conservation laws, formally identical to the nonlinear system ruling the dynamics of the one-dimensional isentropic flow of a pressureless compressible gas. Under appropriate hypotheses, this system is equivalent to a single scalar conservation law [Brenier and Grenier 1998; Bouchut and James 1999]. Then the singularity curves are determined by means of the Rankine–Hugoniot jump condition corresponding to this scalar equation. This method is not directly applicable if distributed loads are present on the lateral sides of the panel or if the determinant of the stress does not vanish.

Following [Lucchesi et al. 2004; 2005a], the present paper uses tensor-valued measures to describe the stress fields in no-tension bodies. On the common range of applicability, the language of measures is essentially equivalent to the method of the conservation law mentioned above. However, the approach via measures is conceptually more direct in accounting for the singularities of the stress field and in the way the balance of forces is taken into account. The former means that we consider generally measures that are not absolutely continuous with respect to the Lebesgue measure (= volume); the latter means that the balance of forces is interpreted in the weak sense. This in turn means that the distributional divergence of the stress measure is equal to the body force in the interior of the body, and that the boundary trace of the measure equals the external boundary loads, given by a prescribed measure.

Using the divergence theorem, we first calculate the weak divergence and the trace of some elementary stress fields:

(i) those distributed over volumes or concentrated on surfaces and lines, and

(ii) stress fields given by some specific expression (see Equation (3–8)) that is encountered in some solutions for panels given below.

The stress fields encountered in the applications to panels are linear combinations of the elementary stress fields in items (i) and (ii) of Proposition 1. We then consider briefly the general balance equation, and show, among other things, that for the stress field consisting of the bulk stress and of the stress concentrated on a surface, one obtains the classical form of the balance equations.

The rest of the paper illustrates the general notions on the rectangular panels made of a no-tension material. We assume that the panel is free from body forces, clamped at its bottom and subjected to loads prescribed on the boundary; applications to three-dimensional bodies under gravity will be treated in a future work. The stress field in our solutions is plane and negative semidefinite and characterized by the presence of one or more curves of concentrated stress. This feature, which is allowed by supposing the material to have infinite compressive strength, seems to be paradoxical at first sight. On the other hand, this simplifying hypothesis is frequently used in the study of masonry structures, at least when the collapse is believed to take place for small values of the compressive stress [Heyman 1966]. Moreover, these singular equilibrated stress fields look like a formalization of the rough idea that a masonry building is 'safe' if its interior contains an equilibrated (and compressed) structure, an idea that was probably already in the mind of Leonardo da Vinci [Benvenuto 1991].

Even though the present paper makes no mention of the displacement fields, our use of measures to describe the stresses raises the question of the appropriate duality between the stresses and strains. We are especially interested in the expression

$$\mathsf{T} \cdot \boldsymbol{\epsilon}(\boldsymbol{u}) \tag{1–1}$$

for the virtual work of the stress field T against the virtual displacement $\boldsymbol{u}$ with the small strain tensor $\boldsymbol{\epsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathsf{T}})$. Generally, the 'wilder' T is, the smoother $\boldsymbol{u}$ must be, and vice versa. The issues are well understood in Hencky's plasticity, where $\boldsymbol{u}$ is generally a vector field with bounded deformation. The papers [Témam and Strang 1980; 1980; Anzellotti 1983] and [Kohn and Témam 1983] provide a variety of results pertaining to that case. Roughly, T must be a Lebesgue measurable function with divergence measure and with some natural integrability properties. Thus no concentrations in T are allowed. Our situation is the opposite: the stress has concentrations and the strain must be 'tamer.' The results of Whitney's theory of flat chains [Whitney 1957; Federer 1969] apply here (see [Šilhavý 2005c]). For symmetric tensor-valued stress measures T with divergence measure, the expression in Equation (1–1) is a well defined measure provided the displacement $\boldsymbol{u}$ is Lipschitz continuous (for stress measures with additional properties the class of displacements may be wider). The result is not immediate because $\boldsymbol{\epsilon}(\boldsymbol{u})$ may be undefined on the surface of concentration of T; a substantial use has been made of the fact that the divergence of T is a measure. We will return to these issues in a separate paper.

2. Vector-valued measures

This section introduces measures with values in a finite-dimensional inner product space V. Such measures can be identified with an m-tuple of (scalar-valued) signed measures where $m := \dim V$ is the dimension of V. We refer to [Rudin 1974, Chapters 1 and 6] for the details of scalar-valued signed measures. The notation in Equations (2–2) and (2–3), below, will be used systematically throughout the paper.

Throughout the paper, Lin denotes the space of all linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^n$ with the scalar product $\boldsymbol{A} \cdot \boldsymbol{B} = \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}^{\mathsf{T}})$, $\boldsymbol{A}, \boldsymbol{B} \in$ Lin, and Sym is a subspace of Lin consisting of all symmetric transformations. We interpret Lin as the space of all second order tensors, and use vector and tensor notations and conventions from [Gurtin 1981] and [Šilhavý 1997].

If V is a finite-dimensional real inner product space then a function $\boldsymbol{\mu}$, defined on the system of all Borel subsets of $\mathbb{R}^n$, is said to be a *V-valued measure* on $\mathbb{R}^n$ if

$$\boldsymbol{\mu}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \boldsymbol{\mu}(A_i), \tag{2–1}$$

for every pairwise disjoint sequence A_i of Borel sets. We recall that the system of all Borel sets is the smallest σ algebra of subsets of $\mathbb{R}^n$ that contains all open subsets of $\mathbb{R}^n$. The application of Equation (2–1) to the sequence $A_i = \varnothing$, $i = 1, \dots,$ gives $\boldsymbol{\mu}(\varnothing) = \mathbf{0}$. If $U \subset \mathbb{R}^n$ is a Borel set, we say that a V-valued measure $\boldsymbol{\mu}$ is *supported* on U if $\boldsymbol{\mu}(B) = \mathbf{0}$ for every Borel set $B \subset \mathbb{R}^n$ such that $U \cap B = \varnothing$. We denote by $\mathcal{M}(U, V)$ the set of all V-valued measures supported on U. We call the elements of $\mathcal{M}(U, \mathbb{R})$ *signed measures* on U and the elements of $\mathcal{M}(U, \mathrm{Lin})$ *tensor measures.* In the special case $V = \mathbb{R}^m$ one has

$$\boldsymbol{\mu}(A) = (\mu^1(A), \dots, \mu^m(A)),$$

for each Borel set $A \subset \mathbb{R}^n$ where μ^i, $1 \le i \le m$, are signed measures. A similar reduction of $\boldsymbol{\mu}$ applies to any V equipped with a basis.

If $\boldsymbol{\mu}$ is a V-valued measure, we say that a Borel set $A \subset \mathbb{R}^n$ is a $\boldsymbol{\mu}$ null set if $\boldsymbol{\mu}(B) = \mathbf{0}$ for each Borel set $B \subset A$. We say that a map $\boldsymbol{f}$ is defined as $\boldsymbol{\mu}$ *almost everywhere* (a.e.) on a set M if the set of all $\boldsymbol{x} \in M$ for which $\boldsymbol{f}(\boldsymbol{x})$ is not defined forms a $\boldsymbol{\mu}$ null set. Similarly, we say that a given property holds a.e. on M if the set of all $\boldsymbol{x}$ for which the property is violated forms a $\boldsymbol{\mu}$ null set.

If $\boldsymbol{\alpha} : U \to V$ is a bounded Borel function and $\boldsymbol{\nu} \in \mathcal{M}(U, V)$, then

$$\int_U \boldsymbol{\alpha} \cdot d\boldsymbol{\nu}$$

is a well defined number. If $\boldsymbol{a} \in U$, we denote by $\delta_{\boldsymbol{a}} \in \mathcal{M}(U, \mathbb{R})$ the *Dirac measure* at $\boldsymbol{a}$, defined by

$$\delta_{\boldsymbol{a}(B)} = \begin{cases} 1 & \text{if} \quad \boldsymbol{a} \in B, \\ 0 & \text{if} \quad \boldsymbol{a} \notin B, \end{cases}$$

for any Borel set $B \subset \mathbb{R}^n$, and note that if $f : U \to \mathbb{R}$ is a (bounded) Borel function then

$$\int_U f \, d\delta_{\boldsymbol{a}} = f(\boldsymbol{a}).$$

We denote by $\mathscr{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ [Federer 1969, Subsection 2.6.5] and if k is an integer, $0 \le k \le n$, we denote by $\mathscr{H}^k$ the k-dimensional Hausdorff measure in $\mathbb{R}^n$ [Federer 1969, Subsections 2.10.2–2.10.6]; recall that $\mathscr{H}^n = \mathscr{L}^n$. If

A is a Borel set, we denote by $\mathcal{H}^k \llcorner A$ the *restriction* of $\mathcal{H}^k$ to A, which is the measure defined by

$$(\mathcal{H}^k \llcorner A)(B) = \mathcal{H}^k(A \cap B), \tag{2–2}$$

for each Borel set $B \subset \mathbb{R}^n$. If $A \subset U$ is a Borel set and $\boldsymbol{f}$ is a V-valued Borel map defined a.e. on A, integrable with respect to $\mathcal{H}^k$ on A, then $\boldsymbol{f}\mathcal{H}^k \llcorner A$ denotes the V-valued measure on $\mathbb{R}^n$ defined by

$$(\boldsymbol{f}\mathcal{H}^k \llcorner A)(B) = \int_{A\cap B} \boldsymbol{f}\, d\mathcal{H}^k, \tag{2–3}$$

for each Borel set $B \subset \mathbb{R}^n$. The definitions (2–2) and (2–3) also apply to $k = n$, that is, to $\mathcal{L}^n \equiv \mathcal{H}^n$, resulting in $\mathcal{L}^n \llcorner A$ and $\boldsymbol{f}\mathcal{L}^n \llcorner A$. If $\boldsymbol{\alpha} : A \to V$ is a bounded Borel function then

$$\int_A \boldsymbol{\alpha} \cdot d(\boldsymbol{f}\mathcal{H}^k \llcorner A) = \int_A \boldsymbol{\alpha} \cdot \boldsymbol{f}\, d\mathcal{H}^k.$$

The construction (2–3) will be used to introduce stresses concentrated on surfaces. In that case $A \equiv \mathcal{S}$ is a k-dimensional surface with boundary (see the Appendix for the summary of differential-geometric notions), $\boldsymbol{f} \equiv \boldsymbol{T}_s$ is a $\mathcal{H}^k$ integrable map on $\mathcal{S}$ with values in Lin and

$$\mathsf{T}_s := \boldsymbol{T}_s \mathcal{H}^k \llcorner \mathcal{S} \tag{2–4}$$

is a stress field concentrated on $\mathcal{S}$. Similarly, if $A = U$ is an open subset of $\mathbb{R}^n$ and $\boldsymbol{T}_r$ is an $\mathcal{L}^n$ integrable map on U with values in Lin then

$$\mathsf{T}_r := \boldsymbol{T}_r \mathcal{L}^n \llcorner U \tag{2–5}$$

is a distributed stress field on U. Only combinations of measures of type (2–4) and (2–5) are of real use in Sections 5–7. The corresponding equilibrium equations are considered in Sections 3 and 4.

3. Divergence measure tensor fields

If V is a finite-dimensional real inner product space and if U is an open subset of $\mathbb{R}^n$, we denote by $C_0^\infty(U, V)$ the set of all infinitely differentiable functions

$$\boldsymbol{\alpha} : \mathbb{R}^n \to V$$

whose support $\operatorname{spt}\boldsymbol{\alpha}$ is contained in U. We say that a tensor-valued measure

$$\mathsf{T} \in \mathcal{M}(U, \mathrm{Lin})$$

is a *divergence measure tensor field* if there exists a measure

$$\mathbf{div}\, \mathsf{T} \in \mathcal{M}(U, \mathbb{R}^n),$$

called the *divergence* of T, such that

$$\int_U \nabla \boldsymbol{v} \cdot d\mathsf{T} = -\int_U \boldsymbol{v} \cdot d\,\mathbf{div}\,\mathsf{T}, \tag{3–1}$$

for each $\boldsymbol{v} \in C_0^\infty(U, \mathbb{R}^n)$. We note that vector-valued measures with divergence measure have been introduced in [Chen and Frid 2001; 2003]; vector or tensor-valued functions with divergence measure have been considered in [Anzellotti 1983; Kohn and Témam 1983; Chen and Frid 1999; Degiovanni et al. 1999; Marzocchi and Musesti 2001; Šilhavý 2005a; Chen and Torres 2005]. (For the application of divergence measure tensor fields to masonry structures, see [Lucchesi et al. 2005a].) A measure $\mathsf{T} \in \mathcal{M}(U, \mathrm{Lin})$ is said to be an *equilibrated tensor field* if there exist measures $\mathsf{b} \in \mathcal{M}(U, \mathbb{R}^n)$ and $\mathsf{t} \in \mathcal{M}(\partial U, \mathbb{R}^n)$ such that

$$\int_U \nabla \boldsymbol{v} \cdot d\mathsf{T} = \int_U \boldsymbol{v} \cdot d\mathsf{b} + \int_{\partial U} \boldsymbol{v} \cdot d\mathsf{t}, \tag{3–2}$$

for each $\boldsymbol{v} \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ [Podio-Guidugli 2005; Šilhavý 2005a; 2005b; 2005c]. Here ∂U denotes the topological boundary of U. Since the measures b and t are supported on the disjoint sets U and ∂U, respectively, they are uniquely determined (provided they exist). We call the pair (b, t) the *load* corresponding to T and the measure t the *normal trace* of T; we use the notation $\mathrm{N}(\mathsf{T}) := \mathsf{t}$ for the normal trace. Equation (3–2) then reads

$$\int_U \nabla \boldsymbol{v} \cdot d\mathsf{T} = -\int_U \boldsymbol{v} \cdot d\,\mathbf{div}\,\mathsf{T} + \int_{\partial U} \boldsymbol{v} \cdot d\mathrm{N}(\mathsf{T}), \tag{3–3}$$

for each $\boldsymbol{v} \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Clearly, any equilibrated tensor field T is a divergence measure tensor field, and if (b, t) is the load, then $\mathbf{div}\mathsf{T} = -\mathsf{b}$. There are divergence measure tensor fields that are not equilibrated [Šilhavý 2005b, Chapter 9; Šilhavý 2005c, Example 9.1], that is, for which the measure t does not exist.

