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EXPERIMENTAL EVALUATION OF TWO MULTIPHASE CONSTITUTIVE MODELS APPLICABLE TO METAL MATRIX COMPOSITES UNDER NONPROPORTIONAL VARIABLE AMPLITUDE LOADING

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In a previous research investigation, using the Mróz model and the endochronic theory of plasticity as their bases, two sets of elastic-plastic constitutive relations were identified that account for the interaction in stress fields between adjacent particles in particulate metal matrix composites (PMMCs). In this paper the ability of the two models to predict the behavior of PMMCs under variable amplitude nonproportional cyclic loading paths is evaluated by comparing the models predictions with experimental results obtained from a series of biaxial (tension-torsion) cyclic tests performed on tubular specimens made from 6061-T6 aluminum with 10 and 20% volume fractions of alumina particles. For most of the investigated loading paths, both models predict satisfactorily the amplitudes of the experimental strains. However, the endochronic theory-based constitutive model generally gives better predictions of the measured strains.

Notation

m	matrix	i, j, k, l, r, s	indices = 1,2,3 (summation convention)
f	reinforcement	S_{ij}	components of deviatoric stress tensor
C_f	reinforcement stiffness tensor	S_{ijkl}	components of Eshelby's tensor
C_m	matrix stiffness tensor	V_f	reinforcement volume fraction
F	yield function	V_m	matrix volume fraction
I_{ijkl}	identity tensor	C_r	material constants
δ	Kronecker delta	α_{ij}^{ℓ}	components of back stress tensor of surface ℓ
Δ	increment of a variable	ξ_{ij}	deviatoric components of α_{ij}
ε_{ij}	components of strain tensor	$\mathrm{d}(\cdot)$	differential form of variable or constant
$\mathrm{d}\mu$	magnitude of translation	$(\cdot)^e$	elastic components of a variable
$\rho()$	hereditary (memory) function	$(\cdot)_{(f)}$	reinforcement component
σ_{ij}	components of stress tensor	$(\cdot)\ell$	component with reference to surface ℓ
σ_o^{ℓ}	yield stress of surface ℓ	$(\cdot)_{(m)}$	matrix components of a variable or constant
$\mathrm{d}\mu$	magnitude of translation	$(\cdot)^p$	plastic components of a variable
K_p	hardening modulus	$(\cdot)_q$	q-th step component of a variable
z	intrinsic time scale	$(\cdot)^T$	total components of a variable

Keywords: cyclic plasticity, nonproportional loading, particulate reinforced material, Mróz model, endochronic theory.

1. Introduction

Due to their material properties, particulate metal matrix composites (PMMCs) are increasingly finding applications in aerospace, automotive, and related industries. However, due to the complexity of the geometry of the reinforcements and the nonlinear (inelastic) behavior of the matrix, the behavior of PMMCs particularly when subjected to complex service loading conditions is still not fully understood [Ji and Wang 2003; Lease et al. 1996; Ju and Chen 1994]. Thus, with increasing interest in the application of these materials in both structural and nonstructural applications, the need to understand, model, and predict the mechanical response and fatigue life of the materials to applied service loads also increases.

Most research studies that have described the constitutive behavior of PMMCs assume that the composites remain elastic as the loads are applied. Goodier [1933] worked on modeling a spherical and a cylindrical inclusion in a matrix, providing the first detailed analysis of the elastic stress-strain response of composite materials. Subsequently, Sadowsky and Sterberg [1952] derived relations defining the elastic stress-field around an ellipsoidal cavity under plane stress conditions. The general problem of elastic field inside and at the interface of an ellipsoidal inclusion was solved by Eshelby [1957]. His theory of equivalent inclusions is perhaps the most widely accepted elastic constitutive model applicable to composite materials. However, Eshelby's method is only applicable to composites with very small volume fraction of the reinforcement — at most 5%. For higher volume fractions of reinforcements, the method has to be modified to account for the interaction in the stress field between the reinforcements.

The concept of average stress and strain in the composite and its constituents has been used extensively in the past to account for the interactions in the stress field between the particles. The self-consistent theory [Hershey 1954; Kröner 1961] and the mean-field theory of Mori and Tanaka [1973], both based on the concept of average stress and strain, are the commonly used models. The self-consistent theory was originally developed to model the average constitutive behavior of polycrystals; it has been used to estimate the macroscopic elastic moduli of two-phase composites and also the average internal nonuniformity of strain and stresses in the matrix and particles of a composite system. Hutchison [1970] used this theory to estimate the elastic-plastic incremental relations of polycrystals and composites. Although straightforward, the method does not provide acceptable results when the matrix contains either voids or perfectly rigid inclusions [Mori and Wakashima 1990]. Consequently, Mori–Tanaka mean-field theory has been more widely used. The original approach of [Mori and Tanaka 1973] has been used mainly to evaluate the elastic behavior of composite materials. The emphasis on elasticity shows its importance in applications, and provides the means for developing practical design methods. It is, however, unrealistic to assume that all components being modeled behave linearly.

Tandon and Weng [1988] considered the elastic-plastic stress-strain behavior of spherical particle reinforced composites under multiaxial loading using the secant moduli of the ductile, work-hardening matrix. Specifically, they analyzed the elastic-plastic behavior of the composites by applying the concept of secant properties to the elastic mean field theory. In the Tandon–Weng model, the secant modulus of the matrix in the plastic range changes with an increase in plastic deformation. Since the change is not known beforehand, at each state of stress or strain an initial value has to be assumed for the matrix effective plastic strain. The model has received attention from the research community, but it is only applicable if the loads are applied in a proportional manner and not if there is load reversal.

In an attempt to study the elastic-plastic deformation of multiphase composite materials under nonproportional loading, Li and Chen [1990] reformulated the mean field theory in an incremental form. Their method is meant to be applied to multiphase materials in which the components exhibit different elastic-plastic material behavior under nonproportional or reverse loading. Theoretically, the model could be used to study the elastic-plastic deformation of composite materials under nonproportional monotonic and cyclic loadings; however, Li and Chen only implemented and validated the model for uniaxial monotonic loading. That is, in [Li and Chen 1990], the model was not validated for the case of multiaxial cyclic loading. Generally, their model can be used in conjunction with any cyclic plasticity model developed for homogeneous materials to predict the matrix plastic strain components.