The integration by parts shows that if $\boldsymbol{T} : U \to \mathrm{Lin}$ is a continuously differentiable tensor field with integrable gradient on an open set $U \subset \mathbb{R}^n$, then the tensor-valued measure

$$\mathsf{T} = \boldsymbol{T}\mathcal{L}^n \llcorner U$$

is a divergence measure tensor field and

$$\mathbf{div}\,\mathsf{T} = \operatorname{div} \boldsymbol{T}\mathcal{L}^n \llcorner U.$$

Here div is the classical divergence given by the usual differential expression; a particular case of the surface divergence introduced in the Appendix, while **div** denotes the divergence as a measure, defined above. The reader is also referred to the Appendix for the differential-geometric concepts employed in the subsequent discussion. If, additionally, U is an open region with Lipschitz boundary, and $\boldsymbol{T}$

has a continuous extension to the closure $\mathrm{cl}\, U$ of U, again denoted by $\boldsymbol{T}$, that is $\mathcal{H}^{n-1}$ integrable on ∂U, then the divergence theorem shows that T is an equilibrated tensor field and

$$\mathrm{N}(\mathsf{T}) = \boldsymbol{T}\boldsymbol{m}\, \mathcal{H}^{n-1} \llcorner \partial U,$$

where $\boldsymbol{m}$ is the outer normal to U. This justifies the term 'normal trace' for $\mathrm{N}(\mathsf{T})$.

The following proposition generalizes the above considerations to singular tensor fields concentrated on k-dimensional surfaces, $1 \le k \le n$. If $\boldsymbol{T} : \mathcal{S} \to \mathrm{Lin}$ is a tensor field on a k-dimensional surface $\mathcal{S}$ with boundary, we say that $\boldsymbol{T}$ is *superficial* [Gurtin 2000, p. 94] if $\boldsymbol{T}(\boldsymbol{x})\boldsymbol{v} = \boldsymbol{0}$ for every $\boldsymbol{x} \in S$ and every $\boldsymbol{v} \in \mathbb{R}^n$ perpendicular to $\mathbf{T}_{\boldsymbol{x}}(\mathcal{S})$. This is equivalent to each of the following two statements:

(a) $\mathbf{T}_{\boldsymbol{x}}(\mathcal{S})^{\perp} \subset \ker \boldsymbol{T}(\boldsymbol{x})$ for every $\boldsymbol{x} \in S$;

(b) $\mathrm{ran}\, \boldsymbol{T}^{\mathsf{T}}(\boldsymbol{x}) \subset \mathbf{T}_{\boldsymbol{x}}(\mathcal{S})$.

Here ker and ran denote the kernel and range of a linear transformation.

Proposition 1. *Let U be an open subset of $\mathbb{R}^n$, let $k \ge 1$ be an integer, let $\mathcal{S}$ be a compact orientable k-dimensional surface with boundary such that* $\mathrm{int}\, \mathcal{S} \subset U$*, let $\boldsymbol{T} : \mathcal{S} \to \mathrm{Lin}$ be a continuous map with a continuous and $\mathcal{H}^k$ integrable derivative in* $\mathrm{int}\, \mathcal{S}$*, and put*

$$\mathsf{T} := \boldsymbol{T}\mathcal{H}^k \llcorner \mathcal{S}.$$

Then

(i) T *is equilibrated* $\Leftrightarrow$ T *is a divergence measure tensor field* $\Leftrightarrow$ $\boldsymbol{T}$ *is superficial;*

(ii) *if $\boldsymbol{T}$ is superficial then*

$$\mathbf{div}\, \mathsf{T} = \mathrm{div}\, \boldsymbol{T}\mathcal{H}^k \llcorner (\mathcal{S} \cap U) - \boldsymbol{T}\boldsymbol{m}\mathcal{H}^{k-1} \llcorner (\partial\mathcal{S} \cap U) \tag{3–4}$$

and

$$\mathrm{N}(\mathsf{T}) = \boldsymbol{T}\boldsymbol{m}\, \mathcal{H}^{k-1} \llcorner (\partial\mathcal{S} \cap \partial U), \tag{3–5}$$

where $\boldsymbol{m}$ is the outer normal to $\mathcal{S}$.

Item (1) says that a measure concentrated on a k-dimensional surface is a divergence measure tensor field only if it is superficial. If this is the case then the divergence of T consists of the (surface) divergence of $\boldsymbol{T}$ concentrated on $\mathcal{S} \cap U$ and of the normal component of $\boldsymbol{T}$ concentrated on $\partial\mathcal{S} \cap U$ while the normal trace of T is the remaining part of the normal component of $\boldsymbol{T}$, that is, the part concentrated on $\partial\mathcal{S} \cap \partial U$. We emphasize that $k \ge 1$ is arbitrary. If $k = 1$, that is, if $\mathcal{S}$ is a curve, then the measure

$$\boldsymbol{T}\boldsymbol{m}\, \mathcal{H}^{k-1} \llcorner (\partial\mathcal{S} \cap U)$$

reduces to

$$\sum_{\boldsymbol{a}\in\partial\mathcal{S}\cap U} \boldsymbol{T}(\boldsymbol{a})\boldsymbol{m}(\boldsymbol{a})\delta_{\boldsymbol{a}},$$

where the set $\partial\mathcal{S}\cap U$ is the set of all endpoints of $\mathcal{S}$ in U and $\boldsymbol{m}(\boldsymbol{a})$ are outward tangents to $\mathcal{S}$ at the endpoints. A similar interpretation applies to the right side of Equation (3–5). We note that the argument used below to prove Item (i) can also be applied to show that there are no nontrivial divergence measure tensor fields concentrated at points, that is, surfaces of dimension 0.

Proof. (ii): Assume that $\boldsymbol{T}$ is superficial and prove Equations (3–4) and (3–5). Let $\boldsymbol{v}\in C_0^\infty(\mathbb{R}^n,\mathbb{R}^n)$; since $\boldsymbol{T}$ is superficial, $\boldsymbol{T}^\top\boldsymbol{v}$ is tangential and thus the surface divergence theorem 1, the identity (A–7), and the hypothesis $\operatorname{int}\mathcal{S}\subset U$ give

$$\int_{\mathcal{S}\cap U}(\boldsymbol{v}\cdot\operatorname{div}\boldsymbol{T}+\boldsymbol{T}\cdot\nabla\boldsymbol{v})\,d\mathcal{H}^k=\int_{\partial\mathcal{S}}\boldsymbol{T}\boldsymbol{m}\cdot\boldsymbol{v}\,d\mathcal{H}^{k-1}.$$

Rearranging, we obtain

$$\int_{\mathcal{S}\cap U}\boldsymbol{T}\cdot\nabla\boldsymbol{v}\,d\mathcal{H}^k=-\int_{\mathcal{S}\cap U}\boldsymbol{v}\cdot\operatorname{div}\boldsymbol{T}\,d\mathcal{H}^k \\ +\int_{\partial\mathcal{S}\cap U}\boldsymbol{T}\boldsymbol{m}\cdot\boldsymbol{v}\,d\mathcal{H}^{k-1}+\int_{\partial\mathcal{S}\cap\partial U}\boldsymbol{T}\boldsymbol{m}\cdot\boldsymbol{v}\,d\mathcal{H}^{k-1};$$

comparing this with Equation (3–3) and invoking the uniqueness of $\mathbf{div}\,\mathsf{T}$ and $\mathrm{N}(\mathsf{T})$ we see that Equation (3–4) and Equation (3–5) hold.

(i): We shall prove the cycle of implications: T is equilibrated $\Rightarrow$ T is a divergence measure tensor field $\Rightarrow$ $\boldsymbol{T}$ is superficial $\Rightarrow$ T is equilibrated. The first of these implications is automatic, as mentioned above, while the last implication has been proved in (ii). Thus it remains to be proved that if T is a divergence measure tensor field then $\boldsymbol{T}$ is superficial. Let T be a divergence measure tensor field, let $\boldsymbol{x}\in\mathcal{S}\cap U$ and assume first additionally that $\boldsymbol{x}\notin\partial\mathcal{S}$. Let Z and $\boldsymbol{\omega}$ be as in Condition (ii) of Subsection A.1, and assume, as we can, that $Z\subset U$. Let $\lambda\in C_0^\infty(\mathbb{R}^n,\mathbb{R})$ be such that $\operatorname{spt}\lambda\subset Z$. Let $\boldsymbol{b}\in\mathbb{R}^{n-k}$ and $\boldsymbol{a}\in\mathbb{R}^n$. If $\epsilon>0$ then there exists $h\in C^\infty(\mathbb{R}^{n-k},\mathbb{R})$ such that

$$h(\mathbf{0})=0,\quad \nabla h(\mathbf{0})=\boldsymbol{b} \tag{3–6}$$

and $|h|_\infty<\epsilon$. Here $|\cdot|_\infty$ is the maximum norm; that is, if M is a set and $f:M\to V$ then

$$|f|_\infty:=\sup\{|f(\boldsymbol{x})|:\boldsymbol{x}\in M\}.$$

Let $\boldsymbol{v}\in C_0^\infty(U,\mathbb{R}^n)$ be defined by

$$\boldsymbol{v}=\begin{cases}\boldsymbol{a}\lambda\,(h\circ\boldsymbol{\omega}) & \text{on } Z,\\ 0 & \text{on } \mathbb{R}^n\setminus Z.\end{cases} \tag{3–7}$$

Using $|h \circ \boldsymbol{\omega}|_\infty = |h|_\infty < \epsilon$ one finds that $|\boldsymbol{v}|_\infty \le |\boldsymbol{a}||\lambda|_\infty \epsilon$. Applying the product rule to calculate $\nabla \boldsymbol{v}$ from Equation (3–7), using Equation (3–6), and noting that $h \circ \boldsymbol{\omega} = 0$ on $\mathcal{S}$, one finds that

$$\nabla \boldsymbol{v} = \begin{cases} \lambda \boldsymbol{a} \otimes \nabla \boldsymbol{\omega}^\top \boldsymbol{b} & \text{on } \mathcal{S} \cap Z, \\ 0 & \text{on } \mathcal{S} \setminus Z. \end{cases}$$

Equation (3–1) thus reads

$$\int_{\mathcal{S} \cap Z} \lambda \, \nabla \boldsymbol{\omega} \, \boldsymbol{T}^\top \boldsymbol{a} \cdot \boldsymbol{b} \, d\mathcal{H}^k = -\int_U \boldsymbol{v} \cdot d\,\mathbf{div}\, \mathsf{T}.$$

Denoting by $\mathbf{M}(\boldsymbol{\mu})$ the total variation of a vector-valued measure $\boldsymbol{\mu} \in \mathcal{M}(U, V)$,

$$\mathbf{M}(\boldsymbol{\mu}) := \sup\{ \int_U \boldsymbol{\alpha} \cdot d\boldsymbol{\mu} : \boldsymbol{\alpha} \in C_0^\infty(U, V), |\boldsymbol{\alpha}|_\infty \le 1 \},$$

we note that the inequality $|\boldsymbol{v}|_\infty \le |\boldsymbol{a}||\lambda|_\infty \epsilon$ gives

$$\Big| \int_U \boldsymbol{v} \cdot d \;\mathbf{div}\; \mathsf{T} \Big| \le \mathbf{M}(\;\mathbf{div}\; \mathsf{T}) |\boldsymbol{a}||\lambda|_\infty \epsilon,$$

and hence

$$\Big| \int_{\mathcal{S} \cap Z} \lambda \, \nabla \boldsymbol{\omega} \, \boldsymbol{T}^\top \boldsymbol{a} \cdot \boldsymbol{b} \, d\mathcal{H}^k \Big| \le \mathbf{M}(\mathbf{div}\; \mathsf{T}) |\boldsymbol{a}||\lambda|_\infty \epsilon.$$

As $\epsilon > 0$ is arbitrary, we have

$$\int_{\mathcal{S} \cap Z} \lambda \, \nabla \boldsymbol{\omega} \, \boldsymbol{T}^\top \boldsymbol{a} \cdot \boldsymbol{b} \, d\mathcal{H}^k = 0.$$

As this must hold for any $\lambda, \boldsymbol{a}, \boldsymbol{b}$ subject to the conditions above, we have

$$\nabla \boldsymbol{\omega} \, \boldsymbol{T}^\top = \mathbf{0}$$

on $\mathcal{S} \cap Z$, and since $\ker \nabla \boldsymbol{\omega}(\boldsymbol{x}) = \mathbf{T}_{\boldsymbol{x}}(\mathcal{S})$, we deduce that $\operatorname{ran} \boldsymbol{T}^\top(\boldsymbol{x}) \subset \mathbf{T}_{\boldsymbol{x}}(\mathcal{S})$. Thus the restriction of $\boldsymbol{T}$ to $\mathcal{S} \cap (U \setminus \partial \mathcal{S})$ is superficial. Since the closure of the last set is $\mathcal{S}$, the continuity of $\boldsymbol{T}$ implies that $\boldsymbol{T}$ is superficial on $\mathcal{S}$. □

A subset C of $\mathbb{R}^n$ is said to be a *cone* if $r\boldsymbol{v} \in C$ for every $r > 0$ and $\boldsymbol{v} \in C$. For each $\boldsymbol{x} \in \mathbb{R}^n$ and $r > 0$, let $\mathbf{B}(\boldsymbol{x}, r)$ denote the open ball in $\mathbb{R}^n$ of center $\boldsymbol{x}$ and radius r, and let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$.

Proposition 2. *Let $U \subset \mathbb{R}^n$ be an open region with Lipschitz boundary containing the origin, and let C be an open cone such that $\partial C \setminus \{\mathbf{0}\}$ is an $(n-1)$-dimensional surface. Let $\alpha : \operatorname{cl} C \cap \mathbb{S}^{n-1} \to \mathbb{R}$ be a continuous function which is continuously differentiable in $C \cap \mathbb{S}^{n-1}$, and let $\boldsymbol{T} : (U \cap \operatorname{cl} C) \setminus \{\mathbf{0}\} \to \mathrm{Lin}$ be given by*

$$\boldsymbol{T}(\boldsymbol{x}) = |\boldsymbol{x}|^{-n-1} \alpha\Big(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\Big) \boldsymbol{x} \otimes \boldsymbol{x}, \quad \boldsymbol{x} \in (U \cap \operatorname{cl} C) \setminus \{\mathbf{0}\}. \tag{3–8}$$

Then $\boldsymbol{T}$ *is continuous in* $(U \cap \operatorname{cl} C) \setminus \{\mathbf{0}\}$ *and continuously differentiable in* $C \cap U$ *and we have*

$$\operatorname{div} \boldsymbol{T} = \mathbf{0} \quad in \quad C \cap U. \tag{3–9}$$

Moreover, if we put

$$\mathsf{T} = \boldsymbol{T} \mathscr{L}^n \llcorner (C \cap U),$$

then T *is an equilibrated tensor field and*

$$\mathbf{div}\, \mathsf{T} = \boldsymbol{c}\, \delta_{\mathbf{0}}, \tag{3–10}$$

$$\mathrm{N}(\mathsf{T}) = \boldsymbol{T}\boldsymbol{m}\, \mathscr{H}^{n-1} \llcorner (\partial U \cap C), \tag{3–11}$$

where

$$\boldsymbol{c} = \int_{C \cap \mathbb{S}^{n-1}} \alpha(\boldsymbol{e})\boldsymbol{e}\, d\mathscr{H}^{n-1}(\boldsymbol{e}), \tag{3–12}$$

and $\boldsymbol{m}$ *is the outer normal to* $\partial U \cap C$.

For $n = 2$, the stress field $\boldsymbol{T}$ as in Equation (3–8) falls within the class studied in [Podio-Guidugli 2005, Section 4], from where also (3–9) can be deduced. In (3–12) we denote by $\boldsymbol{e} \in C \cap \mathbb{S}^{n-1}$ the integration variable. Using the divergence theorem and (3–9) as in the proof below, one also finds that

$$\boldsymbol{c} = \int_{C \cap \partial U} \boldsymbol{T}\boldsymbol{m}\, d\mathscr{H}^{n-1}. \tag{3–13}$$

Proof. The continuity and differentiability of $\boldsymbol{T}$ follows directly from the assumptions on α and (3–9) is a straightforward calculation which we omit. To prove (3–10) and (3–11), we denote by B_r the closed ball with center $\mathbf{0}$ and radius $r > 0$ and consider the set $(C \setminus B_r) \cap U$. This is an open set and if $r > 0$ is small enough to satisfy $B_r \subset U$ (recall that $\mathbf{0} \in U$), then

$$\partial[(C \setminus B_r) \cap U] = (C \cap \partial U) \cup (C \cap \partial B_r) \cup [(\partial C \setminus B_r) \cap U] \cup T \tag{3–14}$$

to within a set of $\mathscr{H}^{n-1}$ measure 0 where

$$T := \{\boldsymbol{x} \in (\partial C \setminus \{\mathbf{0}\}) \cap \partial U : \boldsymbol{n}_{\partial C \setminus \{\mathbf{0}\}}(\boldsymbol{x}) = \boldsymbol{m}(\boldsymbol{x})\}, \tag{3–15}$$

where $\boldsymbol{n}_{\partial C \setminus \{\mathbf{0}\}}$ is the outer normal to $C \setminus \{\mathbf{0}\}$ and $\boldsymbol{m}$ is the outer normal to U. Equation (3–14) can be deduced from the general formula in [Marzocchi and Musesti 2001, Proposition 2.2]. We note that $(C \setminus B_r) \cap U$ is an open region with Lipschitz boundary with the outer normal given by

$$\boldsymbol{n}(\boldsymbol{x}) = \begin{cases} \boldsymbol{m}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in C \cap \partial U, \\ -\boldsymbol{x}/r & \text{if } \boldsymbol{x} \in C \cap \partial B_r\ , \\ \boldsymbol{n}_{\partial C \setminus \{\mathbf{0}\}}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in [(\partial C \setminus B_r) \cap U] \cup T, \end{cases} \tag{3–16}$$

to within a change on a $\mathcal{H}^{n-1}$ null set. Furthermore, we note that

$$\boldsymbol{T}\boldsymbol{n}_{\partial C\setminus\{\boldsymbol{0}\}} = \boldsymbol{0} \quad \text{on} \quad \partial C \setminus \{\boldsymbol{0}\} \tag{3–17}$$

because $\boldsymbol{x}\cdot\boldsymbol{n}_{\partial C\setminus\{\boldsymbol{0}\}} = 0$ since C is a cone with vertex $\boldsymbol{0}$. If $\boldsymbol{v}\in C_0^\infty(\mathbb{R}^n,\mathbb{R}^n)$ is a vector field, then

$$\int_U \nabla\boldsymbol{v}\cdot d\mathsf{T} = \int_C \nabla\boldsymbol{v}\cdot\boldsymbol{T}\,d\mathcal{L}^n = \lim_{r\to 0}\int_{C\setminus B_r}\nabla\boldsymbol{v}\cdot\boldsymbol{T}\,d\mathcal{L}^n.$$

Theorem 1 and formula (A–7) yield

$$\int_{C\setminus B_r}\nabla\boldsymbol{v}\cdot\boldsymbol{T}\,d\mathcal{L}^n = -\int_{C\setminus B_r}\boldsymbol{v}\cdot\operatorname{div}\boldsymbol{T}\,d\mathcal{L}^n + \int_{\partial[(C\setminus B_r)\cap U]}\boldsymbol{v}\cdot\boldsymbol{T}\boldsymbol{n}\,d\mathcal{H}^{n-1}.$$

Combining with Equations (3–14)–(3–16) we obtain

$$\begin{aligned}\int_{C\setminus B_r}\nabla\boldsymbol{v}\cdot\boldsymbol{T}\,d\mathcal{L}^n = -\int_{C\setminus B_r}&\boldsymbol{v}\cdot\operatorname{div}\boldsymbol{T}\,d\mathcal{L}^n\\ &-\int_{C\cap\partial B_r}\boldsymbol{v}(\boldsymbol{x})\cdot\boldsymbol{T}(\boldsymbol{x})\frac{\boldsymbol{x}}{r}\,d\mathcal{H}^{n-1}(\boldsymbol{x})\\ &\qquad+\int_{C\cap\partial U}\boldsymbol{v}\cdot\boldsymbol{T}\boldsymbol{m}\,d\mathcal{H}^{n-1},\end{aligned} \tag{3–18}$$

where we have used

$$\int_{[(\partial C\setminus B_r)\cap U]\cup T}\boldsymbol{v}\cdot\boldsymbol{T}\boldsymbol{n}_{\partial C\setminus\{\boldsymbol{0}\}}\,d\mathcal{H}^{n-1} = 0,$$

which follows from Equation (3–17). A combination of (3–18) with (3–9) provides

$$\int_U \nabla\boldsymbol{v}\cdot d\mathsf{T} = -\lim_{r\to 0}\int_{C\cap\partial B_r}\boldsymbol{v}(\boldsymbol{x})\cdot\boldsymbol{T}(\boldsymbol{x})\frac{\boldsymbol{x}}{r}\,d\mathcal{H}^{n-1}(\boldsymbol{x}) + \int_{C\cap\partial U}\boldsymbol{v}\cdot\boldsymbol{T}\boldsymbol{m}\,d\mathcal{H}^{n-1}. \tag{3–19}$$

Moreover, the continuity of $\boldsymbol{v}$ gives

$$\lim_{r\to 0}\int_{C\cap\partial B_r}\boldsymbol{v}(\boldsymbol{x})\cdot\boldsymbol{T}(\boldsymbol{x})\frac{\boldsymbol{x}}{r}\,d\mathcal{H}^{n-1}(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{0})\cdot\lim_{r\to 0}\int_{C\cap\partial B_r}\boldsymbol{T}(\boldsymbol{x})\frac{\boldsymbol{x}}{r}\,d\mathcal{H}^{n-1}(\boldsymbol{x}), \tag{3–20}$$

provided the limit on the right side exists. On the other hand, we have

$$\int_{C\cap\partial B_r}\boldsymbol{T}(\boldsymbol{x})\frac{\boldsymbol{x}}{r}\,d\mathcal{H}^{n-1}(\boldsymbol{x}) = \int_{C\cap\mathbb{S}^{n-1}}\alpha(\boldsymbol{e})\boldsymbol{e}\,d\mathcal{H}^{n-1}(\boldsymbol{e}), \tag{3–21}$$

for scaling reasons, because C is a cone. Therefore, in view of Equations (3–19), (3–20), (3–21) and (3–12),

$$\int_U \nabla \boldsymbol{v} \cdot d\mathsf{T} = -\boldsymbol{v}(\mathbf{0}) \cdot \boldsymbol{c} + \int_{C \cap \partial U} \boldsymbol{v} \cdot \boldsymbol{T}\boldsymbol{m}\, d\mathcal{H}^{n-1}.$$

A comparison with Equation (3–2) gives (3–10) and (3–11). □

4. Balance equations

If an equilibrated tensor field $\mathsf{T} \in \mathcal{M}(U, \mathrm{Lin})$ is interpreted as the stress field in a continuous body under the action of a body force given by a prescribed measure $\mathsf{b}_0 \in \mathcal{M}(U, \mathbb{R}^n)$ and the boundary traction given by a prescribed measure

$$\mathsf{t}_0 \in \mathcal{M}(\partial U, \mathbb{R}^n),$$

then the equations of equilibrium read

$$\mathbf{div}\ \mathsf{T} + \mathsf{b}_0 = \mathbf{0}, \tag{4–1}$$

$\mathrm{N}(\mathsf{T}) = \mathsf{t}_0$. In particular, if b_0 is absolutely continuous with respect to the Lebesgue measure (e.g., the gravity), that is,

$$\mathsf{b}_0 = \boldsymbol{b}_0 \mathcal{L}^n \llcorner U, \tag{4–2}$$

where $\boldsymbol{b}_0 : U \to \mathbb{R}^n$ is an $\mathcal{L}^n$ integrable function, then $\mathbf{div}\ \mathsf{T}$ must be absolutely continuous with respect to $\mathcal{L}^n$ as well.

We now illustrate these balance equations on various linear combinations of the fields described in Propositions 1 and 2. Let $U \subset \mathbb{R}^n$ be an open set.

(i) Let K^+ and K^- be two regions with Lipschitz boundary contained in $\mathrm{cl}\, U$, which are complementary in the sense that the following relations hold:

$$\mathrm{int}\, K^+ \cap \mathrm{int}\, K^- = \varnothing, \quad K^+ \cup K^- = \mathrm{cl}\, U, \quad \partial K^\pm \cap U = K^+ \cap K^- \cap U.$$

Let furthermore $\boldsymbol{T}^\pm : K^\pm \to \mathrm{Lin}$ be continuous maps which have a continuous and $\mathcal{L}^n$ integrable derivative in $\mathrm{int}\, K^\pm$. Denoting by $\mathcal{I} := \partial K^\pm \cap U$ the common interface in U we see that the exterior normals $\boldsymbol{n}^\pm$ to $K^\pm$ satisfy $\boldsymbol{n}^+ = -\boldsymbol{n}^-$ on $\mathcal{I}$ and we denote this common value by $\boldsymbol{n} : \mathcal{I} \to \mathbb{S}^{n-1}$. Denoting further by $\boldsymbol{T} : \mathrm{int}\, K^+ \cup \mathrm{int}\, K^- \to \mathrm{Lin}$ the field given by $\boldsymbol{T}^\pm$ on $\mathrm{int}\, K^\pm$ and noting that $\boldsymbol{T}$ is defined $\mathcal{L}^n$ a.e. on U, we define the measure T by

$$\mathsf{T} = \boldsymbol{T} \mathcal{L}^n \llcorner U.$$

Applying Proposition 1 to $\boldsymbol{T} \equiv \boldsymbol{T}^\pm$, $\mathcal{G} \equiv K^\pm$, we see that $\boldsymbol{T}^\pm \mathcal{L}^n \llcorner K^\pm$ is an equilibrated tensor field and

$$\mathrm{div}(\boldsymbol{T}^\pm \mathcal{L}^n \llcorner K^\pm) = \mathrm{div}\, \boldsymbol{T}^\pm \mathcal{L}^n \llcorner K^\pm - \boldsymbol{T}^\pm \boldsymbol{n}^\pm \mathcal{H}^{n-1} \llcorner \mathcal{I},$$

$$\mathrm{N}(\boldsymbol{T}^{\pm}\mathcal{L}^n \llcorner K^{\pm}) = \boldsymbol{T}^{\pm}\boldsymbol{m}\,\mathcal{H}^{n-1} \llcorner (\partial K^{\pm} \cap \partial U),$$

where $\boldsymbol{m}$ is the outer normal to U. Adding the results, we conclude that T is an equilibrated tensor field and

$$\mathbf{div}\ \mathsf{T} = \operatorname{div} \boldsymbol{T}\mathcal{L}^n \llcorner U - [\boldsymbol{T}]\boldsymbol{n}\mathcal{H}^{n-1} \llcorner \mathcal{I},$$

$$\mathrm{N}(\mathsf{T}) = \boldsymbol{T}\boldsymbol{m}\mathcal{H}^{n-1} \llcorner \partial U,$$

where $\operatorname{div} \boldsymbol{T}$ is the divergence of $\boldsymbol{T}$ on $U \setminus \mathcal{I}$ and for every $\boldsymbol{x} \in \mathcal{I}$,

$$[\boldsymbol{T}](\boldsymbol{x}) := \boldsymbol{T}^{+}(\boldsymbol{x}) - \boldsymbol{T}^{-}(\boldsymbol{x})$$

is the jump of $\boldsymbol{T}$ across $\mathcal{I}$. Any map $\boldsymbol{T}$ which arises in the above way is called a *piecewise smooth tensor field.* If T is a stress field under the action of the body force as in Equation (4–2), then the equilibrium equation (4–1) is equivalent to the following two equations:

$$\operatorname{div} \boldsymbol{T} + \boldsymbol{b}_0 = \boldsymbol{0} \text{ in } U \setminus \mathcal{I}, \quad [\boldsymbol{T}]\boldsymbol{n} = \boldsymbol{0} \text{ on } \mathcal{I}.$$

(ii) If $\boldsymbol{T}_r$ is a piecewise smooth tensor field with the interface $\mathcal{I}$ as in (i) above and $\boldsymbol{T}_s : \mathcal{I} \to \mathrm{Lin}$ is a superficial tensor field satisfying the hypothesis of Proposition 1 (ii) with $k = n-1$ and $\mathcal{S} := \mathcal{I}$, then the tensor field

$$\mathsf{T} := \boldsymbol{T}_r\mathcal{L}^n \llcorner U + \boldsymbol{T}_s\mathcal{H}^{n-1} \llcorner \mathcal{I}$$

is equilibrated and

$$\mathbf{div}\ \mathsf{T} = \operatorname{div} \boldsymbol{T}_r\mathcal{L}^n \llcorner U + \big(\operatorname{div} \boldsymbol{T}_s - [\boldsymbol{T}_r]\boldsymbol{n}\big)\mathcal{H}^{n-1} \llcorner \mathcal{I},$$

$$\mathrm{N}(\mathsf{T}) = \boldsymbol{T}_r\boldsymbol{m}\mathcal{H}^{n-1} \llcorner \partial U + \boldsymbol{T}_s\,\boldsymbol{p}\mathcal{H}^{n-2} \llcorner (\partial U \cap \partial \mathcal{I}),$$

where $\boldsymbol{p}$ is the outer normal to $\mathcal{I}$. With the body force as in Equation (4–2) the equilibrium equation (4–1) is equivalent to the pair of standard equations

$$\operatorname{div} \boldsymbol{T}_r + \boldsymbol{b}_0 = \boldsymbol{0} \text{ in } U \setminus \mathcal{I}, \quad [\boldsymbol{T}_r]\boldsymbol{n} - \operatorname{div} \boldsymbol{T}_s = \boldsymbol{0} \text{ on } \mathcal{I}; \tag{4–3}$$

see, for example, [Gurtin and Murdoch 1975; Podio-Guidugli and Caffarelli 1990; Gurtin 2000].