Inelastic constitutive models for homogeneous metals subjected to cyclic loads (cyclic plasticity models) are still evolving. Chaboche [1986] describes two classes of such models. Those in the first class are based on thermodynamic concepts and assume that the present state of the material depends on the present values of observable variables and a set of internal variables. Included in this category are the Ziegler model [1959], the Mróz model [1967], the two-surface plasticity model [Dafalias and Popov 1975; Krieg 1975], and the Armstrong and Frederick model [1966], modified by Chaboche et al. [1979]. The second class of plasticity models is based on the assumption that the present state of material depends on the present values and the past history of observable variables only (total strains, temperature etc.), giving rise to hereditary theories. The endochronic theory of plasticity [Valanis 1971; 1980] is based on this concept. Several of these models and their various modifications have been used to predict the elastic-plastic, creep and ratcheting behaviors of homogeneous materials and are well documented in the literature, including recent work such as [Kang et al. 2003; Chiou and Yip 2003; Tong et al. 2004; Vincent et al. 2004; Chen et al. 2005; Hashiguchi et al. 2005].

Only a few cyclic plasticity models have been incorporated into a formulation targeted at describing the constitutive behavior of metal matrix composites under multiaxial loading conditions. Ogarevic [1992] formulated a composite constitutive model based on Li and Chen's incremental mean field theory, the incremental theory of plasticity, and a linear kinematic hardening rule to study the uniaxial cyclic deformation of discontinuously reinforced metal matrix composites (MMCs) both at room and elevated temperatures. The model was demonstrated only analytically for externally applied uniaxial cyclic loading. A major relevant model that addresses cyclic biaxial proportional external loading was developed in [Lease 1994; Lease et al. 1995], where the Li and Chen model was used together with Chaboche's incremental plasticity theory to simulate the constitutive behavior of the composite system. The model was demonstrated both analytically and experimentally for cyclic axial and biaxial proportional loading. Although the axial and torsional elastic-plastic strain and stress seemed to accurately simulate the monotonic tests, the elastic loading/unloading portions of the cyclic uniaxial and biaxial tests show obvious differences that increase with increasing strain range [Lease 1994]. Fleming and Temis [2002] used cyclic strain plasticity relationships based on the classical strain plasticity theory hypotheses in [Johnson and Mellor 1975] to predict cyclic stress-strain response of the matrix material. The constitutive equations lead to a nonlinear finite element problem that was solved using a special iterative procedure at every half-cycle of the loading or unloading. However, the model has only been demonstrated for monotonic and uniaxial cyclic loading conditions.

In [Owolabi and Singh 2003], two constitutive models capable of predicting the elastic-plastic strain-stress response of the matrix, reinforcement, and the PMMC under multiaxial cyclic loading conditions were developed. Specifically, the elastic components of the matrix and reinforcement strains and stresses were obtained from the applied incremental stresses or strains by implementing known relations specific to composites. The matrix plastic strain components were obtained using two alternative cyclic plasticity routines. Specifically, the Mróz multisurface model [1967] and the endochronic theory of plasticity [Valanis 1971; 1980], originally developed for homogeneous materials, were used to model the matrix cyclic plasticity. The elastic and plastic (matrix only) strains were superimposed to obtain the constituents constitutive relations. These models are capable of predicting the constitutive behavior of PMMCs under a variety of loading conditions and are dependent on the properties of the matrix and the properties, volume fraction and geometry of the reinforcing particles. The constitutive models account for the interactions in stress fields between adjacent particles in PMMCs. In [Owolabi and Singh 2003] we compared the models predictions to limited experimental results given in [Lease 1994] for PMMCs under biaxial cyclic strain-controlled (proportional loading) tests with various strain amplitudes. However in our 2003 paper only preliminary comments could be made regarding the model's suitability in defining the constitutive behavior of composites, based on a comparison with the limited experimental results in [Lease 1994]. That is, more testing was required to make more meaningful conclusions, particularly when the material is subjected to more complex loadings. Investigations of the ability of the cyclic constitutive models to predict the elastic-plastic response of PMMCs under more complex loading conditions such as variable amplitude nonproportional cyclic loading is critical in engineering design.

To date, no cyclic plasticity model applicable to PMMCs has been validated experimentally for cyclic nonproportional variable amplitude external loading. In addition, experimental data that can be used to validate existing and newly developed models for PMMCs subjected to variable amplitude loads are very rare in the open literature. Thus, this paper is designed primarily to provide a complete formulation of the two models that completely define the constitutive response of the matrix, reinforcement, and PMMCs; and validate the ability of the models to predict the elastic-plastic behavior of PMMCs under complex external loading. The theoretical and experimental results of the constitutive response of PMMCs to such complex loading paths are reported. Note that the experimental results presented here were all conducted under load-controlled tests. In addition to the variable amplitude loads, one load-controlled biaxial proportional path was also included.

Section 2 presents a summary of the constitutive models. The experimental procedure and load paths used are presented in Section 3. In Section 4, the model's predictions are compared with experimental results. Conclusions are given in Section 5.

2. Constitutive models

The details of the elastic-plastic constitutive model appear in [Owolabi and Singh 2003], so only a summary is given here. For PMMCs, we assume that the matrix, consisting of a homogeneous metallic material, is initially isotropic and strain hardens at the onset of plastic deformation. Since it is additionally assumed to be cyclically stable, transient effects are not considered. The reinforcement, consisting of ceramic particles, behaves elastically throughout the loading paths and has higher stiffness relative to the matrix.

For small deformations, the total incremental matrix strain tensor $d\varepsilon^{T}_{ij(m)}$ can be decomposed into elastic and plastic components, denoted by superscript e and p:

$$d\varepsilon^{T}_{ij(m)} = d\varepsilon^{e}_{ij(m)} + d\varepsilon^{p}_{ij(m)}. \tag{1}$$

The reinforcement strain, $d\varepsilon^{T}_{ij(f)}$, only has an elastic component, since the reinforcement (ceramic particles) deforms elastically:

$$d\varepsilon^{T}_{ij(f)} = d\varepsilon^{e}_{ij(f)}. \tag{2}$$

The approach used to determine the incremental elastic stress and strain components in the matrix, reinforcement, and composite was presented in [Owolabi and Singh 2003] on the same basis as Li and Chen's [1990] incremental formulations of mean field theory. This approach is suitable for PMMCs with high volume fractions of reinforcement, since it accounts for interactions in the stress fields between reinforcing particles; it is summarized in Section 2.1. Once the matrix stress exceeds the material yield stress, a suitable cyclic plasticity model must be used to obtain the matrix strains. Two such cyclic plasticity models are presented in Section 2.2.