(iii) Let $\mathcal{S}_j$, $j = 1, \dots, p$, be curves with endpoints such that $\operatorname{int} \mathcal{S}_j \subset U$. Assume that $\operatorname{int} \mathcal{S}_i \cap \operatorname{int} \mathcal{S}_j = \varnothing$, $\partial\mathcal{S}_i \cap \partial\mathcal{S}_j = \{\boldsymbol{a}\}$ for all $i \neq j$ and some $\boldsymbol{a} \in U$, and that $\partial\mathcal{S}_i \cap \partial U \neq \varnothing$ for all i. Let the measure T be defined by

$$\mathsf{T} := \sum_{j=1}^{p} \boldsymbol{T}_j\mathcal{H}^1 \llcorner \mathcal{S}_j,$$

where $\boldsymbol{T}_j$ is a superficial tensor field on $\mathcal{S}$ which satisfies the hypotheses of Proposition 1. Then T is an equilibrated tensor field and

$$\mathbf{div}\,\mathsf{T} = \sum_{j=1}^{p} \operatorname{div} \boldsymbol{T}_j \mathcal{H}^1 \llcorner \mathcal{S}_j + \Big[\sum_{j=1}^{p} \boldsymbol{T}_j(\boldsymbol{a})\boldsymbol{m}_j(\boldsymbol{a})\Big]\delta_{\boldsymbol{a}},$$

$$\mathrm{N}(\mathsf{T}) = \Big[\sum_{j=1}^{p} \boldsymbol{T}_j(\boldsymbol{a}_j)\boldsymbol{m}_j(\boldsymbol{a}_j)\Big]\delta_{\boldsymbol{a}_j},$$

where the points $\boldsymbol{a}_j$ are defined by $\partial\mathcal{S}_j \cap \partial U = \{\boldsymbol{a}_j\}$ and for any $\boldsymbol{b} \in \partial\mathcal{S}_i$ the symbol $\boldsymbol{m}_i(\boldsymbol{b})$ denotes the outer tangent to $\mathcal{S}_i$ at $\boldsymbol{b}$. If $n \geq 2$, the equilibrium equation (4–1) with $\mathbf{b}_0$ as in (4–2) reads

$$\operatorname{div} \boldsymbol{T}_j = \mathbf{0} \;\text{ on } \mathcal{S}_j \;\text{ for } j = 1, \dots, p, \qquad \sum_{j=1}^{p} \boldsymbol{T}_j(\boldsymbol{a})\boldsymbol{m}_j(\boldsymbol{a}) = \mathbf{0}, \tag{4–4}$$

while if $n = 1$ then the first equation of (4–4) must be replaced by $\operatorname{div} \boldsymbol{T}_j + \boldsymbol{b}_0 = \mathbf{0}$.

(iv) Let $\boldsymbol{T}_r : (U \cap \operatorname{cl} C) \setminus \{\mathbf{0}\} \to \mathrm{Lin}$ be a tensor field of the form described in Proposition 2, let $\mathcal{S}$ be a curve with endpoints such that one endpoint is in ∂U and another coincides with $\mathbf{0}$, and let $\boldsymbol{T}_s : \mathcal{S} \to \mathrm{Lin}$ be a continuous superficial tensor field that is continuously differentiable in $\operatorname{int}\mathcal{S}$. Let T be given by

$$\mathsf{T} = \boldsymbol{T}_r \mathcal{L}^n \llcorner (C \cap U) + \boldsymbol{T}_s \mathcal{H}^1 \llcorner \mathcal{S};$$

then T is an equilibrated tensor field and

$$\mathbf{div}\,\mathsf{T} = \operatorname{div} \boldsymbol{T}_s \mathcal{H}^1 \llcorner \mathcal{S} + [\boldsymbol{c} + \boldsymbol{T}_s(\mathbf{0})\boldsymbol{m}(\mathbf{0})]\delta_{\mathbf{0}},$$

$$\mathrm{N}(\mathsf{T}) = \boldsymbol{T}_s(\boldsymbol{a})\boldsymbol{m}(\boldsymbol{a})\delta_{\boldsymbol{a}} + \boldsymbol{T}_r \boldsymbol{n}\, \mathcal{H}^{n-1} \llcorner (C \cap \partial U),$$

where $\boldsymbol{c}$ is given by Equation (3–12), $\boldsymbol{a}$ is defined by $\partial U \cap \partial\mathcal{S} = \{\boldsymbol{a}\}$, the symbols $\boldsymbol{m}(\mathbf{0}), \boldsymbol{m}(\boldsymbol{a})$ denote the outer tangents to $\mathcal{S}$ at $\mathbf{0}$, $\boldsymbol{a}$, respectively, and $\boldsymbol{n}$ is the outer normal to U. If $n \geq 2$, the equilibrium equation (4–1) with $\mathbf{b}_0$ as in (4–2) can be satisfied only if $\boldsymbol{b}_0 = \mathbf{0}$ (see (3–9)) and if this is the case, it is equivalent to the following pair of equations:

$$\operatorname{div} \boldsymbol{T}_s = \mathbf{0}, \quad \boldsymbol{c} + \boldsymbol{T}_s(\mathbf{0})\boldsymbol{m}(\mathbf{0}) = \mathbf{0}.$$

5. Dimension two

Let $U \subset \mathbb{R}^2$ be an open set and $\mathcal{S} \subset \operatorname{cl} U$ be a smooth curve, let s be the natural (arc) parameter of $\mathcal{S}$ and let $\boldsymbol{t}(s)$, $\boldsymbol{n}(s)$, and $\kappa(s)$ be the unit tangent, the unit normal, and the curvature, respectively. If $\boldsymbol{T}_s : \mathcal{S} \to \mathrm{Lin}$ is a symmetric tensor field on $\mathcal{S}$ then $\boldsymbol{T}_s$ is superficial if and only if we have

$$\boldsymbol{T}_s(s) = \sigma(s)\, \boldsymbol{t}(s) \otimes \boldsymbol{t}(s), \tag{5–1}$$

with σ a scalar field on $\mathcal{S}$. Then

$$\operatorname{div} \boldsymbol{T}_s = \frac{d}{ds}(\sigma \boldsymbol{t}) = \frac{d\sigma}{ds}\boldsymbol{t} + \kappa\sigma\boldsymbol{n}, \tag{5–2}$$

where the first equality follows from Equation (A–9) by noting that the natural parameterization of $\mathcal{S}$ has unit Jacobian, and the second equality follows from Frenet's formula. Alternatively, if we suppose that the curve $\mathcal{S}$ is the graph of a function $y = \omega(x)$, with $x \in [x_0, x_1]$, that is, $\mathcal{S} = \{(x, \omega(x)) \in U : x \in [x_0, x_1]\}$, then

$$\operatorname{div} \boldsymbol{T}_s = J^{-1} \frac{d}{dx}\left(\frac{\sigma}{J}\, \boldsymbol{e}_1 + \frac{\sigma}{J}\, \omega' \boldsymbol{e}_2\right), \tag{5–3}$$

where the prime denotes the differentiation with respect to x,

$$J = \frac{ds}{dx} = \sqrt{1 + (\omega')^2}, \tag{5–4}$$

and $\boldsymbol{e}_1, \boldsymbol{e}_2$ is the standard basis of $\mathbb{R}^2$. This follows from the application of Equation (A–9) to $\boldsymbol{\phi} : (x_0, x_1) \to \mathbb{R}^2$ given by $\boldsymbol{\phi}(x) = (x, \omega(x))$. Note also that the unit tangent $\boldsymbol{t}$ and the unit normal $\boldsymbol{n}$ of $\mathcal{S}$ are given by

$$\boldsymbol{t} = J^{-1}(\boldsymbol{e}_1 + \omega' \boldsymbol{e}_2), \quad \boldsymbol{n} = J^{-1}(-\omega' \boldsymbol{e}_1 + \boldsymbol{e}_2). \tag{5–5}$$

Remark 1. Let $[\boldsymbol{T}_r]\boldsymbol{n}$ be the jump of the normal component of $\boldsymbol{T}_r$ across $\mathcal{S}$. Equations (5–2) and (4–3) yield

$$[\boldsymbol{T}_r]\boldsymbol{n} - \frac{d\sigma}{ds}\boldsymbol{t} - \kappa\sigma\boldsymbol{n} = \boldsymbol{0}. \tag{5–6}$$

If we multiply this relation by $\boldsymbol{t}$ and $\boldsymbol{n}$ and put $\boldsymbol{t} \cdot [\boldsymbol{T}_r]\boldsymbol{n} = -q$ and $\boldsymbol{n} \cdot [\boldsymbol{T}_r]\boldsymbol{n} = -p$, we obtain, respectively,

$$\frac{d\sigma}{ds} + q = 0, \qquad \kappa\sigma + p = 0.$$

These equations coincide with the equilibrium equations of a planar curved beam when the bending moment and the shear force are null, if we interpret q and p as the tangential and normal component of the load, respectively, and $\sigma\boldsymbol{t} = \boldsymbol{T}_s\boldsymbol{t}$ as the axial force [Love 1944].

From (5–1) and the first equation in (5–5) we get

$$\boldsymbol{T}_s = \sigma J^{-2}\big\{\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + 2\omega' \boldsymbol{e}_1 \odot \boldsymbol{e}_2 + (\omega')^2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2\big\}, \tag{5–7}$$

where

$$e_1 \odot e_2 = \tfrac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1).$$

Defining $\delta_{11}, \delta_{12}, \delta_{22}$ by

$$[T_r] = \delta_{11} e_1 \otimes e_1 + 2\delta_{12} e_1 \odot e_2 + \delta_{22} e_2 \otimes e_2, \tag{5–8}$$

we obtain from the second equation in (5–5)

$$[T_r]n = J^{-1}\big\{(\delta_{12} - \omega'\delta_{11})e_1 + (\delta_{22} - \omega'\delta_{12})e_2\big\}. \tag{5–9}$$

With

$$\beta := \frac{\sigma}{J}, \tag{5–10}$$

we deduce from Equations (4–3), (5–9) and (5–3) the system of ordinary differential equations

$$\beta' + \omega'\delta_{11} - \delta_{12} = 0, \tag{5–11}$$

$$(\beta\omega')' + \omega'\delta_{12} - \delta_{22} = 0, \tag{5–12}$$

some applications of which are illustrated in the following sections. We observe that, in view of Equations (5–1), the first equation in (5–5) and (5–10), we have

$$\beta = \sigma(t \cdot e_1) \quad \text{and} \quad \beta\omega' = \sigma(t \cdot e_2), \tag{5–13}$$

which are the horizontal and vertical components of the axial force, respectively.

6. Panels: vertical top loads

In the rest of the paper we apply the considerations of Sections 3–5 to study the statics of rectangular panels in two dimensions made of a no-tension material, with infinite compressive strength [Del Piero 1989]. The panel is free from body forces, clamped at its bottom and subjected to loads prescribed on the boundary. The stress is supposed to be symmetric, plane and negative semidefinite, with singularities along a finite number of curves in the interior $U \subset \mathbb{R}^2$ of the panel; we use equilibrated tensor fields to describe the stress. If $\mathcal{S}$ is the union of these curves, the stress field T is the sum of a measure absolutely continuous with respect to Lebesgue's measure with a smooth density T_r in $U \setminus \mathcal{S}$, and a measure concentrated on $\mathcal{S}$, whose density is a smooth superficial tensor field T_s. The equilibrium requires that T_r has null divergence outside $\mathcal{S}$, and that the surface divergence of T_s be balanced by the jump of the normal component of T_r across $\mathcal{S}$, as required by Equation (4–3). In the examples presented in this paper, the form of the singularity curves and the superficial stress field T_s are obtained by means of this relation, once T_r has been determined.

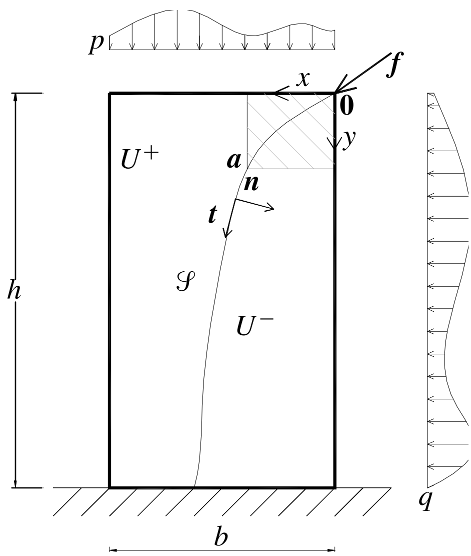

Figure 1. The panel under general load conditions.

We shall deal with either solid rectangular panels or rectangular panels with an opening. In all cases we place the origin of the coordinate system (x, y) in the upper right corner of the panel, with the axis x along the upper side of the panel pointing to left and the y axis along the right side pointing downwards, see Figure 1.

Consider first a solid rectangular panel of width b and height h, clamped at its base $y = h$ and subjected to a vertical load, p, distributed on its top, $y = 0$, a horizontal load, q, distributed along its right side, $x = 0$, and a force,

$$\boldsymbol{f} = f_1\boldsymbol{e}_1 + f_2\boldsymbol{e}_2,$$

concentrated at the upper right corner (Figure 1). Denoting by U the inner part of the panel,

$$U = \{(x, y) \in \mathbb{R}^2 : 0 < x < b,\ 0 < y < h\},$$

we aim to determine a curve $\mathscr{S}$ in U

$$y = \omega(x), \quad \text{with} \quad \omega(0) = 0, \tag{6–1}$$

and a continuously differentiable, negative-semidefinite superficial stress field $\boldsymbol{T}_s$ on $\mathscr{S}$, such that the tensor field T, defined by

$$\mathsf{T} := \boldsymbol{T}_r \mathscr{L}^2 \llcorner U + \boldsymbol{T}_s \mathscr{H}^1 \llcorner \mathscr{S},$$

is balanced and in equilibrium with the external loads, where $\boldsymbol{T}_r$ is given by

$$\boldsymbol{T}_r = \begin{cases} -p(x)\boldsymbol{e}_2 \otimes \boldsymbol{e}_2, & \text{in } U^+, \\ -q(y)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 & \text{in } U^-, \end{cases} \tag{6–2}$$

with $U^- = \{(x, y) \in U : 0 < x < \omega^{-1}(y)\}$ and $U^+ = \{(x, y) \in U : \omega^{-1}(y) < x < b\}$ denoting the two regions into which U is divided by $\mathcal{S}$. Since, according to (6–2), $\boldsymbol{T}_r$ is equilibrated with the distributed loads p and q and satisfies the first equation in (4–3), it is sufficient to determine $\mathcal{S}$ and $\boldsymbol{T}_s$ to satisfy the second equation in (4–3) and the equilibrium boundary condition

$$\boldsymbol{T}_s(\boldsymbol{0})\boldsymbol{t}(\boldsymbol{0}) = -\boldsymbol{f}. \tag{6–3}$$

To this end, note that in this case Equations (5–8) and (6–2) give

$$\delta_{11} = q(\omega(x)), \quad \delta_{22} = -p(x), \quad \delta_{12} = 0$$

and therefore from Equations (5–11) and (5–12) we deduce

$$\beta' + q(\omega(x))\omega' = 0 \tag{6–4}$$

and

$$(\beta\omega')' + p(x) = 0. \tag{6–5}$$

Denoting by P and Q the respective primitives of p and q with $P(0) = 0$ and $Q(0) = 0$, we get

$$\beta(x) = \beta(0) - Q(\omega(x)), \tag{6–6}$$

$$\beta(x)\omega'(x) = \beta(0)\omega'(0) - P(x). \tag{6–7}$$

With the help of (6–6), (6–7) becomes

$$(Q(\omega(x)) - \beta(0))\,\omega'(x) = P(x) - \beta(0)\omega'(0), \tag{6–8}$$

and, in view of Equation (5–13), the equilibrium boundary condition (6–3) becomes

$$\beta(0) = -f_1, \quad \beta(0)\omega'(0) = -f_2. \tag{6–9}$$

Then, (6–8) implies

$$(Q(\omega(x)) + f_1)\,\omega'(x) = P(x) + f_2, \tag{6–10}$$

which can be integrated under the boundary condition in the second equation of (6–1). This is equivalent to the requirement for equilibrium of all rectangular regions, like the one shaded in Figure 1, with respect to the rotation about the point $\boldsymbol{a}$ [Lucchesi and Zani 2002]. Since $\boldsymbol{T}_s$ is negative-semidefinite, we can see that $\sigma(0) \le 0$ in view of Equation (5–1), and from Equations (5–4) and (5–10) we

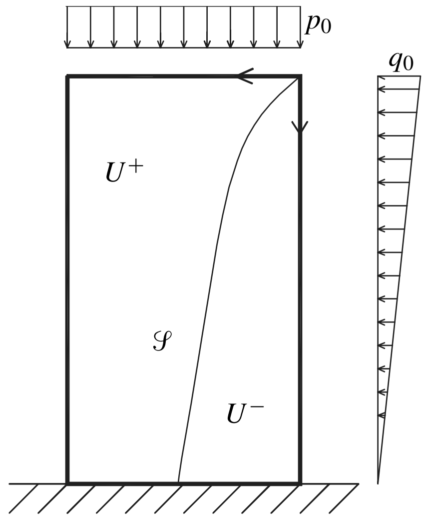

Figure 2. Load-distribution laws on the boundary of the panel; Example 1.

obtain $\beta \le 0$. Moreover, since the curve $\mathscr{S}$ (except for its ends) is wholly contained within U, we have $\omega'(0) \ge 0$. From Equation (6–9), it follows that both f_1 and f_2 must be non-negative, that is to say, the force $\boldsymbol{f}$ must be directed towards the inside of the panel [Di Pasquale 1984].

Example 1. In this example, we suppose that the vertical distributed load is uniform, the horizontal one is linear and the concentrated force is zero (Figure 2), that is,

$$p(x) = p_0, \quad q(y) = q_0\left(1 - \frac{y}{h}\right), \quad \boldsymbol{f} = \boldsymbol{0}.$$

Under such conditions

$$Q(\omega) = q_0\omega\left(1 - \frac{\omega}{2h}\right), \quad P(x) = p_0 x \tag{6–11}$$

and from Equations (6–10) and the second equation in (6–1) we obtain for $\mathscr{S}$ the implicit equation

$$q_0\omega^2\left(1 - \frac{\omega}{3h}\right) = p_0 x^2. \tag{6–12}$$

It can be seen that $\mathscr{S}$ intersects the panel base at

$$x = h\sqrt{\frac{2q_0}{3p_0}}.$$

In order for such a solution to be valid, the intersection point must be within the panel's base, that is to say,

$$b \geq h\sqrt{\frac{2q_0}{3p_0}},$$

and this requires that q_0 does not exceed the value

$$q_m = \frac{3}{2}p_0\left(\frac{b}{h}\right)^2.$$

The attainment of this value would cause the panel to overturn around the corner at coordinates $x = b$, $y = h$. From (6–12) we deduce

$$x = \omega\sqrt{\frac{q_0}{p_0}\left(1 - \frac{\omega}{3h}\right)},$$

and then

$$\omega' = \left(\frac{2h}{2h-\omega}\right)\sqrt{\frac{p_0}{q_0}\left(1 - \frac{\omega}{3h}\right)}$$

by Equations (6–10) and (6–11). The expression for σ can be obtained from Equations (5–10), (6–6), the first equation of (6–11), and (5–4), and it is easy to verify that σ is an increasing function of x. In particular, at the panel bottom we have $\omega = h$ and therefore, taking Equations (5–4) and (6–6) into account, we obtain

$$\omega' = 2\sqrt{\frac{2p_0}{3q_0}}, \qquad J = \sqrt{1 + \frac{8p_0}{3q_0}}, \qquad \beta = -\frac{1}{2}q_0 h,$$

from which we obtain the reaction $\boldsymbol{T}_s\boldsymbol{t} = \sigma\boldsymbol{t}$ at the end of $\mathcal{S}$ of magnitude

$$\frac{1}{2}q_0 h\sqrt{1 + \frac{8p_0}{3q_0}}$$

by Equations (6–4) and (6–9).

Example 2. In this example, we again suppose that the vertical load is uniform, while the horizontal one is zero. Moreover, we assume a concentrated force to be acting (Figure 3),

$$p = p_0, \quad q = 0, \quad \boldsymbol{f} = f_1\boldsymbol{e}_1 + f_2\boldsymbol{e}_2,$$

so that $P(x) = p_0 x$ and $Q(\omega) = 0$. Therefore, we assume $\boldsymbol{T}_r$ as in Equation (6–2) with $p(x) = p_0$ and $q(x) = 0$, and from Equations (6–10) and the second equation in (6–1) we deduce

$$\omega(x) = \frac{p_0 x^2}{2f_1} + \nu x, \tag{6–13}$$

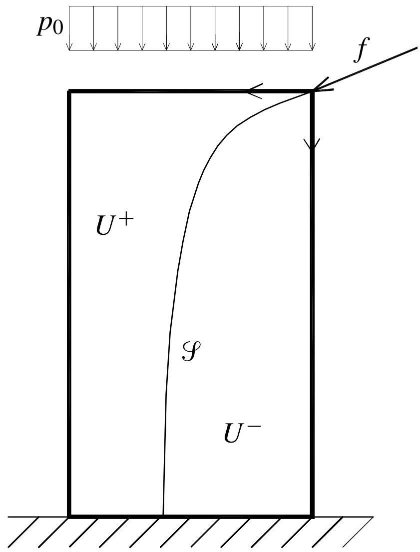

Figure 3. Load-distribution laws on the boundary of the panel; Example 2.

with $\nu = f_2/f_1$. Thus, in view of Equations (5–4), (6–6), the first equation in (6–9) and (5–10) we obtain

$$J = \sqrt{1 + \left(\frac{p_0 x}{f_1} + \nu\right)^2}, \quad \beta = -f_1, \quad \sigma = -f_1\sqrt{1 + \left(\frac{p_0 x}{f_1} + \nu\right)^2}, \qquad (6–14)$$

from which we can determine $\boldsymbol{T}_s$ with the help of Equation (5–7).

From (6–13), for $x = b$, $\omega = h$ and $\nu < h/b$, we get the maximum magnitude of force $\boldsymbol{f}$ compatible with the equilibrium,

$$|\boldsymbol{f}_m| = \frac{p_0 b^2}{2(h - \nu b)}\sqrt{1 + \nu^2}.$$

Moreover, in view of Equations (6–13) and the third equation in (6–14) the intensity of the concentrated reaction at the panel's base is

$$f_1\sqrt{1 + \frac{2p_0 h}{f_1} + \nu^2}.$$

This result can be generalized to the situation where, besides the vertical load p_0, there are two forces, $\boldsymbol{f}$ and $\boldsymbol{g}$, applied to the panel's corners. For simplicity, we limit ourselves to the case in which $\boldsymbol{f}$ and $\boldsymbol{g}$ are horizontal, as shown in Figure 4. Let us suppose $f \leq g$. Proceeding as in the previous case it can be verified that,

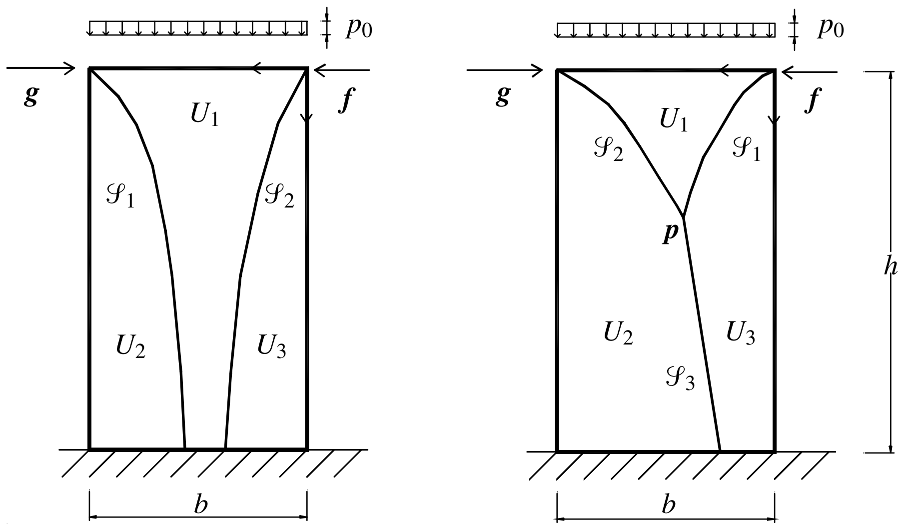

Figure 4. Stress states.

for

$$\sqrt{f} + \sqrt{g} \leq \frac{1}{2} b \sqrt{\frac{2p_0}{h}},$$

the panel is subdivided into three regions, U_1, U_2 and U_3, by parabolas $\mathcal{S}_1$ and $\mathcal{S}_2$ with equations

$$\omega_1(x) = \frac{p_0 x^2}{2f} \quad \text{with} \quad \sigma_1 = -f \sqrt{1 + \left(\frac{p_0 x}{f}\right)^2} \tag{6–15}$$

and

$$\omega_2(x) = \frac{p_0 (b-x)^2}{2g} \quad \text{with} \quad \sigma_2 = -g \sqrt{1 + \left(\frac{p_0 (b-x)}{g}\right)^2}, \tag{6–16}$$

and, moreover,

$$\boldsymbol{T}_r = \begin{cases} -p_0 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 & \text{in } U_1 \\ & \text{in } U_2 \cup U_3. \end{cases} \tag{6–17}$$

For

$$\sqrt{f} + \sqrt{g} > \frac{1}{2} b \sqrt{\frac{2p_0}{h}}$$

and

$$g - f \leq \frac{p_0 b^2}{2h},$$

the curves $\mathcal{S}_1$ and $\mathcal{S}_2$ intersect at a point $\boldsymbol{p}$ in the interior of U. In this case, an equilibrated tensor field is determined by supposing that the the panel is further subdivided by the curve $\mathcal{S}_3$ (Figure 4 (b)). For $\boldsymbol{T}_r$ defined by Equation (6–17), we have $[\boldsymbol{T}_r] = \mathbf{0}$ across $\mathcal{S}_3$ and, in view of (5–6), this implies the vanishing of both the curvature κ and $d\sigma/ds$. Thus, $\mathcal{S}_3$ is a straight line whose equation can be determined with the help of (6–15) and (6–16), by observing that at the point $\boldsymbol{p}$ we have

$$\sigma_1 \boldsymbol{t}_1 + \sigma_2 \boldsymbol{t}_2 - \sigma_3 \boldsymbol{t}_3 = \mathbf{0},$$

in view of the second equation in Equation (4–4), which expresses the "equilibrium of the node $\boldsymbol{p}$". Finally, we obtain

$$\omega_3(x) = \frac{p_0 b(b-2x)}{2(g-f)} \quad \text{with} \quad \sigma_3 = -(g-f)\sqrt{1+\left(\frac{p_0 b}{g-f}\right)^2}.$$

For

$$g - f = \frac{p_0 b^2}{2h},$$

the panel is free to rotate around the point with coordinates $(0, h)$.