2.1. ***Elastic model.*** In [Owolabi and Singh 2003; Lease 1994; Lease et al. 1995] it was shown that the average incremental stress tensors in the matrix and the reinforcements, due to externally applied load tensor, can be obtained using the incremental form of Mori and Tanaka's mean field theory [1973]. A summary of this approach is presented here. Consider an elastic component subjected to an increment in external load or displacement tensor. In the absence of reinforcement, the external load would give rise to an increase in the uniform stress field, $\Delta\sigma_{ij}$, which can be related to the increment in the strain field, $\Delta\varepsilon_{ij}$. The average incremental stress in the matrix, $\Delta\sigma_{ij(m)}$, differs from the applied incremental stress by a perturbed incremental stress, $\Delta\tilde{\sigma}_{ij(m)}$:

$$\Delta\sigma_{ij(m)} = \Delta\sigma_{ij} + \Delta\tilde{\sigma}_{ij(m)} = C_{ijkl(m)}(\Delta\varepsilon_{kl} + \Delta\tilde{\varepsilon}_{kl(m)}), \tag{3}$$

where $C_{ijkl(m)}$ is the matrix stiffness tensor and $\Delta\tilde{\varepsilon}_{ij(m)}$ is the matrix incremental strain disturbance that results from the presence of the particles. The reinforcement average incremental stress, $\Delta\sigma_{ij(f)}$, and strain, $\Delta\varepsilon_{ij(f)}$ are also different from those of the matrix. The average incremental stress in the reinforcement is

$$\Delta\sigma_{ij(f)} = C_{ijkl(m)}(\Delta\varepsilon_{kl} + \Delta\tilde{\varepsilon}_{kl(m)} + \Delta\varepsilon^{c}_{kl} - \Delta\varepsilon^{t}_{kl}), \tag{4}$$

where $\Delta\varepsilon^{c}_{kl}$ is a constrained strain set up at all points in the matrix and the reinforcement, and $\Delta\varepsilon^{t}_{kl}$ is a transformation strain having a finite value within the reinforcements and zero outside them. Although the solution for the constrained strain field in the matrix is quite complex, an approximate relation between the constrained strain, the stress free transformation strain, and the 6×6 Eshelby tensor S is given by

$$\Delta\varepsilon^{c}_{ij} = S_{ijkl}\Delta\varepsilon^{t}_{kl}. \tag{5}$$

The incremental strain disturbance in the matrix can be found using Equations (3)–(5) and the rule of mixture as

$$\Delta\tilde{\varepsilon}_{ij(m)} = (1 - V_m)(\boldsymbol{I}_{ijkl} - \boldsymbol{S}_{ijkl})(\Delta\varepsilon^{t}_{kl}), \tag{6}$$

where V_m is the matrix volume fraction. and $\boldsymbol{I}$ is the identity tensor.

The incremental transformation strain, $\Delta\varepsilon^t_{ij}$, is found in [Owolabi and Singh 2003] to be

$$\Delta\varepsilon^t_{ij} = L^{-1}_{ijkl}(C_{klrs(f)} - C_{klrs(m)})(C^{-1}_{klrs(m)}\Delta\sigma_{kl}), \tag{7}$$

where

$$L_{ijkl} = (V_f - 1)C_{klrs(m)}(\boldsymbol{I}_{ijrs} - \boldsymbol{S}_{ijrs}) + C_{klrs(f)}\big(V_f(\boldsymbol{S}_{ijrs} - \boldsymbol{I}_{ijrs}) - \boldsymbol{S}_{ijrs}\big). \tag{8}$$

Substituting (7) into (6), and using the result in (3), we get the average incremental stress in the matrix:

$$\Delta\sigma_{ij(m)} = \big(\boldsymbol{I}_{ijkl} - V_f C_{ijrs(m)}(\boldsymbol{S}_{klrs} - \boldsymbol{I}_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)} - C_{ijkl(m)})C^{-1}_{klrs(m)}\big)\Delta\sigma_{kl}, \tag{9}$$

where V_f is the volume fraction of reinforcement.

The incremental matrix elastic strain, $\Delta\varepsilon^e_{ij(m)}$, can be obtained from the incremental stress using the generalized Hooke's law:

$$\Delta\varepsilon^e_{ij(m)} = C^{-1}_{ijkl(m)}\Delta\sigma_{kl(m)}. \tag{10}$$

The mean incremental stress tensor in the reinforcement, $\Delta\sigma_{ij(f)}$, is obtained from (4), (5) and (7):

$$\Delta\sigma_{ij(f)} = \big(\boldsymbol{I}_{ijkl} + V_m C_{ijrs(m)}(\boldsymbol{S}_{klst} - \boldsymbol{I}_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)} - C_{ijkl(m)})C^{-1}_{klrs(m)}\big)\Delta\sigma_{kl}. \tag{11}$$

The incremental elastic strain in the reinforcement can be obtained from the incremental stress using the generalized Hooke's law:

$$\Delta\varepsilon^e_{ij(f)} = C^{-1}_{ijkl(f)}\Delta\sigma_{kl(f)}. \tag{12}$$

The increment in average strains in the composite can be estimated using an approximate technique proposed by [Li and Chen 1990] for a multiphase system. The technique assumes that the work done by the average stress of the composite is equal to the weighted sum of the work done by the local stresses of the inclusions and the matrix. Under this assumption, the following expression is obtained for a two-phase composite material:

$$\Delta\sigma_{ik}\Delta\varepsilon_{kj} = V_m\Delta\sigma_{ik(m)}\Delta\varepsilon_{kj(m)} + V_f\Delta\sigma_{ik(f)}\Delta\varepsilon_{kj(f)}, \tag{13}$$

from which $\Delta\varepsilon_{ij}$ can be obtained in terms of other stress and strain increments. A similar expression of this work-based rule of mixture has been used in [Lease 1994; Lease et al. 1995] and is valid both in the elastic and the elastic-plastic regions.

2.2. *Cyclic plasticity models.* In [Owolabi and Singh 2003] we identified two sets of elastic-plastic constitutive relations as applicable to PMMCs. These are the based respectively on the Mróz model (Section 2.2.1) and the endochronic theory of plasticity (Section 2.2.2), and are used below to describe the increments in the matrix plastic strain given in (1).

2.2.1. *The Mróz model.* For a plastically deforming material, Mróz [1967] describes a field of ℓ initially concentric work hardening surfaces and prescribing a translation rule for the surfaces moving with respect to one another. The model assumes that each surface can be described by the same relationship as the yield criterion. Using the von Mises yield criterion on the matrix gives

$$F^\ell\big(S_{ij(m)}, \xi_{ij(m)}\big) = \tfrac{3}{2}\big(S_{ij(m)} - \xi^\ell_{ij(m)}\big)\big(S_{ij(m)} - \xi^\ell_{ij(m)}\big) - {\sigma^\ell_{o(m)}}^2 = 0, \tag{14}$$

where $S_{ij(m)}$ and $\xi^\ell_{ij(m)}$ are the deviatoric components of the current matrix stress tensor $\sigma_{ij(m)}$ and the backstress tensor α^ℓ_{ij}; F^ℓ is the yield function of the active surface (the one on which the stress state is located during elastic-plastic loading) at higher stress level; and σ^ℓ_o is the material yield stress.