Example 3. Consider the case in which the panel is subjected only to the sole action of the uniform distributed vertical load $p = p_0$. Using the results of the previous example, we wish to verify that, beyond the regular stress state, $\boldsymbol{T} = -p_0 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2$, defined throughout U, it is possible to determine infinitely many equilibrated and compatible stress fields, each of which characterized by

(i) a superficial stress $\boldsymbol{T}_s$ defined on a curve $\mathcal{S}$ with equation $y = \omega(x)$ that is symmetric with respect to the axis $x = b/2$ (Figure 5), which intersects this axis for $y = \lambda$, $(0 \leq \lambda < h)$ and that also intersects the panel bottom for $|x - b/2| = \mu$, $(0 < \mu \leq b/2)$,

$$\omega(b/2) = \lambda, \quad \omega(b/2 \pm \mu) = h, \quad \omega'(b/2) = 0; \tag{6–18}$$

(ii) a stress field

$$\boldsymbol{T}_r = \begin{cases} -p_0 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2, & \text{in } U^+, \\ & \text{in } U^-, \end{cases} \tag{6–19}$$

where $U^- = \{(x, y) \in U : |x - b/2| < \mu,\ \omega(x) < y < h\}$ is the region below $\mathcal{S}$ and U^+ is the interior of its complement in U.

In fact, using Equations (6–4) and (6–5) with $p = p_0$, $q = 0$, and combining with (6–18), we see that $\mathcal{S}$ is a parabola given by

$$\omega(x) = \lambda + \frac{h-\lambda}{\mu^2}(x - b/2)^2 \tag{6–20}$$

and that β is given by

$$\beta = -\frac{p_0 \mu^2}{2(h-\lambda)},$$

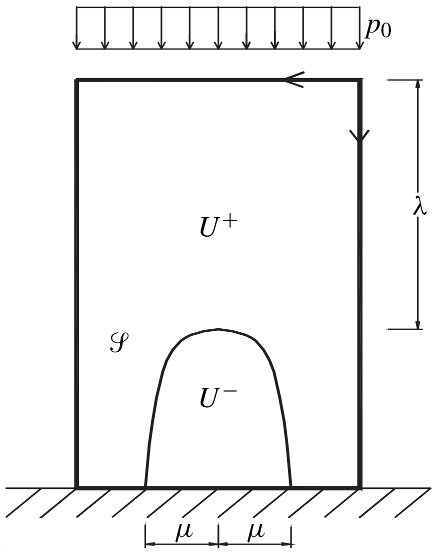

Figure 5. Stress field; Example 3.

from which we can calculate J, σ and $\boldsymbol{T}_s$ by Equations (5–4), (5–10) and (5–7). It can be seen that the interaction between the two parts of the panel, separated by the symmetry axis, $x = b/2$, consists solely of a horizontal force concentrated at the apex of $\mathcal{S}$, whose intensity $-\beta$ is an increasing function of λ which becomes unbounded when λ tends to h.

Example 4 (panels with openings). Let us consider a rectangular panel with base $b = b_1 + 2b_2$ and height $h = h_1 + h_2$, with a symmetric opening with dimensions b_1 and h_1 (Figure 6), clamped at its base and subjected to a vertical load p_0, uniformly distributed on its top. Clearly, the stress field from the preceding example is appropriate here also provided that the parabola in Equation (6–20) is entirely contained inside the panel. It is easy to see that the most favorable situation is attained when the apex of the parabola belongs to the top of the panel (Figure 6). Then $\boldsymbol{T}_r$ is as in (6–19), where regions U^- and U^+ are divided by parabola $\mathcal{S}$

$$\omega(x) = \frac{p_0(b/2 - x)^2}{2g},$$

with

$$\sigma = -g\sqrt{1 + \left(\frac{p_0(b/2 - x)}{g}\right)^2},$$

where

$$g = -\beta$$

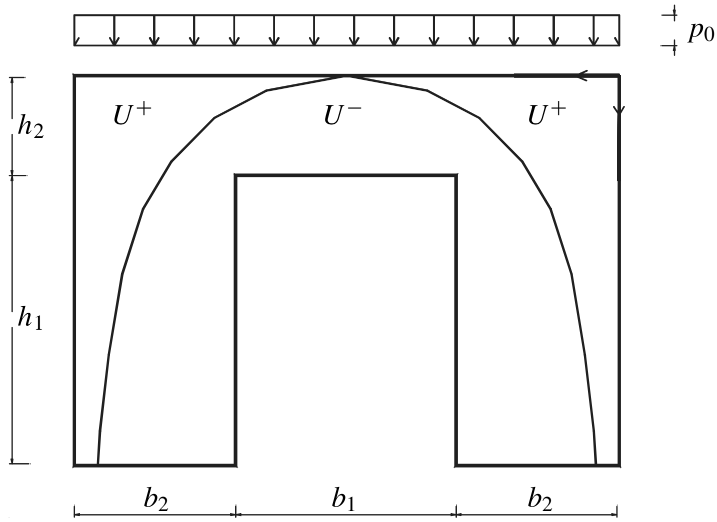

Figure 6. Panel with a symmetric opening.

is the interaction between the two parts of the panel across the symmetry axis. For $g = p_0 b_1^2/(8h_2)$ the parabola $\mathcal{S}$ contains the points (b_2, h_2) and $(b_1 + b_2, h_2)$, whereas for $g = p_0 b^2/(8h)$, $\mathcal{S}$ meets the corners of the panel with coordinates $(0, h)$ and (b, h); hence the equilibrium is possible only for $p_0 b^2/(8h) \geq p_0 b_1^2/(8h_2)$, that is, for

$$\zeta \leq 4\xi(\xi + 1), \tag{6–21}$$

with $\xi = b_2/b_1$ and $\zeta = h_1/h_2$. We observe that when the equality holds in Equation (6–21), $\mathcal{S}$ meets the four points $(0, h)$, (b_2, h_2), $(b_1 + b_2, h_2)$, and (b, h) and thus, apart from the value of p_0, the panel can be considered to be a kinematically indeterminate structure, made of four hinged bodies.

Assuming that $\zeta < 4\xi(\xi + 1)$, we now want to determine an equilibrated stress field when the panel is subjected to a horizontal force $\boldsymbol{f}$ applied to the upper right corner $\mathbf{0}$ in addition to the vertical load p_0. First, let us consider the case in which parabola $\mathcal{S}_1$ with equation $\omega(x) = p_0 x^2/(2f)$ is contained inside the panel as shown in Figure 7 (see also Example 2). For this, f has to satisfy the inequality

$$\frac{p_0(b_1 + b_2)^2}{2h_2} \leq f \leq \frac{p_0 b^2}{2h}, \tag{6–22}$$

because for

$$f = p_0(b_1 + b_2)^2/(2h_2),$$

ω meets the point $\boldsymbol{b}$, whereas for

$$f = p_0 b^2/(2h),$$

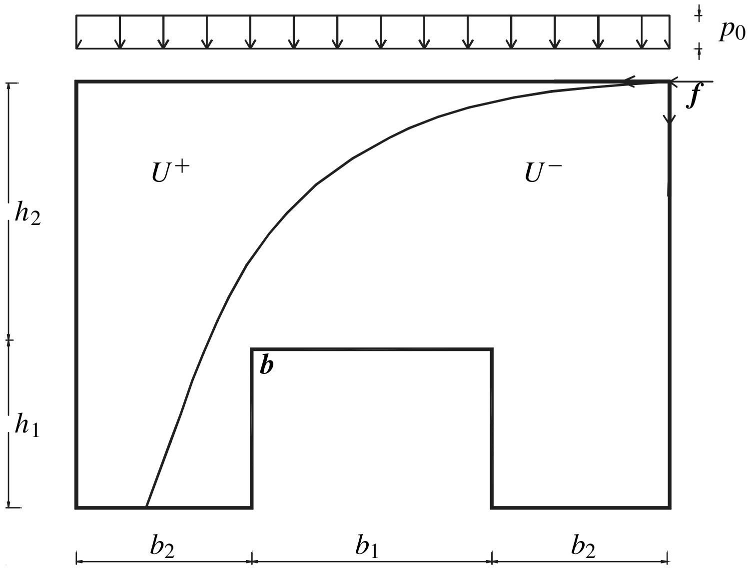

Figure 7. Stress field with a horizontal force $\boldsymbol{f}$, for $\zeta \leq \frac{\xi(3\xi+2)}{(\xi+1)^2}$.

ω meets the point $\boldsymbol{c}$. Inequalities (6–22) are verified only if

$$p_0 b^2/(2h) \geq p_0(b_1+b_2)^2/(2h_2),$$

that is, if

$$\zeta \leq \frac{\xi(3\xi+2)}{(\xi+1)^2}. \tag{6–23}$$

When Equations (6–23) and (6–22) are satisfied, we can assume that $\boldsymbol{T}_r$ and $\boldsymbol{T}_s$ are given in the same way as in the first part of Example 2 and then we see that the maximum value of f compatible with the equilibrium is

$$f_m = \frac{p_0 b^2}{2h}. \tag{6–24}$$

Let us now consider the case

$$\frac{\xi(3\xi+2)}{(\xi+1)^2} < \zeta < 4\xi(\xi+1). \tag{6–25}$$

In view of the previous discussion (see the second part of Example 2 and Figure 4 (b), the equilibrated tensor field can be obtained with the following three singularity curves:

(i) an arc of parabola $\mathscr{S}_1$, with equation

$$\omega_1(x) = \frac{p_0(x-a)^2}{2g}$$

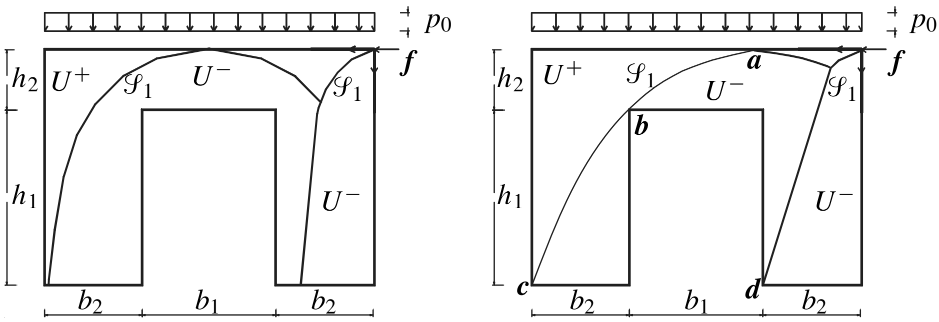

Figure 8. Stress field in the panel. (a) Apex of S_1 on the symmetry axis. (b) The collapse state.

and apex in $\boldsymbol{a} \equiv (a, 0)$ with $a \in (b/2, 0)$;

(ii) an arc of parabola $\mathcal{S}_2$ with the equation

$$\omega_2(x) = \frac{p_0 x^2}{2f},$$

and apex in $(0, 0)$;

(iii) a line $\mathcal{S}_3$, starting from the intersection point of $\mathcal{S}_1$ and $\mathcal{S}_2$ (Figure 8).

These curves subdivide the panel into three regions, where $\boldsymbol{T}_r$ is defined as in (6–17).

With the aim to obtain the maximum admissible intensity of f, we first determine the values of g and a so that the parabola $\mathcal{S}_1$ meet the points $\boldsymbol{b} \equiv (b_1 + b_2, h_2)$ and $\boldsymbol{c} \equiv (b, h)$ (Figure 8 (b)); this is always possible in view of Equation (6–25). In this way we obtain

$$g = \frac{p_0 b_1^2 \xi^2 \big(2 + \zeta + 2\sqrt{1+\zeta}\,\big)}{2\zeta^2 h_2}, \qquad a = b_1 \left(1 + \frac{\xi}{\zeta}\big(\zeta - 1 - \sqrt{1+\zeta}\,\big)\right). \quad (6\text{–}26)$$

Once g and a are determined, we impose the requirement that the segment $\mathcal{S}_3$ meets the point $\boldsymbol{d} \equiv (b_2, h)$ and then we get

$$f_m = \frac{p_0 b_1^2}{2h_2} \cdot \frac{2\xi\,(\xi+1)\sqrt{\zeta+1} + 2\xi^2\,(\zeta+1) + 2\xi - \zeta}{\zeta(\zeta+1)}. \quad (6\text{–}27)$$

When this value of f is reached, the panel behaves as a kinematically indeterminate structure made of three bodies, hinged at points $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

We observe that a, as given by the second equation in (6–26), is positive in view of the first inequality of (6–25), and vanishes for

$$\zeta = \frac{\xi(3\xi+2)}{(\xi+1)^2}.$$

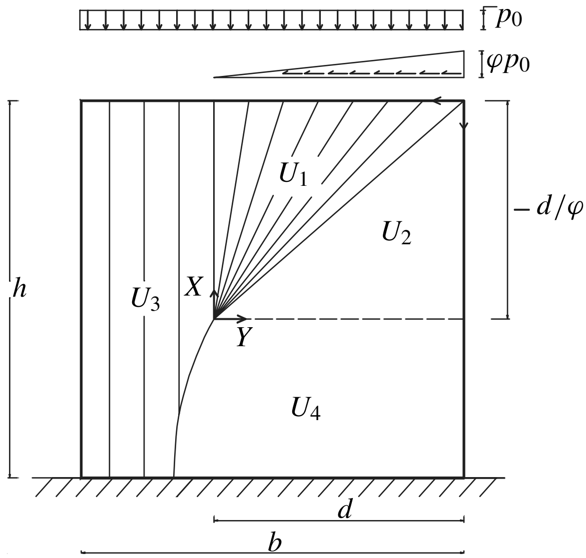

Figure 9. Load-distribution laws on the top of the rectangular panel and corresponding stress field.

In this particular circumstance, the value of f_m given by (6–27) coincides with the value of f_m given by (6–24) and with the value of g given by the first equation in (6–26).