For the active surface, the increment in the plastic strain tensor is related to the increment in the stress by the flow rule:

$$d\varepsilon^{p}_{ij(m)} = \frac{1}{K_p}\frac{\partial F^{\ell}}{\partial\sigma_{ij(m)}}\left(\frac{\partial F^{\ell}}{\partial\sigma_{kl(m)}}\partial\sigma_{kl(m)}\right), \tag{15}$$

where K_p is the hardening modulus obtained from the matrix uniaxial stress-strain curve. Mróz [1967] prescribed a translation rule for the determination of the active surface; in our case it takes the form

$$d\xi^{\ell}_{ij(m)} = d\mu\big(S^{\ell+1}_{ij(m)} - S^{\ell}_{ij(m)}\big), \tag{16}$$

where $d\mu$ is a scalar parameter of the active surface translation, which can be determined using the consistency condition, and the term $(S^{\ell+1}_{ij(m)} - S^{\ell}_{ij(m)})$ governs the direction of its translation. The quantity $S^{\ell+1}_{ij(m)}$ is the point on the surface $\ell+1$ immediately enclosing the active surface; it has the same unit normal as the active surface ℓ at the actual current stress state, $S^{\ell}_{ij(m)}$. It is obtained from the Mróz translation rule as

$$S^{\ell+1}_{ij(m)} = \xi^{\ell+1}_{ij(m)} + \frac{\sigma^{\ell+1}_{o(m)}}{\sigma^{\ell}_{o(m)}}\big(S_{ij(m)} - \xi^{\ell}_{ij(m)}\big). \tag{17}$$

2.2.2. *Endochronic theory.* The ability of the endochronic theory to model certain phenomena in cyclic plasticity and creep of homogeneous materials was demonstrated in [Wu and Yang 1983; Khan and Wang 1988; Watanabe and Atluri 1986; Hsu et al. 1991]. In [Owolabi and Singh 2003], the theory was used along with the incremental mean field theory to predict the constitutive behavior of PMMCs under biaxial proportional loading conditions. We give a brief description of the theory. For a plastically incompressible and time independent matrix material, the deviatoric stress is related to the matrix incremental plastic strain by the equation

$$S_{ij(m)} = 2\int_0^{z}\rho(z-z')\frac{d\varepsilon^{p}_{ij(m)}(z')}{dz'}dz'. \tag{18}$$

Here z is the intrinsic time scale and $\rho(z)$ is the material function called the hereditary function, and given by

$$\rho(z) = \sum_{r=1}^{r} C_r e^{-\alpha_r z}, \tag{19}$$

where C_r and α_r are material constants determined from the uniaxial cyclic stress-strain curve of the matrix.

As described in [Hsu et al. 1991; Owolabi and Singh 2003], for a stress-controlled loading condition, the incremental matrix plastic strain tensor is related to the incremental intrinsic time scale by

$$\Delta\varepsilon^{p}_{ij(m)} = \frac{a_{ij}\Delta z}{b}, \tag{20}$$

where

$$a_{ij} = \frac{1}{2}\left((\Delta S_{ij(m)})_q + \sum_{r=1}^{r}(S^{r}_{ij(m)})_{q-1}(1-e^{-\alpha_r\Delta z})\right) \tag{21}$$

and

$$b = \left(\sum_{r=1}^{r} C_r\frac{(1-e^{-\alpha_r\Delta z})}{\alpha_r}\right). \tag{22}$$

In these equations q and $q-1$ denote the current and the previous loading steps, and Δ is the difference between the current and previous steps. For a given increment of composite stress tensor, the increment in the intrinsic time scale, Δz, can be obtained, using the secant method or the Newton–Raphson method, as a root of the equation

$$b^2 - a_{ij}a_{ij} = 0 = R(\Delta z). \tag{23}$$

2.2.3. *Complete formulation of the constitutive models.* We now present two models that completely define the constitutive response of the matrix, reinforcement, and PMMCs. The first combines the elastic constitutive response defined by the incremental mean field theory with the Mróz model. The second model differs from the first in that the endochronic theory of plasticity is used to define the increments in the matrix plastic strain components.

Constitutive relations based on the Mróz model. For multiaxial cyclic loading, using the Mróz-based model, incremental mean field theory, and the work relation, the following constitutive relations can be developed to predict the constituents and the composite elastic-plastic strain and stress increments.

Matrix constitutive model. The matrix constitutive relation can be finalized by substituting (9), (10) and (15) into (1). Changing the differential to small increments for numerical implementation yields

$$\begin{aligned}\Delta\varepsilon_{ij(m)} = {}& \frac{1+\upsilon_m}{E_m}\big(I_{ijkl} - V_f C_{ijrs(m)}(\boldsymbol{S}_{klrs} - \boldsymbol{I}_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)} - C_{ijkl(m)})C^{-1}_{klrs(m)}\big)\Delta\sigma_{kl} \\ & - \frac{\upsilon_m}{E_{(m)}}\big(\Delta\sigma_{kk} - V_f C_{ijrs(m)}(\boldsymbol{S}_{klrs} - \boldsymbol{I}_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)} - C_{ijkl(m)})C^{-1}_{ijrs(m)}\Delta\sigma_{kk}\big)\delta_{ij} \\ & + \frac{\hat{n}_{ij}}{K_{p(m)}}\hat{n}_{kl}\big(I_{ijkl} - V_f C_{ijrs(m)}(\boldsymbol{S}_{klrs} - \boldsymbol{I}_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)} - C_{ijkl(m)})C^{-1}_{klrs(m)}\big)\Delta\sigma_{kl}.\end{aligned} \tag{24}$$

Reinforcement constitutive model. The reinforcement, being relatively stiff, only has elastic strain components. The reinforcement constitutive relation can be finalized by substituting (11) and (12) into (2). Changing the differential to small increments yields

$$\begin{aligned}\Delta\varepsilon_{ij(f)} = {}& \frac{1+\upsilon_f}{E_f}\big(\boldsymbol{I}_{ijkl} + V_m C_{ijrs(m)}(\boldsymbol{S}_{klrs} - \boldsymbol{I}_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)} - C_{ijkl(m)})C^{-1}_{klrs(m)}\big)\Delta\sigma_{kl} \\ & - \frac{\upsilon_f}{E_{(f)}}\big(\Delta\Delta\sigma_{kk} + V_m C_{ijrs(m)}(\boldsymbol{S}_{klrs} - \boldsymbol{I}_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)} - C_{ijkl(m)})C^{-1}_{ijrs(m)}\Delta\sigma_{kk}\big)\delta_{ij}.\end{aligned} \tag{25}$$

Composite constitutive model. The composite constitutive relation can be reached by substituting Equations (9), (11), (24), and (25) into (13). Changing the differential to small increments yields

$$\begin{aligned}\Delta\varepsilon_{ij} = {}& V_m[\Delta\sigma_{kl}]^{-1}\big(I_{ijkl} - V_f C_{ijrs(m)}(S_{klrs} - I_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)} - C_{ijkl(m)})C^{-1}_{klrs(m)}\big)\Delta\sigma_{kl} \\ & \times\Bigg(\frac{1+\upsilon_m}{E_m}\big(I_{ijkl} - V_f C_{ijrs(m)}(S_{klrs} - I_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)} - C_{ijkl(m)})C^{-1}_{klrs(m)}\big)\Delta\sigma_{kl} \\ & \quad - \frac{\upsilon_m}{E_{(m)}}\big(\Delta\sigma_{kk} - V_f C_{ijrs(m)}(S_{klrs} - I_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)} - C_{ijkl(m)})C^{-1}_{ijrs(m)}\Delta\sigma_{kk}\big)\delta_{ij} \\ & \quad + \frac{\hat{n}_{ij}}{K_{p(m)}}\hat{n}_{kl}\big(I_{ijkl} - V_f C_{ijrs(m)}(S_{klrs} - I_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)} - C_{ijkl(m)})C^{-1}_{klrs(m)}\big)\Delta\sigma_{kl}\Bigg) +\end{aligned}$$

$$+V_f[\Delta\sigma_{kl}]^{-1}\big(I_{ijkl}+V_m C_{ijrs(m)}(S_{klrs}-I_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)}-C_{ijkl(m)})C^{-1}_{klrs(m)}\big)\Delta\sigma_{kl}$$
$$\times\bigg(\frac{1+\upsilon_f}{E_f}\Big(I_{ijkl}+V_m C_{ijrs(m)}(S_{klrs}-I_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)}-C_{ijkl(m)})C^{-1}_{klrs(m)}\Big)\Delta\sigma_{kl}$$
$$-\frac{\upsilon_f}{E_{(f)}}\big(\Delta\sigma_{kk}+V_m C_{ijrs(m)}(S_{klrs}-I_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)}-C_{ijkl(m)})C^{-1}_{ijrs(m)}\Delta\sigma_{kk}\big)\delta_{ij}\bigg). \quad (26)$$

Constitutive relations based on endochronic theory. For multiaxial cyclic loading, using the endochronic theory-based model, incremental mean field theory, and the work relation, the following constitutive relations are developed to predict the constituents and the composite elastic-plastic strain and stress increments.

Matrix constitutive model. The matrix constitutive relation can be finalized by substituting (9), (10), and (20)–(22) into (1). Changing the differential to small increments yields

$$\Delta\varepsilon_{ij(m)}=\frac{1+\upsilon_m}{E_m}\big(\boldsymbol{I}_{ijkl}-V_f C_{ijrs(m)}(\boldsymbol{S}_{klrs}-\boldsymbol{I}_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)}-C_{ijkl(m)})C^{-1}_{klrs(m)}\big)\Delta\sigma_{kl}$$
$$-\frac{\upsilon_m}{E_{(m)}}\big(\Delta\sigma_{kk}-V_f C_{ijrs(m)}(\boldsymbol{S}_{klrs}-\boldsymbol{I}_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)}-C_{ijkl(m)})C^{-1}_{ijrs(m)}\Delta\sigma_{kk}\big)\delta_{ij}$$
$$+\frac{\frac{1}{2}\Big((\Delta S_{ij(m)})_q+\sum_{r=1}^{n}(S^r_{ij(m)})_{q-1}(1-e^{-\alpha_r\Delta z})\Big)\Delta z}{\sum_{r=1}^{n}C_r\dfrac{(1-e^{-\alpha_r\Delta z})}{\alpha_r}}. \quad (27)$$

Reinforcement constitutive model. The reinforcement, being relatively stiff, has only the elastic strain components. Consequently, the reinforcement constitutive relation is the same as in the Mróz-based PMMCs constitutive model given by Equation (25).

Composite constitutive model. The composite constitutive relation can be finalized by substituting (9), (11), (25) and (27) into (13). Changing the differentials to small increments yields

$$\Delta\varepsilon_{ij}=V_m[\Delta\sigma_{kl}]^{-1}\big(I_{ijkl}-V_f C_{ijrs(m)}(S_{klrs}-I_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)}-C_{ijkl(m)})C^{-1}_{klrs(m)}\big)\Delta\sigma_{kl}$$
$$\times\bigg(\frac{1+\upsilon_m}{E_m}\big(I_{ijkl}-V_f C_{ijrs(m)}(S_{klrs}-I_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)}-C_{ijkl(m)})C^{-1}_{klrs(m)}\big)\Delta\sigma_{kl}$$
$$-\frac{\upsilon_m}{E_{(m)}}\big(\Delta\sigma_{kk}-V_f C_{ijrs(m)}(S_{klrs}-I_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)}-C_{ijkl(m)})C^{-1}_{ijrs(m)}\Delta\sigma_{kk}\big)\delta_{ij}+\frac{a_{ij}\Delta z}{b}\bigg)$$
$$+V_f[\Delta\sigma_{kl}]^{-1}\big(\Delta\sigma_{kl}+V_m C_{klst(m)}(S_{klst}-I_{klst})L^{-1}_{ijkl}(C_{klst(f)}-C_{klst(m)})C^{-1}_{klst(m)}\Delta\sigma_{kl}\big)$$
$$\times\bigg(\frac{1+\upsilon_f}{E_f}\big(I_{ijkl}+V_m C_{ijrs(m)}(S_{klrs}-I_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)}-C_{ijkl(m)})C^{-1}_{klrs(m)}\big)\Delta\sigma_{kl}$$
$$-\frac{\upsilon_f}{E_{(f)}}\big(\Delta\sigma_{kk}+V_m C_{ijrs(m)}(S_{klrs}-I_{klrs})L^{-1}_{ijrs}(C_{ijkl(f)}-C_{ijkl(m)})C^{-1}_{ijrs(m)}\Delta\sigma_{kk}\big)\delta_{ij}\bigg). \quad (28)$$