7. Panels: oblique top loads

Let us consider a rectangular panel of width b and height h, clamped at its bottom and subjected to horizontal and vertical loads distributed on the top of the panel. Assume that the vertical load p_0 is uniform, whereas the horizontal load q has a linear distribution (Figure 9),

$$p(x) = p_0, \quad 0 \le x \le b;$$

$$q(x) = \begin{cases} \frac{\varphi p_0 (d-x)}{d}, & 0 \le x \le d, \\ 0, & d \le x \le b. \end{cases}$$

Let U be the interior of the panel. As proved in [Lucchesi and Zani 2003a], the stress field $\boldsymbol{T}_0$ in the region $0 \le x \le d$ and $0 \le y \le d/\varphi$ is given by

$$\boldsymbol{T}_0 = \begin{cases} \frac{p_0 \varphi^2 d (x-d)^2}{(\varphi y - d)^3} \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \frac{2 p_0 \varphi d (x-d)}{(\varphi y - d)^2} \boldsymbol{e}_1 \odot \boldsymbol{e}_2 + \frac{p_0 d}{\varphi y - d} \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 & \text{in } U_1 \\ & \text{in } U_2, \end{cases}$$

where U_1, U_2 are cones, both with vertex $\boldsymbol{p} = (d, d\varphi)$, given by

$$U_1 = \{(x, y) \in U : 0 \le x \le d,\ 0 \le \varphi y \le x\},$$

$$U_2 = \{(x, y) \in U : 0 \le x \le d,\ x \le \varphi y \le d\}.$$

The stress field $\boldsymbol{T}_0$ is discontinuous along the line $x = \varphi y$ but it is an easy matter to verify that the jump of the normal component of $\boldsymbol{T}_0$ across the discontinuity line is zero. Therefore $x = \varphi y$ differs from the singularity curves considered in the previous examples in that its corresponding superficial stress field vanishes.

In the coordinate system with origin in $\boldsymbol{p}$, $X = d - x$, $Y = d/\varphi - y$, shown in Figure 9, with corresponding unit normal base $\hat{\boldsymbol{e}}_1 = -\boldsymbol{e}_1$ and $\hat{\boldsymbol{e}}_2 = -\boldsymbol{e}_2$, we have

$$\boldsymbol{T}_0(\boldsymbol{x}) = \begin{cases} -\frac{p_0 d}{\varphi}(\boldsymbol{x}\cdot\hat{\boldsymbol{e}}_2)^{-3}\,\boldsymbol{x}\otimes\boldsymbol{x} & \text{in } U_1, \\ \boldsymbol{0} & \text{in } U_2, \end{cases}$$

where we put $\boldsymbol{x} = (X, Y)$. Recalling Proposition 2, we note that in U_1 we have

$$\boldsymbol{T}_0(\boldsymbol{x}) = |\boldsymbol{x}|^{-3}\,\alpha\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\boldsymbol{x}\otimes\boldsymbol{x},$$

with the function α given by

$$\alpha(\boldsymbol{e}) = -\frac{p_0 d}{\varphi}(\boldsymbol{e}\cdot\hat{\boldsymbol{e}}_2)^{-3}, \tag{7–1}$$

$\boldsymbol{e} \in \mathbb{S}^1$. Therefore, writing $\boldsymbol{e} = \hat{\boldsymbol{e}}_1\cos\theta + \hat{\boldsymbol{e}}_2\sin\theta$ and $\psi = \tan^{-1}(1/\varphi)$, from Equation (7–1) we obtain the following value of the vector constant $\boldsymbol{c}$ as in (3–12):

$$\begin{aligned}\boldsymbol{c} &= -\frac{p_0 d}{\varphi}\int_{\psi}^{\pi/2}(\boldsymbol{e}\cdot\hat{\boldsymbol{e}}_2)^{-3}\,\boldsymbol{e}\;d\mathcal{H}^1(\boldsymbol{e}),\\ &= -\frac{p_0 d}{\varphi}\int_{\psi}^{\pi/2}(\sin\theta)^{-3}(\hat{\boldsymbol{e}}_1\cos\theta + \hat{\boldsymbol{e}}_2\sin\theta)\,d\theta,\\ &= -p_0 d(\varphi\hat{\boldsymbol{e}}_1/2 + \hat{\boldsymbol{e}}_2),\\ &= p_0 d(\varphi\boldsymbol{e}_1/2 + \boldsymbol{e}_2),\end{aligned} \tag{7–2}$$

which equals the resultant of the load applied to the top of the panel on the interval $[0, d]$; see Equation (3–13).

To determine the stress field in the remaining parts of the panel, we assume the existence of a singularity curve $\mathcal{S}$ with equation $y = \omega(x)$, starting from the point

$\boldsymbol{p}$, and we proceed as in Example 2, assuming

$$\boldsymbol{T}_r = \begin{cases} \boldsymbol{T}_0 & \text{in } U_1 \cup U_2, \\ -p_0 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 & \text{in } U_3, \\ \boldsymbol{0} & \text{in } U_4. \end{cases}$$

The form of the curve $y = \omega(x)$ is obtained similarly to that in Equation (6–13): one identifies $\boldsymbol{f} = \boldsymbol{c}$ and hence $f_1 = p_0 d\varphi/2$ and $\nu = 2/\varphi$ [see (7–2)] to obtain

$$\omega(x) = \frac{(x-d)^2}{\varphi d} + \frac{2(x-d)}{\varphi} + \frac{d}{\varphi} = \frac{x^2}{\varphi d}.$$

The maximun value of φ compatible with equilibrium is $b^2/(hd)$ and is attained when $\mathcal{S}$ meets the corner (b, h) of the panel.

8. Conclusions

The divergence measure tensor fields presented above constitute a new class of singular stress fields equilibrated with the loads and compatible with the incapability of the material to withstand traction. They appear to be a powerful tool that can be helpful in many applications to assess the safe magnitude of the applied load.

Although the examples presented in this work deal only with the plane problem in the absence of the body forces, the general part of this paper is meant to be applicable to more general situations [Lucchesi et al. 2004; 2005b].

Appendix A. Appendix: surfaces and surface divergence theorem

This appendix gathers the differential geometric notions used above. These are mainly the k-dimensional surfaces in $\mathbb{R}^n$, which we denote generically by $\mathcal{U}$, the k-dimensional surfaces with boundary, which we denote by $\mathcal{S}$, the surface divergence of vector fields and tensor fields on $\mathcal{U}$ or $\mathcal{S}$, and the surface divergence theorem 1. Most of the notions discussed below can be used in their intuitive sense but we prefer to give explicit definitions to avoid misunderstandings.

We start with the notion of surface. We give a definition that is convenient for the proof of Proposition 1. We refer to [Federer 1969, Subsection 3.1.19] for the discussion of surfaces without boundary of arbitrary class C^s and to [Lee 2003] for manifolds with boundary of class C^∞.

A.1. *Surfaces.* If $\mathcal{U}$ is a subset of $\mathbb{R}^n$, k an integer with $0 \le k \le n$, and s a positive integer, then the following conditions are equivalent [Federer 1969, Subsection 3.1.19]:

(i) for each $\boldsymbol{x} \in \mathcal{U}$ there exists a neighborhood Z of $\boldsymbol{x}$ in $\mathbb{R}^n$ and a class C^s injective map $\boldsymbol{\phi}$ from an open set in $\mathbb{R}^k$ into $\mathbb{R}^n$ with continuous inverse such that

$$\mathcal{U} \cap Z = \operatorname{ran} \boldsymbol{\phi}, \quad \operatorname{rank} \nabla \boldsymbol{\phi}(\boldsymbol{p}) = k \quad \text{whenever } \boldsymbol{p} \in \operatorname{dom} \boldsymbol{\phi};$$

(ii) for each $\boldsymbol{x} \in \mathcal{U}$ there exists a neighborhood Z of $\boldsymbol{x}$ in $\mathbb{R}^n$ and a class C^s map $\boldsymbol{\omega} : Z \to \mathbb{R}^{n-k}$ such that

$$\mathcal{U} \cap Z = \boldsymbol{\omega}^{-1}(\mathbf{0}), \quad \operatorname{rank} \nabla \boldsymbol{\omega}(\boldsymbol{y}) = n - k \quad \text{whenever } \boldsymbol{y} \in Z.$$

If these equivalent conditions are satisfied, we say that $\mathcal{U}$ is a *k-dimensional surface* of class C^s. It is not assumed that surfaces are connected. We shall always assume that $s \geq 2$, and omit the qualification "of class C^2" in our terminology. It is easy to see that $\mathcal{U}$ is an n-dimensional surface in $\mathbb{R}^n$ if and only if $\mathcal{U}$ is an open subset of $\mathbb{R}^n$ and $\mathcal{U}$ is a 0-dimensional surface if and only if $\mathcal{U}$ is a set of isolated points. We call one-dimensional surfaces *curves.* We shall encounter zero-dimensional surfaces as boundaries of curves.

If $\boldsymbol{x} \in \mathcal{U}$ we denote by $\mathbf{T}_{\boldsymbol{x}}(\mathcal{U}) \subset \mathbb{R}^n$ the *tangent space* to $\mathcal{U}$ at $\boldsymbol{x}$, defined by

$$\mathbf{T}_{\boldsymbol{x}}(\mathcal{U}) = \operatorname{ran} \nabla \boldsymbol{\phi}(\boldsymbol{\phi}^{-1}(\boldsymbol{x})) = \ker \nabla \boldsymbol{\omega}(\boldsymbol{x}), \tag{A–1}$$

where $\boldsymbol{\phi}$ and $\boldsymbol{\omega}$ are as in (i) and (ii) above; we note that the two expressions in (A–1) are independent of the choices of these two objects. Clearly, $\mathbf{T}_{\boldsymbol{x}}(\mathcal{U})$ is a k-dimensional subspace of $\mathbb{R}^n$.

Next we discuss surfaces with boundary. These occur in the surface divergence theorem, below, and are defined as closed parts $\mathcal{S}$ of surfaces $\mathcal{U}$ without boundary such that the boundary $\partial\mathcal{S}$ is regular enough to have a well defined tangent space for $\mathcal{H}^{k-1}$ a.e. point.

A.2. *Surfaces with boundary.* We say that a subset $\mathcal{S}$ of $\mathbb{R}^n$ is a *k-dimensional surface with boundary* if the following three conditions are satisfied:

(i) $\mathcal{S}$ is closed;

(ii) there exists a k-dimensional surface $\mathcal{U}$ such that $\mathcal{S} \subset \mathcal{U}$;

(iii) for every $\boldsymbol{x}$ in the relative boundary $\partial\mathcal{S}$ of $\mathcal{S}$ in $\mathcal{U}$ there exist a set Z and a map $\boldsymbol{\phi}$ as in Item 1 of Section A.1, an $\epsilon > 0$, and a Lipschitz function $f : \operatorname{dom} f \to \mathbb{R}$ on an open subset of $\mathbb{R}^{k-1}$ such that

$$\boldsymbol{\phi}(D_-) \subset Z \cap (\mathcal{S} \setminus \partial\mathcal{S}), \quad \boldsymbol{\phi}(D_+) \subset Z \cap (\mathcal{U} \setminus \mathcal{S}),$$

where $D_\pm$ are the "$\pm\epsilon$ layers along the graph of f," given by

$$D_\pm = \{(\boldsymbol{y}, a) \in \mathbb{R}^k : \boldsymbol{y} \in \operatorname{dom} f, a = f(\boldsymbol{y}) \pm t, \text{ where } 0 < t < \epsilon\}.$$

We set $\operatorname{int}\mathcal{S} := \mathcal{S} \setminus \partial\mathcal{S}$ and note that $\operatorname{int}\mathcal{S}$ is a surface (without boundary) as defined in Section A.1. If $\boldsymbol{x} \in \mathcal{S}$, we define the *tangent space* $\mathbf{T}_{\boldsymbol{x}}(\mathcal{S})$ to $\mathcal{S}$ at $\boldsymbol{x}$

by $\mathbf{T}_{\boldsymbol{x}}(\mathcal{S}) = \mathbf{T}_{\boldsymbol{x}}(\mathcal{U})$; this definition is independent of the choice of $\mathcal{U}$. Note that if $\boldsymbol{x}$, Z, $\boldsymbol{\phi}$, and f are as in Item (iii) above then

$$Z \cap \partial\mathcal{S} = \boldsymbol{\phi}(\operatorname{graph} f),$$

that is, $\boldsymbol{\phi}$ carries the graph of f into $\partial\mathcal{S}$; in particular $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{y}, f(\boldsymbol{y}))$ for some $\boldsymbol{y} \in \operatorname{dom} f$. Motivated by this, we define the *tangent space* $\mathbf{T}_{\boldsymbol{x}}(\partial\mathcal{S})$ of $\partial\mathcal{S}$ at $\boldsymbol{x}$ for $\mathcal{H}^{k-1}$ a.e. $\boldsymbol{x} \in \partial\mathcal{S}$ as the image of the tangent space $\mathcal{T}$ at $(\boldsymbol{y}, f(\boldsymbol{y}))$ to graph f under $\nabla\boldsymbol{\phi}$ whenever $\mathcal{T}$ exists. By definition, $\mathcal{T}$ exists if and only if f is differentiable at $\boldsymbol{y}$; we then set

$$\mathcal{T} := \operatorname{span}\big\{\nabla_i \boldsymbol{h}(\boldsymbol{y}) : i = 1, \dots, k-1\big\},$$

where $\boldsymbol{h} : \operatorname{dom} f \to \mathbb{R}^k$ is defined by $\boldsymbol{h}(z) = (z, f(z))$, $z \in \operatorname{dom} f$, and

$$\nabla_i, i \le i \le k-1,$$

denotes the partial differentiation in $\mathbb{R}^{k-1}$. We then set

$$\mathbf{T}_{\boldsymbol{x}}(\partial\mathcal{S}) := \operatorname{span}\{\nabla\boldsymbol{\phi}(\boldsymbol{y}, f(\boldsymbol{y}))\nabla_i \boldsymbol{h}(\boldsymbol{y}) : i = 1, \dots, k-1\}$$

where $\boldsymbol{y}$ is defined by $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{y}, f(\boldsymbol{y}))$. Since f is differentiable at $\mathcal{L}^{k-1}$ a.e. point of dom f by Rademacher's theorem, $\mathbf{T}_{\boldsymbol{x}}(\partial\mathcal{S})$ is defined for $\mathcal{H}^{k-1}$ a.e. $\boldsymbol{x} \in \partial\mathcal{S}$.