To obtain numerical results, two MATLAB programs were developed, one for each of the Mróz-based model and the endochronic theory-based model. Each program calculates either the elastic-plastic composite strain or stress history given the composites material properties and the known stress or strain history. The Mróz-based PMMCs constitutive model uses the yield stresses and hardening moduli of the matrix yield surfaces, while the endochronic theory-based PMMCs constitutive model uses the materials constants as additional inputs. The matrix incremental stress-strain increments are obtained separately using (9), (10), and (24) for the Mróz-based model, while for the endochronic theory-based model, the elastic-plastic matrix stress-strain increments are obtained using (9), (10), and (27) after calculating the increment in the intrinsic time using (23). The final state of stress and strain for a loading step can then be obtained by adding the increments to the previous stress-strain history. The increments in the reinforcement stress and strain can be obtained from the elastic analysis using (11) and (25). For each loading step in the loading history the composite elastic-plastic strain or stress state can be obtained using the elastic-plastic matrix strains, the elastic reinforcement strains and stresses, and (26) and (28) respectively for the Mróz-based and the endochronic theory based models.

3. Experimental procedure

To assess the capability of the two constitutive models presented in Section 2.2.3 to predict the elastic-plastic constitutive behavior of PMMCs, biaxial cyclic (proportional and nonproportional) loads were applied to tubular PMMC specimens, machined from round bars as shown in Figure 1. The PMMC materials used are general purpose Duralcan materials made of 6061-T6 aluminum alloy reinforced with 10 and 20% by volume alumina (Al_2O_3). The composite and its constituents properties, obtained from [Bill 2003; Lease 1994], are shown in Table 1. In [Owolabi and Singh 2003], three terms in the series expansion of the Equation (19) were found to accurately model the stress-plastic strain response of the matrix cyclic stress-strain curve. The values of the material constants, C_r and α_r, associated with the endochronic theory-based PMMCs constitutive model for the matrix material 6061-T6 alloy are shown in Table 2. For the Mróz model, the values of the hardening moduli and yield stresses for ten surfaces used in the matrix uniaxial curve are shown in Table 3.

For the verification of any constitutive model, tubular specimens are normally used because they are more suitable for use for torsional and combined tensile-torsional tests than the corresponding smooth solid specimens. That is, the relationship between the applied torque, T, and the shear stress, τ, can easily be made for tubular specimens in both the elastic and the plastic regions than can be for solid specimens.

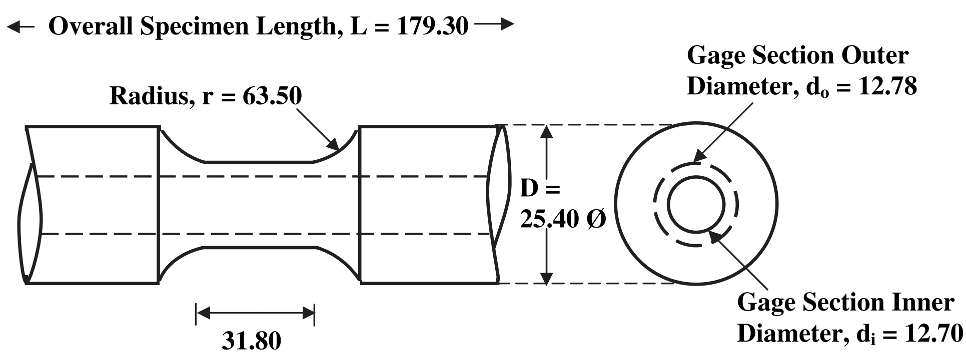

Figure 1. Geometry of the smooth tubular specimen.

material	ultimate strength (MPa)	yield strength (MPa)	elastic modulus (GPa)	elongation (%)
6061-T6	310 (262)	276 (241)	69	20
6061/Al_2O_3/10p	352 (324)	296 (262)	81	10
6061/Al_2O_3/20p	372 (345)	352 (317)	97	4

Table 1. Tensile properties (typical and minimum values) of extruded rods, from [Bill 2003; Lease 1994]. Minimum values based on a 99% confidence interval.

C_1 (MPa)	C_2 (MPa)	C_3 (MPa)	α_1	α_2	α_3
7841132	218948	17115	60172	2097	347

Table 2. Material constants used in the endochronic theory series expansion.

surface	hardening modulus ($\times 10^{-11}$)	yield stress (MPa)
1	0.381	150
2	0.476	175
3	1.150	225
4	1.307	237
5	1.515	275
6	4.167	330
7	5.556	315
8	16.667	325
9	111.111	340
10	222.222	350

Table 3. Discretization of the matrix uniaxial stress strain curve to ten surfaces.

Using the thin-wall assumption, the shear stress distribution in a tubular specimen was obtained in [Lease 1994; Wu et al. 1992] as

$$\tau = \frac{T}{2\pi r_m^2 t}, \tag{29}$$

where r_m and t are the mean radius and wall thickness respectively. Equation (29) is applicable both to elastic and elastic-plastic material behavior in the outer region. The shear stress obtained in Equation (29) is normally considered to correspond to the strain at the mid-surface of the smooth tubular specimen. Consequently, the smaller the wall thickness of the specimen, the higher the accuracy of the results obtained. For the constitutive model verifications, if the smooth tubular specimen is subjected to an axial load P, the stress distribution is uniform. It can be obtained from

$$\sigma = \frac{4P}{\pi(d_o - d_i)^2}, \tag{30}$$

where σ is the uniform axial stress, and d_o and d_i (shown in Figure 1) are the outer and inner diameters of the tubular specimen in the gage section respectively. It should be noted that Equation (30) applies to both elastic and elastic-plastic material behavior.

A servo-hydraulic biaxial load frame [Instron 1992] was used to apply the loads. The frame can apply axial and/or torsional loads both monotonically and cyclically. It has an axial (P) and a torsional (T) load capacity of ± 250 kN and ± 2500 Nm respectively. Experimental tests were conducted using three loading paths: a biaxial proportional loading path (Figure 2) and two variable amplitude nonproportional loading paths (Figure 3).

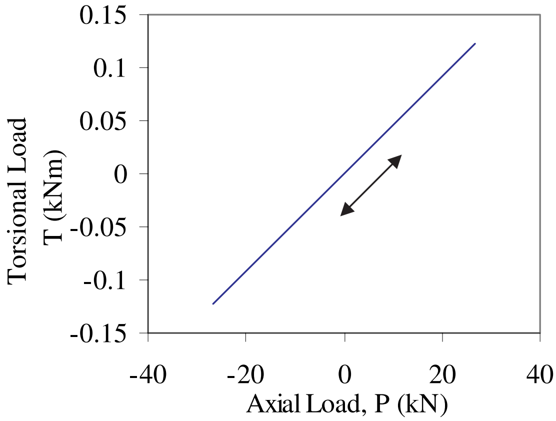

Figure 2. Cyclic proportional tension-torsion $(p\text{-}t)$ loading path.