The tangent space to $\partial\mathcal{S}$ is now used to define an exterior normal to $\partial\mathcal{S}$ as follows. If $\mathcal{S}$ is a k-dimensional surface with boundary then there exists a function $\boldsymbol{m}$, defined on $\mathcal{H}^{k-1}$ almost all of $\partial\mathcal{S}$ and with values in $\mathbb{S}^{n-1}$, which we write $\boldsymbol{m} : \partial\mathcal{S} \to \mathbb{S}^{n-1}$, such that we have the following conditions satisfied for $\mathcal{H}^{k-1}$ a.e. $\boldsymbol{x} \in \partial\mathcal{S}$:

(i) $\boldsymbol{m}(\boldsymbol{x}) \in \mathbf{T}_{\boldsymbol{x}}(\mathcal{S})$;

(ii) $\boldsymbol{m}(\boldsymbol{x})$ is perpendicular to $\mathbf{T}_{\boldsymbol{x}}(\partial\mathcal{S})$;

(iii) $\boldsymbol{m}$ points out of $\mathcal{S}$ in the sense that there exists a continuously differentiable map $\mathcal{G} : (-1, 1) \to \mathbb{R}^n$ with $\mathcal{G}((-1, 0]) \subset \mathcal{S}$, $\mathcal{G}(0) = \boldsymbol{x}$ and $d\mathcal{G}/dt(0) = \boldsymbol{m}(\boldsymbol{x})$.

Any two functions satisfying (i)–(iii) differ at most on a set of $\mathcal{H}^{k-1}$ measure 0; we call any such an $\boldsymbol{m}$ the *exterior normal* of $\mathcal{S}$, and refer to [Lee 2003, Proposition 13.26] for the proof in the case of C^∞ manifolds with C^∞ boundary. If $\mathcal{S}$ is a curve with endpoints and hence $\partial\mathcal{S}$ is a collection of the initial and final endpoints (see below for the definition), then the outer normal coincides with the outer tangents to $\mathcal{S}$ at the endpoints.

In the special case $k = n$, we call the n-dimensional surfaces with boundary in $\mathbb{R}^n$ *regions with Lipschitz boundary;* recall that, contrary to the common terminology, we assume that $\mathcal{S}$ is closed; since in this case $\partial\mathcal{S}$ coincides with the topological boundary of $\mathcal{S}$, it follows that $\operatorname{int}\mathcal{S}$ is a region with Lipschitz boundary in the standard sense [Nečas 1967], which we call *open regions with Lipschitz boundary.*

For the special case $k = 1$ we call surfaces of dimension 1 with boundary *curves with endpoints.*

Referring to [Lee 2003, Chapter 13] for the standard notion of an orientation of a vector space, we say that a k-dimensional surface in $\mathbb{R}^n$ with boundary is *orientable* if there exists a continuous map on $\mathcal{S}$ giving the orientation of each tangent space of $\mathcal{S}$. Each such a map is called an *orientation* of the surface. An *oriented k-dimensional surface with boundary* is a surface $\mathcal{S}$ with boundary together with an orientation of $\mathcal{S}$.

A.3. ***Fields on surfaces, surface gradient and surface divergence.*** With the aim to define the surface divergence of a vector or tensor field defined on a surface, we first introduce a surface derivative via 'theoretical' formulas (A–2) and (A–3), below, and give 'practical' formulas (A–4) and (A–5).

If $\mathcal{S}$ is a k-dimensional surface with boundary, $S \subset \mathcal{S}$, $f : S \to V$ where V is a finite-dimensional inner product space, and $\boldsymbol{x} \in S \setminus \partial\mathcal{S}$, we say that f is *differentiable* at $\boldsymbol{x}$ if $N \cap \mathcal{S} \subset S$ for some neighborhood N of $\boldsymbol{x}$ in $\mathbb{R}^n$ and if there exists a linear transformation $\nabla f(\boldsymbol{x})$ from $\mathbb{R}^n$ to V, called the *surface derivative* of f at $\boldsymbol{x}$, such that

$$\nabla f(\boldsymbol{x})\boldsymbol{P} = \nabla f(\boldsymbol{x}) \tag{A–2}$$

and

$$\lim_{\substack{\boldsymbol{y}\to\boldsymbol{x} \\ \boldsymbol{y}\in S,\, \boldsymbol{y}\neq\boldsymbol{x}}} |f(\boldsymbol{y}) - f(\boldsymbol{x}) - \nabla f(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x})|/|\boldsymbol{y}-\boldsymbol{x}| = 0, \tag{A–3}$$

where $\boldsymbol{P}$ is the orthogonal projection from $\mathbb{R}^n$ onto $\mathbf{T}_{\boldsymbol{x}}(\mathcal{S})$. Note that for $k = n$ this coincides with the usual definition of the (Fréchet) derivative interpreted as a linear transformation [Dieudonné 1960, Chapter VIII]; for $k = n - 1$ this reduces to the surface gradient defined in [Gurtin and Murdoch 1975; Gurtin 2000]. For a general k, we interpret $\nabla f(\boldsymbol{x})$ as a linear transformation from $\mathbb{R}^n$ to V and not as a linear transformation from $\mathbf{T}_{\boldsymbol{x}}(\mathcal{S})$ to V. However Equation (A–2) shows that $\nabla f(\boldsymbol{x}) = \mathbf{0}$ on the orthogonal complement $\mathbf{T}_{\boldsymbol{x}}(\mathcal{S})^{\perp}$. If $V = \mathbb{R}$, we identify the linear transformation $\nabla f(\boldsymbol{x})$ from $\mathbb{R}^n$ to $\mathbb{R}$ with a vector in $\mathbb{R}^n$, equally denoted, via $\nabla f(\boldsymbol{x})\boldsymbol{a} = \nabla f(\boldsymbol{x}) \cdot \boldsymbol{a}$ for any $\boldsymbol{a} \in \mathbb{R}^n$. It is easy to see that if $\boldsymbol{\phi}$ is as in Section A.1 (i) then f is differentiable at $\boldsymbol{x}$ if and only if $f \circ \boldsymbol{\phi}$ is differentiable in the classical sense at $\boldsymbol{p} := \boldsymbol{\phi}^{-1}(\boldsymbol{x})$ and then

$$\nabla f(\boldsymbol{x}) = \nabla(f \circ \boldsymbol{\phi})(\boldsymbol{p})[\nabla\boldsymbol{\phi}(\boldsymbol{p})]^{-1}\boldsymbol{P}, \tag{A–4}$$

where $[\nabla\boldsymbol{\phi}(\boldsymbol{p})]^{-1} : \mathbf{T}_{\boldsymbol{x}}(\mathcal{S}) \to \mathbb{R}^k$ is the inverse of $\nabla\boldsymbol{\phi}(\boldsymbol{p}) : \mathbb{R}^k \to \mathbf{T}_{\boldsymbol{x}}(\mathcal{S})$. Also, if N is a neighborhood of $\boldsymbol{x}$ in $\mathbb{R}^n$ and $g : N \to V$ is an extension of f that is differentiable in the classical sense at $\boldsymbol{x}$ then

$$\nabla f(\boldsymbol{x}) = \nabla g(\boldsymbol{x})\boldsymbol{P}, \tag{A–5}$$

where $\nabla g(\boldsymbol{x})$ is the derivative of g at $\boldsymbol{x}$ in the classical sense. If $T \subset S$, we say that $f : S \to V$ is *continuously differentiable* on T if $\nabla f(\boldsymbol{x})$ exists for every $\boldsymbol{x} \in T$ and the mapping ∇f is continuous on T.

If $\boldsymbol{q} : S \to \mathbb{R}^n$ is differentiable at $\boldsymbol{x} \in S$, we define the *surface divergence* $\operatorname{div} \boldsymbol{q}(\boldsymbol{x}) \in \mathbb{R}$ of $\boldsymbol{q}$ at $\boldsymbol{x}$ by

$$\operatorname{div} \boldsymbol{q}(\boldsymbol{x}) := \operatorname{tr}[\nabla \boldsymbol{q}(\boldsymbol{x})].$$

For $k = n$ this coincides with the standard divergence, while for $k = n - 1$ this reduces to the surface divergence defined in [Gurtin and Murdoch 1975] and [Gurtin 2000].

If $S \subset \mathcal{S}$ and $\boldsymbol{q} : S \to \mathbb{R}^n$, we say that $\boldsymbol{q}$ is *tangential* [Gurtin 2000] if $\boldsymbol{q}(\boldsymbol{x}) \in \mathbf{T}_{\boldsymbol{x}}(\mathcal{S})$ for every $\boldsymbol{x} \in S$. If $\boldsymbol{T} : S \to \mathrm{Lin}$ is differentiable at $\boldsymbol{x} \in S$, we define the *surface divergence* $\operatorname{div} \boldsymbol{T}(\boldsymbol{x}) \in \mathbb{R}^n$ of $\boldsymbol{T}$ at $\boldsymbol{x}$ to be the unique element of $\mathbb{R}^n$ such that

$$\boldsymbol{a} \cdot \operatorname{div} \boldsymbol{T}(\boldsymbol{x}) = \operatorname{div}[\boldsymbol{T}^{\mathsf{T}} \boldsymbol{a}](\boldsymbol{x}), \tag{A–6}$$

for every $\boldsymbol{a} \in \mathbb{R}^n$; see [Gurtin and Murdoch 1975] and [Gurtin 2000]. We note the following standard identity for a smooth tensor field $\boldsymbol{T}$ and a smooth vector field $\boldsymbol{v}$ on an open subset of a surface:

$$\operatorname{div}(\boldsymbol{T}^{\mathsf{T}} \boldsymbol{v}) = \nabla \boldsymbol{v} \cdot \boldsymbol{T} + \boldsymbol{v} \cdot \operatorname{div} \boldsymbol{T}. \tag{A–7}$$

To give formulas for the surface divergence, we assume that $\mathcal{U} \subset \mathbb{R}^n$ is a surface of dimension k, that $\boldsymbol{\phi} : P \to \mathcal{U}$ is a local parameterization of $\mathcal{U}$ on an open set $P \subset \mathbb{R}^k$ of class C^2, (that is, $\boldsymbol{\phi}$ satisfies Item (i) of Section A.1), and that $\boldsymbol{q} : \operatorname{ran} \boldsymbol{\phi} \to \mathbb{R}^n$ is a continuously differentiable tangential vector field on $\operatorname{ran} \boldsymbol{\phi} \subset \mathcal{U}$. We write

$$\boldsymbol{q} \circ \boldsymbol{\phi} = \sum_{i=1}^{k} q^i \boldsymbol{g}_i,$$

where $\boldsymbol{g}_i : P \to \mathbb{R}^n$ are the coordinate vectors of $\boldsymbol{\phi}$, given by $\boldsymbol{g}_i = \nabla \boldsymbol{\phi}\, \boldsymbol{e}_i$, where $\boldsymbol{e}_i$, $i = 1, \dots, k$, is the standard basis in $\mathbb{R}^k$. Then q^i are continuously differentiable functions on P and one has

$$(\operatorname{div} \boldsymbol{q}) \circ \boldsymbol{\phi} = J_{\boldsymbol{\phi}^{-1}} \sum_{i=1}^{k} \nabla_i (J_{\boldsymbol{\phi}} q^i), \tag{A–8}$$

where $J_{\boldsymbol{\phi}: P \to (0,\infty)}$ is the Jacobian of $\boldsymbol{\phi}$, defined by

$$J_{\boldsymbol{\phi}^2} = \det(\nabla \boldsymbol{\phi}^{\mathsf{T}} \nabla \boldsymbol{\phi}),$$

and ∇_i denotes the partial differentiation in $\mathbb{R}^k$. This can be deduced from [Lee 2003, Problem 14-11 (a)] in the case of class C^∞ objects and the generalization to the above smoothness assumptions is straightforward; nevertheless we note that $\boldsymbol{\phi}$

must be of class C^2 to make the right side of Equation (A–8) meaningful. One finds similarly that a continuously differentiable symmetric tensor field $\boldsymbol{T} : \operatorname{ran}\boldsymbol{\phi} \to \mathrm{Lin}$ is superficial if and only if it can be written in the form

$$\boldsymbol{T} \circ \boldsymbol{\phi} = \sum_{i,j=1}^{k} T^{ij} \boldsymbol{g}_i \otimes \boldsymbol{g}_j,$$

where T^{ij} are continuously differentiable functions on P. From Equations (A–6) and (A–8) we can deduce that

$$(\operatorname{div} \boldsymbol{T}) \circ \boldsymbol{\phi} = J_{\boldsymbol{\phi}^{-1}} \sum_{i,j=1}^{k} \nabla_j (J_{\boldsymbol{\phi}} T^{ij} \boldsymbol{g}_i). \tag{A–9}$$

Theorem 1 (Surface divergence theorem). *If $\mathcal{S}$ is an oriented k-dimensional surface with boundary and if $\boldsymbol{q} : \mathcal{S} \to \mathrm{Lin}$ is a continuous tangential vector field with compact support and with a continuous and $\mathcal{H}^k$ integrable derivative in $\operatorname{int}\mathcal{S}$ then*

$$\int_{\mathcal{S}} \operatorname{div} \boldsymbol{q} \, d\mathcal{H}^k = \int_{\partial\mathcal{S}} \boldsymbol{q} \cdot \boldsymbol{m} \, d\mathcal{H}^{k-1} \tag{A–10}$$

where $\boldsymbol{m}$ is the exterior normal to $\mathcal{S}$.

We refer to [Lee 2003, Theorem 14.23] for the proof for C^∞ objects. The proof under the present generality follows by noting that the maps $\boldsymbol{\phi}$ as in Section A.2 item (iii) carry (parts) of $\mathcal{S}$ into (parts) of regions with Lipschitz boundary in $\mathbb{R}^k$ for which the divergence theorem is known to hold [Nečas 1967] for functions from the Sobolev class $W^{1,1}$. In the proof one invokes Equation (A–8) to transform the surface integral of the surface divergence into the volume integral of the 'volume' divergence, invoking the divergence theorem and transforming the resulting integral to the right side of Equation (A–10). The proof is then completed with the help of a partition of unity, see [Šilhavý 2005b, Chapter 5] for details. The use of formula (A–8) requires a class C^2 smoothness of $\mathcal{S}$.

Acknowledgments

The research of M. Šilhavý has been supported by a MIUR grant, "Variational theory of microstructure, semiconvexity, and complex materials." The support is gratefully acknowledged.

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Received 4 Dec 2005.

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