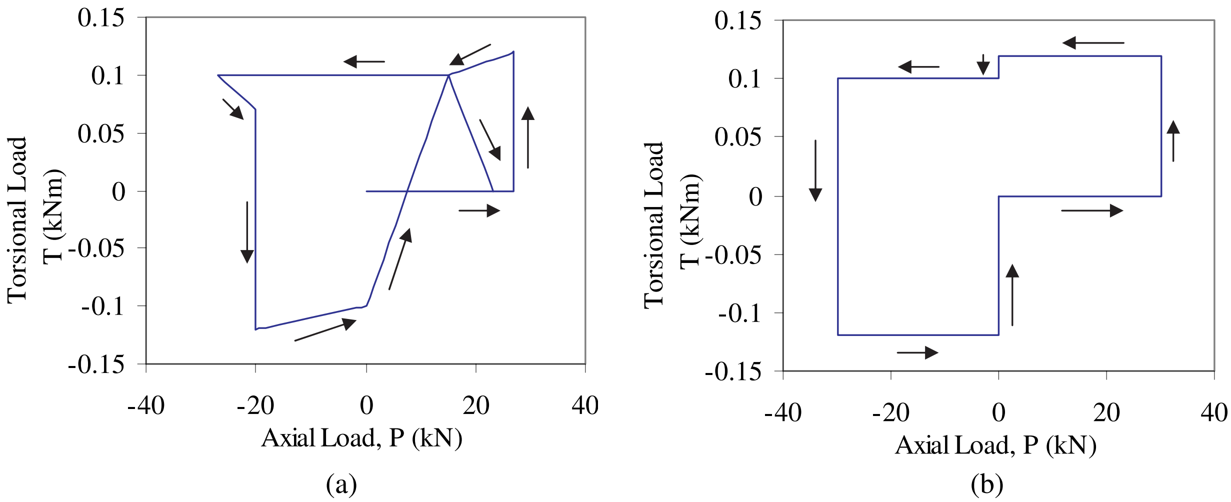

Figure 3. Variable amplitude nonproportional loading path: (a) path 1; (b) path 2.

A 3D image correlation technology system [Aramis 2003] was used to obtain both the axial and the shear strains on the exterior surface close to the middle section of the specimens gage length. The system has the capacity to measure surface strain fields with high resolution; the strain sensitivity is in the range of 50–100 microstrains. The equipment also has a measurement sensitivity of 1/30,000 field of view and thus provides extensive strain measurements at the gage length. To measure the 3D deformations and surface strains, a random or regular pattern is first applied to the area of the specimen under investigation. A typical pattern consists of a white dye penetrant developer, and black spray paint. The white dye is applied first, allowed to dry, then followed with very black spray paint. The pattern should exhibit a high contrast to the surface otherwise the matching of the captured images, from which the displacements and the surface strains are obtained, cannot be carried out correctly [Tyson et al. 2002]. As the loads are applied, the pattern deforms with the test specimen. An image of the deforming pattern is then captured either manually or automatically at desirable load or time intervals by a pair of high resolution digital cameras. The cameras enable the 3D image correlation system to register the 3D shape of the object. The initial image processing defines unique correlation areas known as macro-image facets, typically 5–20 pixels square across the entire imaging area. The center of each facet is a measurement point that can be thought of as an extensometer or a strain rosette. The system processes and visualizes the data gathered in order to obtain an impression of the distribution of strains in the object. It recognizes the surface of the specimen in digital images, and attributes coordinates to every pixel in the image. The system tracks the stochastic pattern applied to the measured surface with subpixel accuracy. Hence localized deformation can be tracked as long as the test specimen remains within the cameras field of view. Using photogrammetric principles [Aramis 2003] and image processing, the specimen's 3D coordinates, 3D displacements, and the surface strain field are automatically calculated using the associated software during the post-processing stage.

4. Results and discussion

The results for the two constitutive models are compared to the experimental results obtained from the tests conducted on the tubular specimens using the tension-torsion load paths in Figures 2 and 3. Note that all the relations presented in the previous sections are valid for cyclic nonproportional (constant or variable amplitude) loading. Either model can be employed knowing the material properties, and either the stress or strain history. Here, the models are demonstrated specifically for load-controlled simulations. In the numerical implementations, equations (29) and (30) are employed to obtain the composite experimental torsional and axial stresses respectively from the applied combined torsion (T) and Tension (P) loads. The elastic-plastic composite strains are subsequently obtained from the known stresses. The strain results obtained are compared with experimentally determined strains.

Figure 4 shows the predicted and experimental results for cyclically stable combined axial/torsional proportional loading (Figure 2) of the tubular specimens. The results of this proportional loading path show the ability of both models to predict the elastic-plastic hysteresis loops associated with cyclic loading. Most of the characteristics exhibited by the experimental results are reflected quite well by both models. However, the major disagreement that can be classified as quantitative is the difference in strain levels prescribed at high plastic strains. The results indicate the Mróz-based PMMCs constitutive model

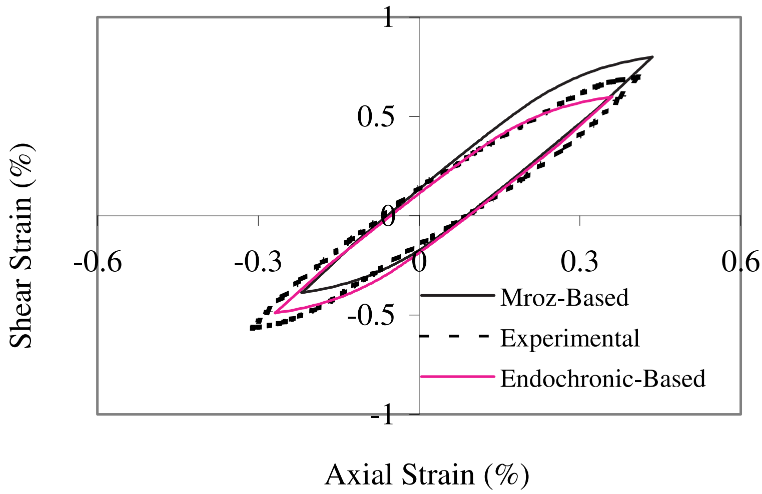

Figure 4. Experimental and simulated shear strains versus axial strains for hollow tube specimen under proportional cyclic load (10% volume fraction).

slightly over-predicts, while the endochronic theory-based PMMCs constitutive model slightly under-predicts the experimental shear strains. The difference between the predicted and experimental results for the endochronic theory-based PMMCs constitutive model may be due to the approximate method used to obtain the material constants. For the Mróz-based PMMCs constitutive model, the number of surfaces used in the multisurface model could have influenced the results. The number of surfaces used may have a significant effect on the direction of translation of the yield surfaces, and thus, affect the predicted results.

Figures 5 and 6 show the strain response of the tubular specimens predicted by the experimental and proposed models to the variable amplitude loading paths shown in Figure 3. Both models predict very similar strain responses for the loading paths and also give reasonable qualitative estimations of the measured strains. However, both models sometimes over-predict and at other times under-predict the measured strains. Generally, the endochronic theory-based PMMCs constitutive model predictions are closer to the experimental results than the Mróz-based PMMCs constitutive model predictions, particularly for the measured peaks and valleys that are essential for fatigue life prediction.

In the load paths considered, both models produce a reasonable qualitative and quantitative response of the composite behavior. The Mróz-based PMMCs constitutive model requires a large number of surfaces and a clearly defined yield point to give good results, while in the endochronic theory-based PMMCs constitutive model, determining the material constants necessary for its implementation is a very difficult task. The Mróz-based PMMCs constitutive model seems to be mathematically more complex than the endochronic theory-based PMMCs constitutive model. In general, the endochronic theory-based PMMCs constitutive model predictions are closer to the experimental results than the Mróz-based PMMCs constitutive model predictions for the load path tested. Thus, the endochronic theory-based PMMC constitutive model was used in [Owolabi and Singh 2006] to provide some of the relations that are necessary in defining notch-root stresses and strains in PMMC components with geometric discontinuities.

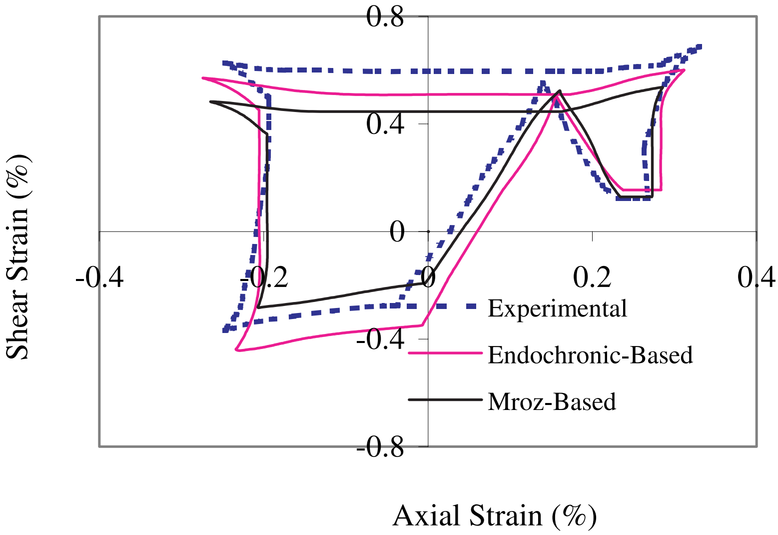

Figure 5. Experimental and simulated shear strains versus axial strains for hollow tube specimen under a variable amplitude load path (Figure 3a, 10% volume fraction).

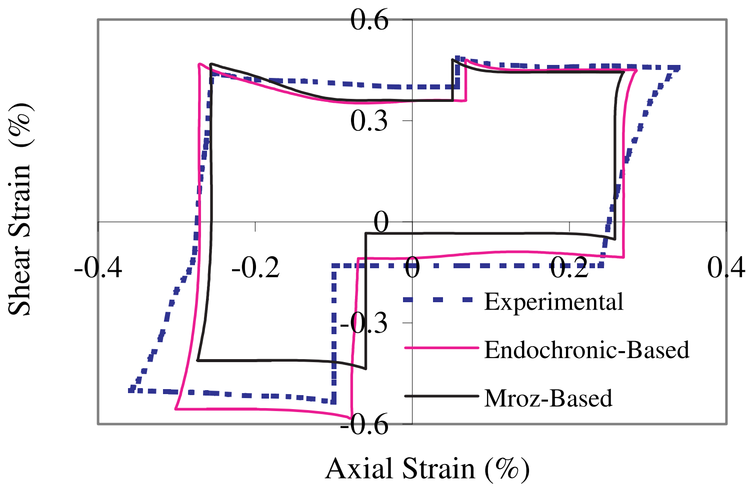

Figure 6. Experimental and simulated shear strains versus axial strains for hollow tube specimen under a variable amplitude load path (Figure 3b, 20% volume fraction).

The differences between the Mróz-based PMMCs constitutive model predictions and the experimental results may be due to the influence of the number of surfaces on the direction of translation of the backstress tensors of the yield surfaces. In addition, the number of load increments used may affect the predicted results. Theoretically, the Mróz-based PMMCs constitutive model can be used with large increments, however, as with other cyclic plasticity models, care must be taken in the specification of the input load increments. Since a quantitative relation between the number of surfaces, load increments, and the predicted results is difficult to formulate, efforts were made in this research to use a combination of the number of yield surfaces and load increments that gave optimal and convergent results. Generally, for small load levels, the number of loading increments and/or yields surfaces used may not significantly affect the predicted results. However, for high load levels, the number of loading increments and/or yield surface may significantly affect the magnitudes of the predicted results. To address this limitation one can use the two-surface model based on the Mróz model and developed in [Dafalias and Popov 1975; McDowell 1985a; 1985b; Itoh et al. 2000].

5. Conclusions

In this paper, two elastic-plastic constitutive models were evaluated for their applicability to model the behavior of PMMCs under complex loading conditions. Details of the experimental and numerical results that demonstrate the basic qualitative and quantitative aspects of the cyclic plasticity models were presented. For most of the investigated loading paths, both models predict satisfactorily the amplitudes of the experimental strains and qualitatively predict reasonably the characteristics features of the experimental results. However, the endochronic theory-based constitutive model generally gives better qualitative and quantitative predictions of the measured strains. This is the first attempt to incorporate two cyclic plasticity routines into the development of elastic-plastic constitutive relations for PMMCs components particularly under multiaxial variable amplitude loading conditions. It is important to state that any plasticity model that incorporates path dependent material behavior may be used. However, the constitutive models used in this study provide a simple to implement explicit numerical algorithm valid for stress-controlled simulations. While the models are associated with some limitations and thus are not expected to be pertinent to all possible cyclic loading conditions, the results obtained, however, provide solid foundation for further and more systematic experimental and theoretical investigations.

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